Research Statement

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Background

As an extended concept of Fourier analysis, wavelet analysis has been developed by pure and applied mathematicians, physicists, and engineers. In Fourier analysis periodic and square integrable functions are expanded in terms of the wave functions e^{inx}. The idea of wavelet analysis is to replace the non-compactly supported wave functions e^{inx} by wavelets, which are wave functions that have compact support or decay fast to zero at infinity. The wave functions e^{inx} of different frequencies are obtained from the function e^{ix} by dilation. Analgously, in wavelet analysis one starts with one wavelet whose dilations and translations can generate any function in a L^2 space. The construction and understanding of wavelet bases are stimulated by the multiresolution analysis (MRA) formulated by Mallat and Meyer in 1986. The example which motivated the formulation of MRA is the Haar function (1910)

$$\psi(x) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}), \\ -1, & \text{if } x \in [\frac{1}{2}, 1), \\ 0, & \text{otherwise,} \end{cases}$$

whose binary dilations and dyadic translations $2^{j/2}\psi(2^j x - k)$ form an orthonormal basis for $L^2(\mathbb{R})$. In 1988, Daubechies constructed the first example of a *continuous* and compactly supported wavelet generating an orthonormal basis for $L^2(\mathbb{R})$.

Wavelet analysis is not only an interesting subject for pure mathematicians, but also draws a lot of attention from researchers who are studying image and signal processing. Wavelet analysis is one of the few areas we can easily see the application of mathematics theory to the real world. When we use wavelet functions to transfer data representing signal or image, at every level the wavelet function catches the difference of the data between the level and the next finer level. This property of a wavelet function gives a high compression rate. FBI fingerprint data compression by using wavelet functions is a well known application. The JPEG algorithm uses wavelet functions for image compression.

A tight wavelet function ψ is a generalized wavelet function whose binary dilations and dyadic translations in $L^2(\mathbb{R}^d)$, but not necessarily orthonormal among its translates and dilations. I am interested in compactly supported tight wavelet frames whose associated refinable function generates MRA and their construction.

By generalizing the orthonormal condition of wavelets, biorthonormal wavelets, prewavelets, tight wavelet framelets and their applications to signal and image processing were also studied by researchers. We can get some advantages in applications and flexibilities of construction of tight wavelet frames over orthonormal and biorthogonal wavelets. For example, unlike biorthnormal wavelets, the self duality of a tight wavelet frame keeps the compact supported property which is important for computation. The redundant representation of tight frame is also useful for the reconstruction of signal from corrupted information. Thus the construction of a tight wavelet frame draws attentions from researchers recently.

My Work

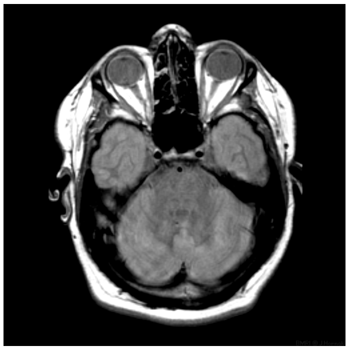
In my dissertation I construct several new box spline tight wavelet framelets by using a method that was introduced by Lai and Stöcler in [10]. This method gives tight wavelet frames for any refinable function whose associated trigonometric function $P(\omega)$ satisfies the condition (called the Quadrature Mirror Filter condition):

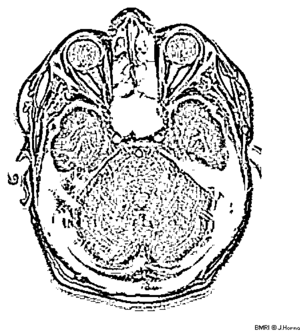
$$1 - \sum_{\ell \in \{0,\pi\}^d} |P(w+\ell)|^2 \geq 0.$$

Solving a system of nonlinear equations I found explicit trigonometric polynomials $\tilde{P}_j(\omega)$ satisfing $1 - \sum_{\ell \in \{0,\pi\}^d} |P(w+\ell)|^2 = \sum_j |\tilde{P}_j(\omega)|^2$. With these explicit trigonometric polynomials $\tilde{P}_j$'s I could apply the constructive scheme in [10]. I obtained compactly supported tight wavelet frames of various order by using bivariate box spline scaling functions on three, four and eight direction meshes [cf. 12].

