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Calculating Group Cohomology: Tests for Completion

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We want to calculate generators and relations for the mod-p cohomology rings of finite groups using computer technology. For this purpose we develop interactive tests to check whether a specific calculation is complete, in the case that the group is a p-group. The method involves checking some conjectures about the nature of the cohomology rings.

1. Introduction

Suppose that G is a finite p-group and that k is a field of characteristic p. In the past few years I have been writing computer programs for performing homological algebra in the category of kG-modules. I am particularly interested in computing projective resolutions of modules and chain maps between resolutions and cup products. A major focus has been the development of programs to calculate generators and relations for the cohomology ring $H^*(G, k)$ of a p-group. This project, which is described in Section 2 is similar to that outlined in Carlson *et al.* (1997). All of my computer programs are written in the MAGMA language and supported by that system (Bosma and Cannon, 1996). However, stand-alone versions of a part of the system have also been written by Green (1997).

The main purpose of this paper is to address another problem which arises in the calculation. It is that the cohomology ring, $H^*(G, k)$, is an infinite object but any calculation of it is necessarily finite. So the question is: when do we know that we are done? One method would be to check the calculated answer against various spectral sequences that might be available. For groups which are not too large, this method is not impractical. Indeed, Rusin's (1989) impressive calculations of the mod-2 cohomology rings of the 51 groups of 32 were performed using the Eilenberg–Moore spectral sequence. However, many of the cohomology rings of the 267 groups of order 64 are much more complicated. Indeed, the method is not very appealing since there are choices to be made in the spectral sequences that always seem ad hoc and the calculations must be completed by hand. It would be much better to have a method which could be implemented on a computer.

The scheme which we propose in this paper is actually based on a couple of conjectures about the nature of the cohomology rings. Several justifications for this approach can be made. First of all, in the course of the calculation the computer checks that the cohomology rings really do satisfy the conjectures. Hence there is no ambiguity in the final result. The conjectures hold in all of the cases that we have looked at to this point. Moreover, the conjectures have been proved for some classes of groups. However, the most important justification is that checking conjectures is one of the aims of the

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project. In fact, some of the technicalities of the conjectures came from looking at the data of these calculations. We hope that the system will be useful for testing and refining other ideas.

The organization of the paper is roughly as follows. In Section 2, we describe the calculation of the cohomology ring. Here, the primary interest is the explicit description of the output in terms of the actual structure of the cohomology ring. In Sections 3 and 4 we present the conditions, one for generators and one for relations, which are conjectured to hold for cohomology rings of p-groups. Finally, in Section 5 we show how the two conditions are related through the hypercohomology spectral sequence. Roughly, the main theorem says that if both conditions are verified to hold out to a certain degree in the cohomology, then they must hold in general and the calculation is complete.

For general references for group cohomology we refer the reader to Benson (1991), Carlson (1996) or Evens (1991). Mostly, the reader will only need to know some standard homological algebra as in Hilton and Stammbach (1971).

The functions for computing cohomology rings have also been used to calculate the cohomology of some sporadic simple groups (Adem *et al.*, 1998; Carlson *et al.*, 1998). We hope that these functions will be included in the MAGMA package in the near future. We are also trying to run the mod-2 cohomology rings of all groups of order 64. Most of this has been accomplished but a few cohomology rings have been particularly difficult to obtain. The outputs from the calculations, that have been made up to this time, have been posted on the author's web page (`http://www.math.uga.edu/~jfc`).

2. Computing Cohomology

In this section we present a brief outline of the steps in computer calculations of a cohomology ring $H^*(G, k)$. Throughout the section and the rest of the paper we assume that G is a p-group and k is a field of characteristic p, for p a prime number. As stated earlier, the methods we use are similar to the ones outlined in Carlson *et al.* (1997). One of the main differences is that in Carlson *et al.* (1997) we replaced the group algebra with its basic algebra in order to reduce the size of the calculation. However, for a p-group the group algebra is isomorphic to its basic algebra and there is nothing to be gained by making the replacement. Another difference is that we have chosen to do the computations numerically to the extent possible. That is, the elements of the group algebra are represented by vectors in a k-vector space rather than as classes in a polynomial ring. Hence products are achieved by matrix multiplication and the noncommutative Gröbner basis algorithms used in Carlson *et al.* (1997) are not required. This should have the advantage of giving faster running times, particularly for small fields. However, more memory may be required for some calculations. It will be interesting to make comparisons once the new implementation of Ed Green's Gröbner basis material is available.

The first step in the computation is the creation of a "standard" free module for the group algebra kG. That is, we construct a representation of kG as a kG-module which is standard in the sense that it is fixed and to be used for all homological calculations for G. Calculations performed with different standard free modules are not comparable.

Basically, we create the module by first choosing for the group a set of PC-generators $x_1, \ldots, x_n$. This is a set of generators with the property that for each i between 1 and n the elements $x_{i+1}, \ldots, x_n$ generate a normal subgroup of order p^{n-i}. Next we create a vector space V of dimension $|G|$, the order of G, and let the jth vector in the standard basis for V represent the element.

$$\Pi(x_i - 1)^{j_i} \in kG,$$

where $j - 1 = j_1 + j_2 p + \cdots + j_n p^{n-1}$, for $0 \leq j_i < p$. That is, we are choosing a specific representation for the group algebra in such a way that each element of kG is represented by a vector of length $|G|$ and determined by the PC-generators. The advantage to this representation is that if (as often happens) $x_i^p = 1$, then $(x_i - 1)^p = 0$. Therefore elements which are contained in a high power of the radical of kG are represented by vectors having a large number of zeros as entries.

Of course, the representation depends on the choice of the PC-generators. In the case that G is not given in PC-form it is important that these data be computed only once. If G is given by generators $y_1, \ldots, y_r$, then the resolution input function, which creates the standard free module, returns the matrices of the actions of the elements $y_1, \ldots, y_r$ as well as the matrices of the computed PC-generators. To save time in other parts of the calculation it is useful to have both a minimal set and a PC-set of generators for G.

