Automorphism Groups of (0,2)-graphs from type A_n Root Systems

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Abstract

It has been shown that one can construct $(0,2)$-graphs from simply laced root systems in [1]. The graph $\Gamma(\alpha_0, A_n)$ corresponding to the root system A_n and target vector $\alpha_0 = \alpha_1 + \cdots + \alpha_n$ is the well-known n-dimensional hypercube, whose automorphism group is isomorphic to $S_n \rtimes \mathbb{Z}_2^n$. In this paper we will describe the generators and structure of the automorphism group of the $(0,2)$-graph $\Gamma(2\alpha_0, A_n) = \Gamma_n$ corresponding to the root system A_n and target vector $2\alpha_0$. This goal will be accomplished by assigning a tableau to each vertex in Γ_n, describing adjacency in terms the tableaux, and describing the generators of $\mathrm{Aut}(\Gamma_n)$ in terms of how they affect the tableaux.

1 Root Systems

In this section we provide the definition of a roots system and state some properties relevant to the construction of Γ_n.

Let V be a n-dimensional Euclidean space with an inner product $(\cdot, \cdot)$.

Definition 1. A **reflection** is a linear transformation $s : V \to V$ such that

(i) there exists $\alpha \in V$ such that $s(\alpha) = -\alpha$, in which case we often write $s = s_\alpha$, and

(ii) every vector in the hyperplane ($(n-1)$-dimensional subspace) H_α, where $H_\alpha \perp \alpha$, is left fixed by s_α.

It is easily checked that

$$s_\alpha(\mathbf{v}) = \mathbf{v} - \frac{2(\mathbf{v}, \alpha)}{(\alpha, \alpha)}\alpha$$

for all $\mathbf{v} \in V$ defines a reflection s_α.

Definition 2. A **root system** is a finite set of nonzero vectors $\Phi \subset V$ such that

(R1) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$

(R2) $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.

The elements of Φ are called **roots**.

For a given basis of V, $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, one can define a total ordering on V using the lexicographic ordering where $\sum_{i=1}^{n} a_i \mathbf{v}_i < \sum_{i=1}^{n} b_i \mathbf{v}_i$ if $a_k < b_k$, where k is the minimal index such that $a_i \neq b_i$ and $a_i, b_i \in \mathbb{R}$ are subject to the usual ordering on $\mathbb{R}$. Under such an ordering, the subset $\Phi^+ = \{\alpha \in \Phi \mid \alpha > 0\} \subset \Phi$ is referred to as the set of **positive roots** of Φ.

Definition 3. A **simple system** $\Delta \subset \Phi$ is a basis for V such that for each $\alpha \in \Phi$, the representation of α as a linear combination of **simple roots** (elements of Δ) has the property that each coefficient has the same sign, *i.e.* all coefficients are negative or all coefficients are positive.

Every simple system Δ is contained in a unique set of positive roots, and any set of positive roots $\Phi^+ \subset \Phi$ contains a unique simple system. In particular, one can always find a simple system for any root system.

Definition 4. A root system Φ is **simply laced** if each root in Φ has the same length, *i.e.* $(\alpha, \alpha)^{1/2}$ is the same for all $\alpha \in \Phi$.

In this paper, we are only interested in the simply laced root system of type A_n, where V is a hyperplane of $\mathbb{R}^{n+1}$. Let $\epsilon_1, \ldots, \epsilon_{n+1}$ be the standard basis vectors for $\mathbb{R}^{n+1}$. Then the simple system for the root system $\Phi = A_n$ is

$$\Delta = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid i = 1, \ldots, n\}$$

and the subset of positive roots is

$$\Phi^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j < n+1\} = \{\alpha_i + \alpha_{i+1} + \cdots \alpha_j \mid 1 \leq i < j < n+1\},$$

i.e. Φ^+ consists of consecutive sums of positive roots. For example, $\alpha_1 + \alpha_2 + \alpha_3 \in \Phi^+$, but $\alpha_1 + \alpha_3 \notin \Phi^+$.

Definition 5. Given a subset $\Psi \subseteq \Phi$, the **weight** of Ψ is the vector $\sum_{\alpha \in \Psi} \alpha$.

For example, the weight of Δ in A_n described above is $\alpha_0 := \alpha_1 + \alpha_2 + \cdots + \alpha_n$, and in this paper we are concerned with subsets of Φ^+ with weight $2\alpha_0$.

For a complete account of these concepts, see [4].