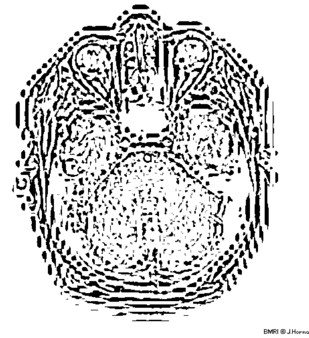
I then use each box spline tight wavelet frame for image decomposition and reconstruction. In particular, I use each box spline tight wavelet frame for image edge detection. The idea of tight wavelet frames for image edge detection is as follows. A low pass filter (a trigonometric polynomial associated with a refinable function) extracts the smooth (low frequency) part of signals or images and high pass filters (trigonometric polynomials associated with wavelet function) picks the changes (high frequency) of signals or images. We can interpret an edge of image as points where the color intensity has sharp changes. If we apply a wavelet function to decompose the image into one low frequency part and several high frequency parts and to reconstruct the image by using only high frequency parts, then we get only edges of the image. Instead of wavelets, I use box spline tight framelets for image edge detection. Numerical experiments to many different images show that a box spline tight frame detects curved and smooth edges better than edge detection by using Haar wavelets, Daubechies wavelets or Laplacian method (see the images on the next page).

Image denoising using box spline tight frames to different images with gaussian noise have slightly higher PSNR number than its denoising using orthonormal wavelets according to numerical experiments. I observed each high pass filter of a box spline tight wavelet frame can catch more different directions of changes than its orthonormal wavelet can and decomposed images by similar looking antisymmetric high pass filters of box spline tight wavelet frame have already less noise.

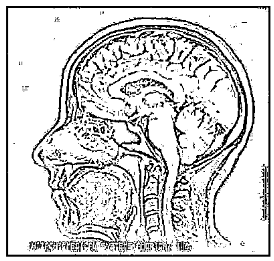




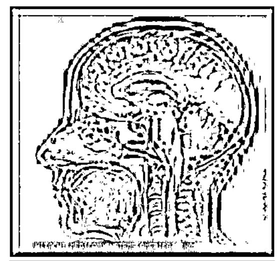
BS 2211 Tight Frame



Daubechies Wavelet D6



BS 2211 Tight Frame



Daubechies Wavelet D6

Further Directions

I want to develop an algorithm of finding trigonometric polynomials such as $\tilde{P}_j$'s mentioned above. The redundancy property of tight wavelet frames could be advantage or disadvantage of image processing over using orthonormal wavelets. Because of the redundancy property of tight wavelet frames, after we decompose an image by a tight wavelet frame we might have some advantages of choosing only part of high frequency images instead of using all of them when we reconstruct images. I will find more applications of using tight wavelet frames.

I will keep working on image denoising experiments using box spline wavelet frames.

Orthonormal wavelets, biorthonormal wavelets and prewavelets construction on Sobolev spaces in one variable setting have been studied [8] and my advisor professor Lai. There are difficulties of generalizing these settings to multivariate case in Sobolev space. I want to study on generalizing these settings to multivariate case. I also interested in working on a tight wavelet frame construction in one variable Sobolev space and further more in multivariate Sobolev spaces.

Currently I am working with my advisor professor Lai on construction of tight wavelet frames over bounded domains. I am using B-spline scaling functions and box spline scaling functions to construct tight wavelet frames over bounded domains. I have result of its construction for linear b-spline scaling function. I am planning further research on bounded domain tight wavelet frame construction for quadratic and cubic B-spline and box spline scaling functions and its application for image processing.

Bibliography

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