For the next stage, let M be a kG-module given by the action of matrices for $y_1, \ldots, y_r$ on the vector space of the module. Our aim is to get a minimal projective resolution of M. First find a basis for the radical

$$\operatorname{Rad} M = \sum_{i=1}^{r} M(y_i - 1).$$

This means finding a basis for the sum of the row spaces of the matrices of $y_1 - 1, \ldots, y_r - 1$. A minimal generating set for M as a kG-module is a basis for a subspace of M complementary to $\operatorname{Rad} M$.

Let $m_1, \ldots, m_s$ be a minimal set of generators for M. Let $P_0 = \oplus \sum_{j=1}^{s} kGu_j$ be a free kG-module with free kG-basis $u_1, \ldots, u_s$. For practical purposes P_0 is just a direct sum of s copies of the standard free module. A projective cover of M is the homomorphism

$$\theta : P_0 \longrightarrow M$$

given by $\theta(u_i) = m_i$, for $i = 1, \ldots, s$. The matrix for θ is easily constructed, since the jth vector in the standard basis for $kG \cdot u_i$ has the image

$$\theta\left(u_i \cdot \sum (x_i - 1)^{j_i}\right) = m_i \cdot \sum (x_i - 1)^{j_i},$$

for $j - 1 = j_1 + j_2 p + \cdots + j_n p^{n-1}$, as before.

Next we compute the kernel of θ, $\Omega(M)$, which as a vector space is the null space of θ. Note here that $\Omega(M) \leq P_0$, so that it is not necessary to actually realize the null space as a kG-module. That is, the action of G is known from the standard free module, and we can avoid the costly process of making matrices for the generators of G on $\Omega(M)$.

Now that we have the null space we can obtain its radical and minimal set of generators by the same process as before. Also, as before, we construct a projective cover for $\Omega(M)$. From the composition we obtain the first step in the projective resolution

$$\partial : P_1 \longrightarrow P_0,$$

where $P_1 = \sum_{i=1}^{t} kGv_i$ is a projective cover of $\Omega(M)$. Note that for each i, $\partial(v_i) = \sum_{j=1}^{s} \alpha_{ij} u_j$ for $\alpha_{ij} \in kG$. The only thing which we record of ∂ is the list of vectors α_{ij}. That is, the output of the projective resolution function consists of list of ranks, (s, t, etc.) of the projective modules and the list of vectors α_{ij} which are stored as the rows of a large matrix.

If the process just described is repeated N times then the output represents a portion

$$P_N \xrightarrow{\partial_N} P_{N-1} \longrightarrow \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \longrightarrow M \longrightarrow 0$$

of the minimal projective resolution of M. The matrices of the boundary maps can be easily recreated from the stored data.

The next stage is the computation of chain maps for the purposes of determining cup products. At present these routines are only implemented for the case that $M = k$ is the trivial kG-module. In this case, because of the minimality of the resolution and the irreducibility of the trivial module, we have that

$$H^n(G,k) \cong \mathrm{Ext}^n_{kG}(k,k) \cong \mathrm{Hom}_{kG}(P_n,k)$$

for all n. That is, a cohomology element $\zeta \in H^n(G,k)$ can be considered to be a homomorphism $\zeta : P_n \to k$. Such an element can always be lifted to a chain map $(\tilde{\zeta}_0, \tilde{\zeta}_1, \ldots)$ as in the diagram:

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & P_{n+1} & \longrightarrow & P_n & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & k & \longrightarrow & 0 \\ & & \tilde{\zeta}_1 \downarrow & & \tilde{\zeta}_0 \downarrow & \searrow \zeta & & & & & & & \\ \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & k. & & & & & & \end{array}$$

The process, which is adequately described in Carlson *et al.* (1997), is similar to the construction of the maps in the projective resolution. The main difference is that constructing $\tilde{\zeta}_i$ so that $\tilde{\zeta}_i\partial = \partial\tilde{\zeta}_{i-1}$ means solving a system of linear equations. Again only the images of the identity element in the copies of kG are saved.

Of course, the point of this exercise is that the chain map of the cup product of two cohomology elements is chain homotopic to the composition of the chain maps. The chain homotopy does not present a problem because its image is always in the radical and this part will be discarded when it comes to computing generators and relations.

Now note that the dimension of the cohomology group $H^n(G,k) = \mathrm{Hom}_{kG}(P_n,k)$ is equal to the number of copies in kG in P_n. In practice, we only compute the chain maps for a minimal set of generators of the ring $H^*(G,k)$. That is, by calculating the chain maps for generators of degrees less than n we can see which elements in degree n are cup products of elements in smaller degrees and how many new generators are needed.

At this point we have a set of generators for cohomology and their chain maps. To obtain the relations we should first note that if $\gamma : P_n \to k$ is a cohomology element, then $\gamma(\mathrm{Rad}\, P_n) = 0$. So it is really only necessary to look at the chain map as a sequence of homomorphisms $(\hat{\gamma}_0, \hat{\gamma}_1, \ldots)$ where

$$P_{n+r}/\,\mathrm{Rad}\, P_{n+r} \xrightarrow{\hat{\gamma}_r} P_r/\,\mathrm{Rad}\, P_r.$$

The actual calculation of the relations is obtained as follows. Let $x_1, \ldots, x_m$ denote the generators of the cohomology. In degree n, form all monomials $x_1^{r_1} \ldots x_m^{r_m}$ whose degree $\sum r_i \deg(x_i) = n$. Each of these is represented uniquely by a homomorphism from $P_n/\,\mathrm{Rad}\, P_n$ to $P_o/\,\mathrm{Rad}\, P_0 \cong k$. That is, each is an element of the dual space $(P_n/\,\mathrm{Rad}\, P_n)^*$. Hence the relations in degree n are the k-dependence relations among these vectors in the dual space. In general, the method produces many more relations than is necessary. Gröbner basis techniques can be used to reduce the set of relations to a manageable collection.