2 (0,2)-Graphs with Target Vector $2\alpha_0$

Let Φ be a simply laced root system. For a given weight $u \in \operatorname{span}(\Phi)$, which we shall refer to as the **target vector**, [?] shows that the following construction will yield a bipartite (0,2)-graph $\Gamma(u, \Phi)$ if $\Gamma(u, \Phi)$ has a nonempty vertex set. The vertices of $\Gamma(u, \Phi)$ are the subsets $A \subseteq \Phi^+$ with weight u, *i.e.* the A such that $\sum_{\alpha \in A} \alpha = u$. Two vertices $A, B \subseteq \Phi^+$ are defined to be adjacent if $|A \Delta B| = 3$, where $A \Delta B = (A \cup B) \backslash (A \cap B)$ is the symmetric difference of A and B.

Lemma 1. *If two vertices $A, B \subseteq \Phi^+$ in $\Gamma(u, \Phi)$ are adjacent, then $||A| - |B|| = 1$.*

Proof. Suppose that $A \sim B$. Then

$$\begin{aligned} |A\Delta B| &= |(A \cup B)\backslash(A \cap B)| \\ &= |A \cup B| - |A \cap B| \\ &= |A| + |B| - |A \cap B| - |A \cap B| = 3 \\ &\Rightarrow |A| + |B| = 3 + 2|A \cap B|. \end{aligned} \tag{1}$$

Since the right side the the last equation is odd, one of $|A|$ and $|B|$ is even and the other is odd. If $|A| = |B| + n$ for some odd $n \geq 1$, then from equation 1 above, we have

$$\begin{aligned} |A\Delta B| &\Rightarrow |A| + |B| = 3 + 2|A \cap B| \\ &\Rightarrow 2|B| + n = 3 + 2|A \cap B| \\ &\Rightarrow |B| - \frac{3-n}{2} = |A \cap B|. \end{aligned}$$

Note that $|A \cap B| < |B|$ since $\sum_{\alpha \in A} \alpha = \sum_{\beta \in B} \beta$ implies that B cannot be a subset of A. Since $|A \cap B| < |B|$ and $3 - n > 0 \Leftrightarrow n < 3$, it must be the case that $n = 1$ and the claim follows. □

3 Tableaux

Now let $\Phi = A_n$ with positive roots Φ^+ and simple roots $\alpha_1, \ldots, \alpha_n$. Now we describe how to assign to each vertex $A \subseteq \Phi^+$ in $\Gamma(2\alpha_0, A_n) = \Gamma_n$ a tableau of integers $0, 1, \ldots, n-1$. First we do this for the vertices of $\Gamma(\alpha_0, A_n) = \Gamma(\alpha_0)$. Let $A \subseteq \Phi^+$ be a vertex of $\Gamma(\alpha_0)$. Then elements of A are finite sums of consecutive simple roots, as noted above at the end of section 1. The tableau τ_A corresponding to A has a single row where i is included if $i < n$ and α_i is the last term in a sum of a root, *i.e.* if $\alpha_{j_1} + \cdots + \alpha_{j_k} + \alpha_i \in A$. For example, if $A = \{\alpha_1, \alpha_2 + \alpha_3, \alpha_4 + \alpha_5\}$ is a vertex in $\Gamma(\alpha_0, A_5)$, then $\tau_A = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array}$. Note that we write $\alpha_0 = \alpha_1 + \cdots + \alpha_n$, so that if $A = \{\alpha_0\}$, then $\tau_A = \begin{array}{|c|}\hline 0 \\ \hline\end{array}$. Given a tableau, we obtain the corresponding subset by listing $\alpha_1, \ldots, \alpha_n$, and then inserting a comma after α_i in the list if i occurs in τ and then inserting a $+$ between any simple roots with no commas in between them. For example, if $\tau = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline\end{array}$ in A_5, then

$$\begin{aligned} \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 &\Rightarrow \alpha_1, \alpha_2, \alpha_3\alpha_4, \alpha_5 \\ &\Rightarrow \alpha_1, \alpha_2, \alpha_3 + \alpha_4, \alpha_5 \\ &\Rightarrow A = \{\alpha_1, \alpha_2, \alpha_3 + \alpha_4, \alpha_5\}. \end{aligned}$$

By convention, we write each row of a tableau in increasing order as in the above examples, even though in principle any permutation of the row entries would return the same set of roots.