So what do we have? First we have a graded-commutative "polynomial ring" $Q = k[x_1, \dots, x_m]$. We also have an ideal I of relations which have been computed. We need to be careful at this point because cohomology rings satisfy the relations that $xy = -yx$ if x and y are elements of odd degree. So if $p > 2$, then the ring Q actually has the form $Q = k[y_1, \dots, y_r] \otimes \Lambda(z_1, \dots, z_s)$ where $y_1, \dots, y_r$ are the generators in even degrees and Λ is an exterior algebra in the generators $z_1, \dots, z_s$ in odd degrees. Here $\{x_1, \dots, x_m\} = \{y_1, \dots, y_r\} \cup \{z_1, \dots, z_s\}$. For convenience of notation we continue to write Q as $k[x_1, \dots, x_m]$ as if it were a polynomial ring. It is still a free object in the category of graded-commutative polynomial rings.

In practice we can avoid some problems by always computing the monomials which are used to obtain the relations with a specific order on the generators. Then a person reading the data must be aware that there is an extra set of relations ($yx = (-1)^{\deg(x)\deg(y)} xy$) arising from the graded commutativity. Of course, if $p = 2$ then none of this matters.

The following should now be clear from the discussion.

Theorem 2.1. *Suppose that R is the result of a computation to degree N of a cohomology ring $H^*(G,k)$. Then R satisfies the Standard Hypothesis given below for $R = R_N$.*

Standard Hypothesis 2.2. *Let N be a fixed integer and let $P = k[x_1, \dots, x_n]$ be a graded-commutative polynomial ring (see the above discussion). Assume that the generators are numbered so that $\deg(x_i) \le \deg(x_j)$ if $i < j$. Let I be the ideal in P so that*

$$H^*(G,k) = P/I.$$

Let $R_N = Q/J$ where the following hold.

(1) *$Q = k[x_1, \dots, x_m] \subseteq P$ is the subring generated by all x_j with $\deg(x_j) \le N$. So $\deg(x_j) > N$ if $j > m$.*
(2) *J is the ideal of Q generated by all homogeneous elements of I of degree at most N.*

Note that $R = R_N$ in the Standard Hypothesis is a graded ring and has the property that

$$R^n = H^n(G,k)$$

for all $n \le N$.

3. Generators

Let $R = \sum_{i \ge 0} R^i$ be a graded-commutative finitely generated k-algebra. Let $\chi = \chi_R$ be its maximal ideal spectrum and for an ideal $I \subseteq R$, let $\chi(I) \le \chi$ be the closed set of all maximal ideals in R which contain I.

Definition 3.1. Let $r = \operatorname{Dim} \chi_R$. A homogeneous set of parameters for R is a sequence $\zeta_1, \dots, \zeta_r$ of homogeneous elements of positive degree with the property that $\chi(I) = \{0\}$ for $I = (\zeta_1, \dots, \zeta_r)$.

Here the zero point of χ is the maximal ideal of all elements of positive degree. An equivalent definition of a homogeneous set of parameters is sequence $\zeta_1, \dots, \zeta_r$ with the property that R is finitely generated module over $k[\zeta_1, \ldots, \zeta_r]$. Note that because $r = \mathrm{Dim}(\chi_R)$, the subring generated by a homogeneous set of parameters is a polynomial subring.

A sequence $\zeta_1, \ldots, \zeta_t \in R$ is a regular sequence provided ζ_1 is a regular element of R (i.e. multiplication by ζ_1 is an injective map from R to R) and for each $i = 2, \ldots, t$, ζ_i is a regular element on $R/(\zeta_1, \ldots, \zeta_{i-1})$. Any regular sequence can be filled out to obtain a homogeneous set of parameters. The depth of R is the length of the longest regular sequence in R. If the depth of R is equal to $r = \mathrm{Dim}(\chi_R)$ the R is said to be Cohen–Macaulay.

In the case that $R = H^*(G, k)$ we let $V_G(k)$ denote the maximal ideal spectrum. We also fix the notation $\tau = \tau(G)$ for the dimension of $V_G(k)$. The following facts are well known.

Proposition 3.2. (i) *(See Benson and Carlson, 1994). A sequence of homogeneous elements $\zeta_1, \ldots, \zeta_\tau \in H^*(G, k)$ is a homogeneous set of parameters iff for every elementary Abelian p-subgroup $E \subseteq G$, the quotient*

$$H^*(E, k)/(\mathrm{res}_{G,E}(\zeta_1), \ldots, \mathrm{res}_{G,E}(\zeta_\tau))$$

is a finite-dimensional algebra. (Here $\mathrm{res}_{G,E}$ is the restriction map.) The latter happens iff the dimension of the ideal $(\mathrm{res}_{G,E}(\zeta_1), \ldots, \mathrm{res}_{G,E}(\zeta_\tau))$ is zero.

(ii) *(See Broto and Hen, 1993; Duflot, 1981). A sequence $\zeta_1, \ldots, \zeta_d \in H^*(G, k)$ is a regular sequence if the sequence $\mathrm{res}_{G,E}(\zeta_1), \ldots, \mathrm{res}_{G,E}(\zeta_d)$ is a sequence of parameters for the cohomology ring $H^*(E, k)$ where E is the unique maximal elementary Abelian p-subgroup of $Z(G)$. In particular, the depth of $H^*(G, k)$ is at least equal to the p-rank of the center of G.*

Remark. One importance of the above conditions is that they are easily checked in a computational situation.

Suppose that $\zeta_1, \ldots, \zeta_r$ is a sequence of homogeneous elements in R. Associated to the sequence is a Koszul complex

$$\mathcal{K}(\zeta_1, \ldots, \zeta_r; R) : 0 \longrightarrow \mathcal{K}_0 \longrightarrow \mathcal{K}_1 \longrightarrow \cdots \longrightarrow \mathcal{K}_r \longrightarrow 0$$

in which each $\mathcal{K}_i = R^t$ is a direct sum of $t = \binom{n}{i}$ copies of the ring R. For our purposes it is sufficient to define the Koszul complex as a tensor product $\mathcal{K}(\zeta_1, \ldots, \zeta_r; R) = \mathcal{K}^{(1)} \otimes \mathcal{K}^{(2)} \otimes \cdots \otimes \mathcal{K}^{(r)}$ where $\mathcal{K}^{(i)}$ is given by

$$\mathcal{K}^{(i)} = \mathcal{K}(\zeta_i; R) : 0 \longrightarrow \mathcal{K}_0^{(i)} \xrightarrow{\hat{\zeta}_i} \mathcal{K}_1^{(i)} \longrightarrow 0$$

for $\mathcal{K}_0 \cong \mathcal{K}_1 \cong R$ and boundary map is multiplication by ζ_i. For notation, let $u_i \in \mathcal{K}_0^{(i)}$, $v_i \in \mathcal{K}_1^{(i)}$ be the identity elements so that $\hat{\zeta}_i(au_i) = a\zeta_i v_i$. Of course, the tensor products are taken over the base field k, and the differential on the whole Koszul complex is the usual one for a tensor product of complexes.

The reader will note here, that we are writing the Koszul complex as a cochain complex rather than the more standard chain complex. The reason for this should become clear in the last section. Because of this notation the "homology" of the Koszul complex should be cohomology and so we write H^n rather than H_n. We hope this causes no confusion.

Proposition 3.3. *Suppose that $\zeta_1, \ldots, \zeta_r$ is a sequence of homogeneous elements in R. Assume that $\zeta_1, \ldots, \zeta_d$ is a regular sequence for R. Then for all i, we have that*

$$H^{i+d}(\mathcal{K}(\zeta_1, \ldots, \zeta_r; R)) \cong H^i(\mathcal{K}(\zeta_{d+1}, \ldots, \zeta_r; R/(\zeta_1, \ldots, \zeta_d))).$$

Proof. We have a chain map of complexes

$$\begin{array}{ccccccccc} \mathcal{K}(\zeta_1, R): & 0 & \longrightarrow & R & \xrightarrow{\zeta_1} & R & \longrightarrow & 0 \\ \downarrow \mu & & & \mu_0 \downarrow & & \mu_1 \downarrow & & \\ C(R/\zeta_1): & 0 & \longrightarrow & 0 & \longrightarrow & R/\zeta_1 & \longrightarrow & 0 \end{array}$$

where $\mathcal{C}(R/\zeta_1)$ is the complex with only one nonzero term R/ζ_1 in degree 1. Here μ_1 is the natural quotient map. Because ζ_1 is regular, μ induces an isomorphism on cohomology and the kernel of μ is an exact complex ($R \xrightarrow{ID} R$) of projective R modules. Then, letting $\mathcal{K}' = \mathcal{K}^{(2)} \otimes \cdots \otimes \mathcal{K}^{(r)}$, we have an exact sequences of complexes

$$0 \longrightarrow \ker \mu \otimes \mathcal{K}' \longrightarrow \mathcal{K}(\zeta_1; R) \otimes \mathcal{K}' \longrightarrow \mathcal{C}(R/\zeta_1) \otimes \mathcal{K}' \longrightarrow 0,$$

which implies a long exact sequence on cohomology. It is straightforward to show that $H^*(\ker \mu \otimes \mathcal{K}') = 0$. Hence it follows that

$$\begin{aligned} H^i(\mathcal{K}(\zeta_1, \ldots, \zeta_r; R)) &= H^i(\mathcal{K}(\zeta_1; R) \otimes \mathcal{K}') \\ &\cong H^i(\mathcal{C}(R/\zeta_1) \otimes \mathcal{K}') \\ &\cong H^{i-1}(\mathcal{K}(\zeta_2, \ldots, \zeta_r); R/\zeta_1). \end{aligned}$$

The rest follows by induction.□

If we assume that the elements $\zeta_1, \ldots, \zeta_r$ are homogeneous, then the Koszul complex $\mathcal{K}(\zeta_1, \ldots, \zeta_r; R)$ and its cohomology are doubly graded. For this paper we grade the complex in such a way that the differentials are degree zero maps (i.e. bidegree (1,0) maps). That is, the identity element in degree 0, $u_1 \otimes \cdots \otimes u_r \in \mathcal{K}^{0,0}$ is in degree (0,0). So for any i, $\zeta_i u_1 \otimes \cdots \otimes u_{i-1} \otimes v_i \otimes \cdots \otimes u_r$ is in degree $(1, 0)$. Thus $u_1 \otimes \cdots \otimes u_{i-1} \otimes v_i \otimes \cdots \otimes u_r \in \mathcal{K}^{1,-m_i}$ and $v_1 \otimes \cdots \otimes v_r \in \mathcal{K}^{r,-\sum m_i}$ where $m_i = \deg(\zeta_i)$. In this regard, the isomorphism of the last proposition requires a double degree shift

$$H^{i+d,j-w}(\mathcal{K}(\zeta_1, \ldots, \zeta_r; R)) \cong H^{i,j}(\mathcal{K}(\zeta_{d+1}, \ldots, \zeta_r; R/(\zeta_1, \ldots, \zeta_d))),$$

where $w = \sum_{i=1}^d m_i$.

The condition which we wish to investigate is the following. The condition holds for $R = H^*(G, k)$ in all of the cases which we have been able to check.

Condition G 3.4. Let $\zeta_1, \ldots, \zeta_r$ be a homogeneous system of parameters for R. Then

$$H^{*,j}(\mathcal{K}(\zeta_1, \ldots, \zeta_r; R)) = 0$$

whenever $j \geq 0$.

Proposition 3.5. *If $R = H^*(G, k)$ is Cohen–Macaulay or if $H^*(G, k)$ has depth $r - 1$, then Condition* **G** *holds for R.*

Proof. (See Benson and Carlson, 1994) The proof there uses the hypercohomology spectral sequence which is discussed in Section 5.□

Remark. For most groups where the group cohomology is in the computable range it is unlikely that one would need to compute a Koszul complex with $r > 3$. This is particularly true in view of Proposition 3.3. In the low-rank cases there is an easy implementation of the complex. If $r = 1$, then $H^1(K(\zeta_1; R)) = R/\zeta_1 = Q/(J + R\zeta_1)$ while

$$\begin{aligned} H^0(K(\zeta_1; R)) &= \mathrm{Ann}_R(\zeta_1) = [J : R\zeta_1]/J \\ &= \{x \in Q | x \cdot \zeta_1 \in J\}/J. \end{aligned}$$

For $r = 2$ things are slightly more complicated but we can filter the cohomology according to lines $w_1 \otimes w_2$ where $w \in \{u, v\}$. That is, suppose the line for w has the form I_1/I_2 for I_1 and I_2 ideals in Q as above. Then the line corresponding to $w_1 \otimes u_2 \cong I_1'/I_2$ where $I_1' = I_n \cap \mathrm{Ann}_{R/I_2}(\zeta_2)$. On the other hand, the line corresponding to $w_1 \otimes v_2$ is I_1/I_2' where $I_2' = I_2 + \zeta_j R$. It is easy to see that this works for $H^0(\mathcal{K})$ and $H^2(\mathcal{K})$. The degree 1 cohomology is actually filtered: $\{0\} \subseteq$ line of $u_1 \otimes u_2 \subseteq H^1(\mathcal{K})$. That is, the quotient of $H^1(\mathcal{K})$ by the line of $u_1 \otimes u_2$, is the line of $v_1 \otimes u_2$ as computed above.