Now to construct the tableaux for weight Γ_n, we partition a subset $A \subseteq \Phi^+$ with weight $2\alpha_0$ into two subsets A_1 and A_2, each with weight α_0. This is always possible because the positive roots in A_n are sums of consecutive simple roots. For each of these disjoint subsets of A, we form the tableau for each with the process described above. The tableau τ_A of

A consists of these two rows. For example, let $A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_3 + \alpha_4\}$, $A_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, and $A_2 = \{\alpha_1 + \alpha_2, \alpha_3 + \alpha_4\}$ in A_4. Then $A = A_1 \cup A_2$ and $\tau_{A_1} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$, and $\tau_{A_2} = \begin{array}{|c|}\hline 2 \\ \hline \end{array}$. Thus,

$$\tau_A = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array}.$$

The convention is to write the the rows in decreasing length, as in the example, and if two rows have the same length, then the row with the smallest entry is written above the other. Once again, this convention does not change the fact that permuting the rows of the tableau yields technically equivalent tableaux. In the previous example, we could have instead chosen $A_1 = \{\alpha_1, \alpha_2, \alpha_3 + \alpha_4\}$, and $A_2 = \{\alpha_1 + \alpha_2, \alpha_3, \alpha + 4\}$, which would yield

$$\tau_A = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array}$$

Thus, a tableau representation for a vertex in Γ_n is not necessarily unique. However, both these tableau representations can be converted back to the original set of roots, and so they are equivalent. To convert from a tableau to a set of roots in the $2\alpha_0$ case, first convert each row as in the α_0 case, and then take the union of these two sets. Observe that if an integer i occurs in both rows, then the strings of numbers (including the empty string) following each i can be interchanged to obtain an equivalent tableau, as seen in the previous example.

Although the tableau τ_A of a given set A is not necessarily unique, the number k of nonzero integers in τ_A is the same for all equivalent tableaux, and $|A| = k + 2$. If $|A| = m$, then there are 2 distinct elements of A which contain α_n as a summand. By the construction of τ_A, only the $m - 2$ elements of A without α_n as a summand correspond to a nonzero integer in τ. Similarly, for a vertex A in $\Gamma(\alpha_0, A_n)$, $|A|$ is one more than the number of nonzero integers in τ_A. We will denote the number of positive integers of a tableau τ by $|\tau|$.

There are also a couple simple rules that a tableau must satisfy. For example, the 1 cannot occur more than once in a tableau because that would mean the simple root α_1 occurs twice in the corresponding set of roots, and similarly for $n-1$ and the root α_n. There also cannot be the same pair of integers in both rows of a tableau, e.g. if $\begin{array}{|c|c|}\hline 2 & 4 \\ \hline \end{array}$ occurred in both rows of a tableau it would imply that the root $\alpha_3 + \alpha_4$ occurs twice in the corresponding set of roots.

Given a tableau τ in Γ_n, we will write $\tau = (\tau_1, \tau_2)$ where τ_1 and τ_2 are the rows of τ. We will denote inserting an integer k into a row τ_i $(i = 1, 2)$ by $\tau_i + k$ and deleting k from τ_i (if it occurs) by $\tau_i - k$. When inserting and deleting multiple integers, we may write $\tau_i + k_1 - k_2$, in which case one cannot actually subtract k_2 from k_1 and insert or delete the result from τ_i. Finally, if the row in which we are inserting or deleting k does not matter, we may write $\tau \pm k$. Note that if there is a row with a single nonzero integer k, then deleting k means replacing it with 0, and if there is a row with a single 0, then inserting k to the row means replacing 0 with k.

Lemma 2. *Two tableaux $\tau = (\tau_1, \tau_2)$ and $\sigma = (\sigma_1, \sigma_2)$ are adjacent* iff *σ can be obtained from τ by inserting or deleting exactly one nonzero integer in τ or a tableau equivalent to τ.*

Proof. Let A and B be adjacent subsets of Φ^+ and let σ be a tableau for B. It follows from Lemma 1 that that the size of A and B differ by one, so without loss of generality, suppose

$|A| = |B| + 1$. Then we have that

$$\begin{aligned}|A\Delta B| &= |A| + |B| - 2|A \cap B| = 3\\ &\Rightarrow 2|B| = 2 + 2|A \cap B|\\ &\Rightarrow |B| = |A \cap B| + 1,\end{aligned}$$

from which it also follows that $|A| = |A \cap B| + 2$. Thus, $A = (A \cap B) \cup \{\alpha, \beta\}$ and $B = (A \cap B) \cup \{\gamma\}$, where $\alpha, \beta, \gamma \in \Phi^+$ are distinct. Moreover, since $\sum A = \sum B$ implies $\sum(A \backslash A \cap B) = \sum(B \backslash A \cap B)$, it follows that $\alpha + \beta = \gamma$. If $\gamma = \alpha_{i_1} + \cdots + \alpha_{i_k}$ for consecutive $1 \leq i_1 < \ldots < i_k \leq n$, then $\alpha = \alpha_{i_1} + \cdots + \alpha_{i_l}$ and $\beta = \alpha_{i_{l+1}} + \cdots + \alpha_{i_k}$ for some $1 \leq l < k$. If $i_k < n$, then i_k will be in σ, and we can obtain a tableau τ for A by adding i_l in front of i_k, which corresponds to replacing $\gamma = \alpha + \beta$ in B with α, β to form A. The rest of σ remains unchanged since $B \backslash \{\gamma\} = A \cap B$. If $i_k = n$, then the last number in one row of σ is $i_1 - 1$. Then τ is obtained from σ by adding i_l after $i_1 - 1$, unless $i_1 - 1 = 0$ when the 0 is simply replaced by i_l. This again just corresponds to replacing γ in B with α, β to form A.

Now suppose that a tableau $\sigma = (\sigma_1, \sigma_2)$ is obtained from a tableau $\tau = (\tau_1, \tau_2)$ by deleting a number $1 \leq i < n$ from τ, and that τ and σ correspond to subsets $A = A_1 \coprod A_2$ and $B = B_1 \coprod B_2$, respectively. Then $\tau = (\sigma_1 + i, \sigma_2)$. Note that the argument will work if a number is deleted from the second row as well since the order of rows is only a convention. Finally, suppose that $|\tau_1| = |\sigma_1 + i| = m$, *i.e.* that $|A_1| = m + 1$ and $|B_1| = m$. From these assumptions, we have that $A_2 = B_2$ and $|A_1| - 1 = |B_1|$. Furthermore, if $\tau_1 = \begin{array}{|c|c|c|c|c|}\hline \alpha & j & i & k & \omega \\ \hline\end{array}$, and $\sigma_1 = \begin{array}{|c|c|c|c|}\hline \alpha & j & k & \beta \\ \hline\end{array}$, where $1 \leq j < i < k < n$ and α and ω are the initial and terminal strings of numbers (possibly empty), then

$$A_1 \cap B_1 = B_1 \backslash \{\alpha_{j+1} + \cdots + \alpha_k\}$$

.

Thus, $|A_1 \cap B_1| = m - 1$.

Therefore,

$$\begin{aligned}|A\Delta B| &= |A| + |B| - 2|A \cap B|\\ &= |A| + |B| - 2|(A_1 \cup A_2) \cap (B_1 \cup B_2)|\\ &= |A| + |B| - 2|(A_1 \cap B_1) \cup (A_1 \cap B_2) \cup (A_2 \cap B_1) \cup (A_2 \cap B_2)|\\ &= |A| + |B| - 2|(A_1 \cap B_1) \cup A_2|\\ &= |A| + |B| - 2|A_1 \cap B_1| - 2|A_2| + 2|A_1 \cap B_1 \cap A_2|\\ &= |A_1| + |A_2| + |B_1| + |B_2| - 2|A_1 \cap B_1| - 2|A_2| + 0\\ &= |A_1| + |A_2| + |A_1| - 1 + |A_2| - 2|A_1 \cap B_1| - 2|A_2|\\ &= 2|A_1| - 1 - 2|A_1 \cap B_1|\\ &= 2(m + 1) - 1 - 2(m - 1)\\ &= 3,\end{aligned}$$

and so A is adjacent to B. □

It is important to consider tableaux equivalent to a given tableau τ when determining the neighbors of τ. For example, $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$ is adjacent to $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ since

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline\end{array} \sim \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Finally, one cannot delete any number from τ and expect to achieve an adjacent tableau because it may not result in a legal tableau, e.g. deleting 3 from the top row of $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$.

4 Automorphisms

Proposition 1. *The following functions defined in terms of the tableau representation of the vertices are automorphisms of* Γ_n*:*

(a) Given a tableau τ and integer $1 \leq i \leq n-1$, define $\varphi_i(\tau)$ to be the tableau obtained by moving i to the opposite row. If i does not occur in τ or i occurs in both rows, then $\varphi_i(\tau) = \tau$. We shall refer this automorphism as slide-i*.*

(b) Given a tableau $\tau = (\tau_1, \tau_2)$, we construct $\varphi_{12}(\tau)$ in the following ways:

  (i) If τ contains $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$ in a row, then delete the 1 and 2. Furthermore, $\boxed{2}$ occurs in the opposite row, then delete the 2, and if not, then insert a 2.