This process can be iterated at least once more to obtain useful information. That is, when $r = 3$, the cohomology of the Koszul complex injects into the modules filtered as above. However, we may not have an isomorphism. Because our interest is in determining when the cohomology is zero, this may still be helpful.

4. Relations

In this section we consider the conjecture on the relations ideal in the cohomology ring. We assume that the ring R satisfies the Standard Hypothesis (i.e. $R = R_N$ as in (2.2)) to degree N with respect to $H^*(G, k)$. Note that for any subgroup $H \subseteq G$ it makes sense to speak of the restriction map $\theta_H : R \longrightarrow H^*(H, k)$. This restriction is the composition of the natural map $R \longrightarrow H^*(G, k)$ followed by the restriction, $\mathrm{res}_{G,H}$, on cohomology.

Condition R 4.1. Let $\mathcal{E}$ be the intersection of the kernels of the restriction maps $\theta_H : R \longrightarrow H^*(H, k)$ for all maximal subgroups H of G. Then $\mathcal{E} = \{0\}$ unless the depth of R is equal to the p-rank of $Z(G)$. Moreover, there exists a regular sequence $y_1, \ldots, y_d$ in $H^*(G, k)$ with $y_1, \ldots, y_d \in R$ and a regular sequence in R and $d = p$-rank $(Z(G))$. If $\mathcal{E} \neq 0$, then $\mathcal{E} = \sum_{i=1}^{s} T\overline{\alpha_i}$ is a free module over $T = k[y_i, \ldots, y_d]$ with a finite number of generators $\overline{\alpha_i} = \alpha_i + J$, $\alpha_i \in Q$ and $\deg(\alpha_i) \leq N$ for $i = 1, \ldots, s$.

In the case that $R = H^*(G, k)$, $\mathcal{E}$ is known as the essential cohomology. In all of the cases that we have checked $H^*(G, k)$ satisfies Condition **R**. Finally, in the case that the depth of R is equal to the p-rank of the center of G we should note that if we choose a regular sequence for R, then we can guarantee that it is regular sequence for $H^*(G, k)$ by making certain that its restriction is a regular sequence for $H^*(Z(G), k)$ by Proposition 3.2.

Theorem 4.2. *Assume that the ring R satisfies Condition* **R**. *Let $\{\mu_i | i = 1, \ldots, t\}$ be a minimal set of generators for the kernel J' of the composition*

$$Q \longrightarrow R \xrightarrow{\theta_H} \ominus \sum_H H^*(H, k),$$

where the sum is over all maximal subgroups H of G. For each $i = 1, \ldots, t$, there exist *elements $f_{i1}, \dots, f_{is} \in T$ such that*

$$\sigma_i = \mu_i - \sum_{j=1}^{s} f_{ij}\alpha_j \in J.$$

For each $i = 1, \dots, m$ and $j = 1, \dots, s$ there exist $g_{ijk} \in T$ such that

$$\tau_{ij} = x_i\alpha_j - \sum_{k=1}^{s} g_{ijk}\alpha_k \in J.$$

Then the set $\{\sigma_i, \tau_{jk} | i = 1, \dots, t, \quad j = 1, \dots, n, \quad k = 1, \dots s\}$ generates the ideal J.

Proof. The existence of the elements f_{ij} and g_{ijk} is a consequence of Condition **R**. In fact, because $\mathcal{E}$ is a free T-module these coefficients are unique.

Now suppose that $y \in J$. Then $y \in J'$ and there exist elements $b_i \in Q$ such that $y = \sum_{i=1}^{t} b_i\mu_i$. Hence,

$$\begin{aligned} y - \sum_{i=1}^{t} b_i\sigma_i &= y - \sum_{i=1}^{t} b_i \left(\mu_i - \sum_{j=1}^{s} f_{ij}\alpha_j \right) \\ &= \sum_{j=1}^{s} \left(\sum_{i=1}^{t} b_i f_{ij} \right) \alpha_j \in J. \end{aligned}$$

Now note that for each j, $\sum_{i=1}^{s} b_i f_{ij} \in Q$ and $Q = T + (x_1, \dots, x_m)$ as vector spaces. So, modulo the ideal $U = (\tau_{ij})$, $x_i\alpha_j \equiv \sum_{k=1}^{s} g_{ijk}\alpha_k$. In particular we must have that

$$\sum_{j=1}^{s} \left(\sum_{i=1}^{t} b_i f_{ij} \right) \alpha_j = \sum_{k=1}^{s} t_k\alpha_k + u$$

for $t_k \in T$, $k = 1, \dots, s$ and $u \in U$. However, $U \subseteq J$ and $\overline{\alpha}_1, \dots, \overline{\alpha}_s$ are generators of a free T-module. Therefore $t_k = 0$ for all k and $y - \sum_{i=1}^{t} b_i\sigma_i = u \in U$ as desired. This proves the theorem.□

Theorem 4.3. *Suppose that R satisfies the Standard Hypothesis to degree N and also satisfies Condition* **R**. *Suppose that N is at least equal to the maximum of the degrees of a set of homogeneous generators for the ideal $J' = \cap J_H \leq Q$, where each J_H is the kernel of the composition $Q \longrightarrow R \xrightarrow{\theta_H} H^*(H, k)$ for H, a maximal subgroup of G. Then $J = 1 \cap (x_1, \dots, x_m)$ and if $\beta \in I$ and $\deg(\beta) < \min\{\deg(x_j) | j = m+1, \dots, n\}$, then $\beta \in J$.*

Proof. The condition which we impose on N relative to the degrees of the generators of J' is meant to insure that the ideal J' has really been computed. That is, we want to know that the relators $\{\sigma_i\}$ of the previous theorem are elements of J. On the other hand, the elements $\overline{\alpha_i}$ are generators for $\mathcal{E} \cong J'/J$ as a T-module and should not be confused with the generators of J' as an ideal of Q.

As noted in (2.2), $N < \deg(x_j)$ for all $j > m$. Suppose that $\beta \in I$ is homogeneous and that $\deg(\beta) < \deg(x_j)$ for all $j > m$. Then $\beta \in Q$. Also, for all maximal subgroups H we have that $\beta \in J_H$ since $Q \cap I \subseteq J_H$. Hence $\beta \in J'$. If $J' = J$ then we are done. Otherwise this implies that there exist elements $t_k \in T$ such that $\beta - \sum t_k\alpha_k \in J$. Now $\beta \in I$ and $J \subseteq I$ and so $\sum t_k\alpha_k \in I$. We need to show that t_k must be zero for all k.

Now recall that $T = k[y_1, \ldots, y_d]$ where $y_1, \ldots, y_d$ is a regular sequence in $H^*(G, k)$. Hence $H^*(G, k)$ is a free module over T (see (10.3.4) of Evens, 1991). By the hypothesis the images of the elements $\alpha_1, \ldots, \alpha_t$ are k-linearly independent in $H^*(G, k)/(y_1, \ldots, y_d)$ $H^*(G, k)$. The last statement is a consequence of the fact that these elements are k-linearly independent in $R/(y_1, \ldots, y_d)R$ which, in degrees up to N, coincides with $H^*(G, k)/(y_1, \ldots, y_d)$. Therefore $\alpha_1, \ldots, \alpha_t$ are free generators of $H^*(G, k)$ as a T-module. Hence $\sum_{k=1}^{s} t_k \alpha_k \in I$ implies that $t_k = 0$ for $k = 1, \ldots, s$.□

5. A Test for Completion

Let R be a finite-dimensional k-algebra satisfying the Standard Hypothesis (2.2). In this section we want to lay down the rules for a test for when $R = H^*(G, k)$. A major ingredient in the proof is the hypercohomology spectral sequence. The spectral sequence is discussed in some detail in Benson and Carlson (1994) and Carlson (1992), though the reader will note some differences in the presentation here. First, we need some notation.

For a kG-module M let

$$\cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

be a minimal projective resolution. For $n > 0$ we define $\Omega^n(M)$ to be $\ker(\partial_{n-1}) = \partial_n(P_n)$. The minimality of the resolution assures that $\Omega^n(M)$ has no projective submodule or quotient modules. Similarly we can define $\Omega^{-n}(M)$ using injective resolutions. If N is another kG-module, then we let $P\operatorname{Hom}_{kG}(M, n)$ be the subset of $\operatorname{Hom}_{kG}(M, n)$ consisting of all homomorphisms which factor through projective modules. Let

$$\underline{\operatorname{Hom}}_{kG}(M, N) = \widehat{\operatorname{Ext}}^0_{kG}(M, N) = \operatorname{Hom}_{kG}(M, N)/P\operatorname{Hom}_{kG}(M, N).$$

The following results are standard (see Benson, 1991; Carlson, 1996; Evens, 1991).

LEMMA 5.1. (i) *For $n > 0$, $m > 0$,*

$$\underline{\operatorname{Hom}}_{kG}(M, \Omega^{-n}(N)) \cong \underline{\operatorname{Hom}}_{kG}(\Omega^n(M), N) \cong \operatorname{Ext}^n_{kG}(M, N),$$

(ii) $\Omega^m(M) \otimes \Omega^n(N) \cong \Omega^{m+n}(M \otimes N) \oplus \mathrm{proj}$,
(iii) $\operatorname{Ext}^m_{kG}(M, \Omega^{-n}(N)) \cong \operatorname{Ext}^{m+n}_{kG}(M, N)) \cong \operatorname{Ext}^n_{kG}(\Omega^m(M), N)$.

Here $\oplus\,\mathrm{proj}$ means the direct sum with some projective module.

Now suppose that $\zeta \in H^n(G, k)$. Then ζ is represented (uniquely) by a cocycle $\zeta : \Omega^n(k) \longrightarrow k$, which we also call ζ. Let $i : \Omega^n(k) \longrightarrow P_{n-1}$ be the inclusion. Because of the minimality of the projective resolution, P_{n-1} is actually the injective hull of $\Omega^n(k)$. Then we obtain an exact sequence

$$0 \longrightarrow \Omega^n(k) \xrightarrow{\alpha} k \oplus P_{n-1} \longrightarrow L_\zeta \longrightarrow 0,$$

where $\alpha(x) = (\zeta(x), i(x))$ for $x \in \Omega^n(k)$ and L_ζ is the natural quotient.

Now suppose that $\zeta_1, \ldots, \zeta_r$ is a sequence of nonzero homogeneous elements of positive degree in $H^*(G, k)$. Let $m_i = \deg(\zeta_i)$ for all i. For each i we can form the exact sequence

$$0 \longrightarrow \Omega^{m_i}(k) \xrightarrow{\alpha_i} k \oplus \mathrm{proj} \longrightarrow L_i \longrightarrow 0$$

as above with $L_i = L_{\zeta_i}$. From this we obtain a complex

$$\mathcal{C}^{(i)} : 0 \longrightarrow \mathcal{C}^{(i)}_1 \xrightarrow{\partial^{(i)}_1} \mathcal{C}^{(i)}_0 \longrightarrow 0,$$

where $\mathcal{C}_1^{(i)} \cong \Omega^{m_i}(k)$, $\mathcal{C}_0^{(i)} = k \oplus \text{proj}$ and $\partial_1^{(i)} = \alpha_i$ as in the exact sequence. Clearly the homology of $\mathcal{C}^{(i)}$ is concentrated in degree 0 and $H_0(\mathcal{C}^{(i)}) \cong L_i$.

Now define $\mathcal{C} = \mathcal{C}^{(1)} \otimes \cdots \otimes \mathcal{C}^{(r)}$ to be the usual tensor product of complexes with $\otimes = \otimes_k$. Note that by the Künneth formula:

$$H_j(\mathcal{C}) = \begin{cases} L_1 \otimes \cdots \otimes L_r & \text{if } j = 0 \\ 0 & \text{if } j \neq 0. \end{cases}$$

Proposition 5.2. *If $\zeta_1, \ldots, \zeta_\tau$ is a homogeneous set of parameters for $H^*(G,k)$, then $H_0(\mathcal{C}) \cong L_1 \otimes \cdots \otimes L_\tau$ is a projective module.*

Proof. This is a standard support variety argument. The point is that $V_G(L_i) = V_G(\zeta_i)$ (for example, see Benson, 1991, (5.9.1) and by the tensor product Benson, 1991, Theorem (5.7.1))

$$V_G(L_1 \otimes \cdots \otimes L_r) = V_G(\zeta_1) \cap \cdots \cap V_G(\zeta_r) = \{0\}.$$

The complex $\mathcal{C}$, has the form

$$0 \longrightarrow \mathcal{C}_r \longrightarrow \cdots \longrightarrow \mathcal{C}_1 \longrightarrow \mathcal{C}_0 \longrightarrow 0,$$

where $\mathcal{C}_0 \cong k \oplus \text{proj}, \mathcal{C}_1 \cong \sum_{i=1}^r \Omega^{m_i}(k) \oplus \text{proj}, \mathcal{C}_2 \cong \sum_{i<j} \Omega^{m_i+m_j}(k) \oplus \text{proj}$, etc. by Lemma 5.1. Also $\mathcal{C}_n \cong \Omega^w(k) \oplus \text{proj}$, where $w = \sum m_i$.

Definition 5.3. Let (P_n, ϵ) be a minimal projective resolution of k. The hypercohomology spectral sequence is the spectral sequence of the double complex

$$E_0^{p,q} = \text{Hom}_{kG}(\mathcal{C}_p \otimes P_{q+1}, \Omega(k))$$

for $p \geq 0$ and $q \geq -1$.

Theorem 5.4. *The spectral sequence converges to the total cohomology of the total complex which in degree s is $\text{Ext}^s_{kG}(L_1 \otimes \cdots \otimes L_r, \Omega(k))$. In particular, if $\zeta_1, \ldots, \zeta_r$ is a homogeneous system of parameters, then $E_\infty^{p,q} = 0$ if either $p > 0$ or $q > -1$, while*

$$E_\infty^{0,-1} = \text{Hom}_{kG}(L_1 \otimes \cdots \otimes L_r, \Omega(k)).$$

Proof. Note that the spectral sequences converges because it has only a finite number of columns and hence $E_{r+1} = E_\infty$. The easiest way to see the theorem is to run the "other" spectral sequence of taking the differentials in the reverse order. That is, first take the differential $(\partial \otimes 1)^*$, i.e. the one induced by the differential on $\mathcal{C}$. Because every P_q is projective, the E_1 page of this complex is

$$E_1^{p,q} = \text{Hom}_{kG}(H_p(\mathcal{C}) \otimes P_{q+1}, \Omega(k)).$$

But $H_p(\mathcal{C}) = 0$ if $p \neq 0$ and $H_0(\mathcal{C}) = L_1 \otimes \cdots \otimes L_r$. Consequently, E_1 has only one nonzero column in $p = 0$ and there is only one further differential which is $(1 \otimes \partial)^*$. But $L_1 \otimes \cdots \otimes L_r \otimes P_*$ is a projective resolution of $L_1 \otimes \cdots \otimes L_r$. Hence,

$$E_\infty^{p,q} = E_2^{p,q} = \begin{cases} \text{Ext}^{q+1}_{kG}(L_1 \otimes \cdots \otimes L_r, \Omega(k)) & \text{if } p = 0 \\ 0 & \text{if } p \neq 0. \end{cases}$$

The last statement follows from Proposition 5.2.□

THEOREM 5.5. *For $q \geq 0$ the E_1-page of the spectral sequence coincides with the Koszul complex $\mathcal{K}(\zeta_1, \ldots, \zeta_n; H^*(G,k))$. That is,*

$$E_1^{p,q} = \mathcal{K}^{p,q}(\zeta_1, \ldots, \zeta_n; H^*(G,k)),$$

for $q \geq 0$ and the differential induced by the boundary map on $\mathcal{C}$, $E_1^{p,q} \longrightarrow E_1^{p+1,q}$ is the same as the differential in the Koszul complex.

PROOF. Note first that for each p, $(\mathcal{C}_p \otimes P_*, 1 \otimes \epsilon)$ is a projective resolution of $\mathcal{C}_p$. Hence

$$E_1^{p,q} \cong \mathrm{Ext}_{kG}^{q+1}(\mathcal{C}_p, \Omega(k)) \cong \mathrm{Ext}_{kG}^{q}(\mathcal{C}_p, k) \qquad \text{for } q \geq 0.$$

Then for $p = 0$, $\mathcal{C}_0 = k \oplus \mathrm{proj}$ and

$$E_1^{0,q} \cong \mathrm{Ext}^q(k,k) \cong H^q(G,k)$$

for $q \geq 0$. For $\mathcal{C}_1 = \sum_{i=1}^{r} \Omega^{m_i}(k) \oplus \mathrm{proj}$ and

$$E_1^{1,q} \cong \mathrm{Ext}_{kG}^{q}\left(\sum_{i=1}^{r} \Omega^{m_i}(k), k\right) \cong \sum_{i=1}^{r} H^{q+m_i}(G,k)$$

etc. We leave it to the reader to check all of the details.□

REMARK. The proof of the first statement of Proposition 3.5 can be derived from the last theorem. That is, if $\zeta_1, \ldots, \zeta_\tau$ is a regular sequence and a homogeneous set of parameters for $H^*(G,k)$, then by Proposition 3.3, $E_2^{p,q} = 0$ for $q \geq 0$ and $p \neq \tau$. But further differentials on the spectral sequence must be zero. Hence $E_2^{\tau,q} = E_\infty^{\tau,q}$ for $q \geq 0$. On the other hand, by Theorem 5.4, $E_\infty^{\tau,q} = 0$ for $q \geq 0$.

The following is our main theorem.

THEOREM 5.6. *Suppose that R satisfies the Standard Hypothesis to degree N, and that R also satisfies both Condition* **G** *and Condition* **R**. *Assume further that N is at least equal to the maximum of the degrees of a set of homogeneous generators for the ideal $J' = \cap J_H \leq Q$, where each J_H is the kernel of the composition $Q \longrightarrow R \xrightarrow{\theta_H} H^*(H,k)$ for H a maximal subgroup of G. Suppose that there exist elements $w_1, \ldots, w_\tau \in Q$, such that $w_1 + I, \ldots, w_\tau + I$ is a homogeneous system of parameters for $H^*(G,k)$ and such that $\deg(w_i) \geq 2$ for all i and $N \geq \sum_{i=1}^{\tau} \deg(w_i)$. Then $R = H^*(G,k)$.*

PROOF. We need to show that $Q = P$; that is, that there are no extra generators in $H^*(G,k)$ and that $I = J$, i.e. that there are no undiscovered relations. Note that all generators and relations for R lie in degree N or less. By Theorem 4.3, any new relation for $H^*(G,k)$ must lie in a degree which is at least equal to the smallest degree of a new generator. Thus, if there are no new generators, then there are no new relations and we are done.

So, assume that there is a new generator $x_{m+1} \in P$ with $\deg(x_{m+1}) = w > N$. That is, w is the least degree in which there is a generator for P which is not in Q. Let $\zeta_1, \ldots, \zeta_\tau \in R$ be a homogeneous set of parameters with the property that $\deg(\zeta_i) \geq 2$ for all i and $\sum_{i=1}^{\tau} \deg(\zeta_i) \leq N$. Now consider the hypercohomology spectral sequence defined by $\zeta_1, \ldots, \zeta_r$. Note that the class of x_{m+1} in $E_1^{\tau,v}, v = w - \sum_{i=1}^{\tau} \deg(\zeta_i) > 0$ survives to the E_2-page of the spectral sequence.

Now remember that $E_\infty^{\tau,v} = 0$ and hence it is necessary that the class of x_{m+1} be hit by something on some earlier page of the spectral sequence. That is, for some $t, 2 \le t \le \tau$, there is an element $\mu \in E_t^{\tau-t,v+t-1}$ such that

$$d_t(\mu) = \gamma,$$

where $\gamma \in E_t^{\tau,v}$ is the class of x_{m+1}. But now

$$E_1^{\tau-t,v+t-1} \cong \sum_{|S|=\tau-t} H^{v+t-1+\deg(S)}(G,k),$$

where the sum is over all subsets $S \subseteq \{1,\dots,\tau\}$ such that the number of elements in S is $\tau - t$ and $\deg(S) = \sum_{i\in S}\deg(\zeta_i)$. From this we obtain that

$$E_1^{\tau-t,v+t-1} = \sum_{|S|=\tau-t} R^{v+t-1+\deg(S)}$$

because

$$\begin{aligned} v+t-1+\deg(S) &= w - \sum_{i=1}^{\tau}\deg(\zeta_i) + t - 1 + \sum_{i\in S}\deg(\zeta_i) \\ &= w + t - 1 - \sum_{i\notin S}\deg(\zeta_i) < w, \end{aligned}$$

since $\deg(\zeta_i) \ge 2$ and there are t of the elements in $\{1,\dots,\tau\}$ which are not in S. The same also hold for $E_1^{\tau-t-1,v+t-1}$ and $E_1^{\tau-t+1,v+t-1}$. Hence, this portion of the E_1-page of the spectral sequence for $H^*(G,k)$ coincides with corresponding portion of the E_1-page of the hypercohomology spectral sequence for R, which is the Koszul complex $K(\zeta_1,\dots,\zeta_r;R)$. Hence, $E_2^{\tau-t,v+t-1} = E_t^{\tau-t,v+t-1} = 0$. Therefore, we have a contradiction.

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References

Adem, A., Carlson, J. F., Karagreuzian, D., Milgram, R. J. The cohomology of the Higman-Sims group, (in preparation).

Benson, D. J. (1991). *Representations and Cohomology II: Cohomology of Groups and Modules,* volume 31 of Cambridge Studies in Advanced Mathematics, Cambridge, Cambridge University Press.

Benson, D. J., Carlson, J. F. (1994). Projective resolutions and Poincaré duality complexes. *Trans. Am. Math. Soc.*, **342**, 488–447.

Bosma, W., Cannon, J. (1996). *Handbook of Magma Functions*, Sydney, Magma Computer Algebra.

Broto, C., Henn, H.-W. (1993). Some remarks on central elementary Abelian p-subgroups and cohomology of classifying spaces. *Quart. J. Math.*, **44**, 155–163.

Carlson, J. F. (1992). Projective resolutions and degree shifting for cohomology of group rings. In Tachikawa, H., Brenner, S. eds, *Representations of Algebras and Related Topics,* LNS **168**, pp. 80–126. Cambridge, Cambridge University Press.

Carlson, J. F. (1996). In *Modules and Group algebras,* ETH Lecture Notes, Basel, Birkhäuser.

Carlson, J. F., Green, E., Schneider, G. J. A. (1997). Computing the ext algebras for the group algebras of finite groups. *J. Symb. Comput.*, **24**, 317–325.

Carlson, J. F., Maginnis, J., Milgram, R. J. (1999). The cohomology of the sporadic groups J_2 and J_3. *J. Algebra*, **214**, 143–173.

Duflot, J. (1981). Depth and equivariant cohomology. *Comm. Math. Helvetici*, **56**, 627–637.

Evens, L. (1991). *The Cohomology of Groups*, New York, Oxford University Press.
Green, D. (1997). In *Constructing Projective Resolutions for p-groups,* LNM **24**, Germany, University of Essen.
Hilton, P., Stammbach, U. (1971). *A Course in Homological Algebra*, New York, Springer Verlag.
Rusin, D. (1989). The cohomology of groups of order 32. *Math. Comput.*, **53**, 359–385.

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