  (ii) If τ does not satisfy the above conditions and τ has a $\boxed{1}$ in a row, then insert 2 into the opposite row if it does not already occur or delete 2 if it does occur.

  (iii) If τ does not meet above conditions and has a single 2, then delete the 2 and insert $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$ in the opposite row.

  (iv) Finally, if τ contains no 1s or 2s, then insert $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$ in τ_1 and insert $\boxed{2}$ in τ_2.

  We will refer to $\varphi_{1,2}$ as change-$(1,2)$*.*

(c) Define $\varphi_{n-2,n-1}$ analogously to φ_{12}, with $n-1$ in place of 1 and $n-2$ in place of 2 above. We will refer to $\varphi_{n-2,n-1}$ as change-$(n-2, n-1)$*.*

(d) The final automorphism we refer to as the palindromic automorphism *and we denote it by ρ_n (it depends on A_n). Given a tableau τ, $\rho_n(\tau)$ is obtained from τ by replacing every nonzero integer i with $n-i$.*

Before proving that these are actually automorphisms, we provide some examples of how they act on tableaux for clarity. In each case, we rearrange the image to match the conventions for writing tableaux.

Example. Let $\tau_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$, $\tau_2 = \begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline\end{array}$, and $\tau_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$

(a) We have that $\varphi_1(\tau_1) = \begin{ytableau} 1 & 4 \\ 2 & 5 \end{ytableau}$, $\varphi_3(\tau_1) = \tau_1$, $\varphi_1(\tau_2) = \tau_2$. Similarly, $\varphi_2(\tau_3) = \tau_3$ since both 2s are moved to the opposite row.

(b) We have that $\varphi_{12}(\tau_1) = \begin{ytableau} 2 & 4 \\ 5 \end{ytableau}$, $\varphi_{12}(\tau_2) = \begin{ytableau} 1 \\ 2 \end{ytableau}$, and $\varphi_{12}(\tau_3) = \begin{ytableau} 3 \\ 0 \end{ytableau}$. Similarly, $\varphi_{45}(\tau_1) = \begin{ytableau} 1 & 2 & 5 \\ 0 \end{ytableau}$ in A_6, $\varphi_{34}(\tau_2) = \begin{ytableau} 1 & 3 & 3 \\ 4 \end{ytableau}$ in A_5, and $\varphi_{34}(\tau_3) = \begin{ytableau} 2 & 3 & 4 \\ 1 & 2 \end{ytableau}$, or equivalently $\begin{ytableau} 1 & 2 & 3 & 4 \\ 2 \end{ytableau}$ in A_5.

(c) Finally, $\rho_6(\tau_1) = \begin{ytableau} 1 & 4 & 5 \\ 2 \end{ytableau}$, $\rho_5(\tau_2) = \begin{ytableau} 4 \\ 0 \end{ytableau}$, and $\rho_5(\tau_3) = \begin{ytableau} 2 & 3 & 4 \\ 3 \end{ytableau}$.

Proof of Proposition. In each case we must show that the functions are a permutation of the vertices of Γ_n and that they preserve adjacency.

(a) To see that φ_i is a bijection, we will show that $\varphi_i^{-1} = \varphi_i$. Given a tableau $\tau = (\tau_1, \tau_2)$, if i does not occur in τ or i occurs in both rows of τ, then φ_i leaves τ fixed, so $\varphi_i^2(\tau) = \tau$. If i occurs in exactly one row of τ, say row 1, then $\varphi_i^2(\tau) = \varphi_i((\tau_1 - i, \tau_2 + i)) = (\tau_1 - i + i, \tau_2 + i - i) = \tau$.

Now suppose that $\tau = (\tau_1, \tau_2)$ is adjacent to $\sigma = (\sigma_1, \sigma_2)$. By Lemma 2, σ can be obtained by either inserting or deleting exactly one number from τ. Suppose without loss of generality that σ is obtained by deleting j from τ for some $1 \leq j \leq n-1$, so $\tau = (\sigma_1 + j, \sigma_2)$ (once again, which row is chosen does not matter). If $j \neq i$ and i does not occur or occurs in both rows of σ, then $\sigma = \varphi_i(\sigma) \sim \varphi_i(\tau) = \tau$. If i occurs in σ_1, then $\varphi_i(\sigma) = (\sigma_1 - i, \sigma_2 + i)$ and $\varphi_i(\tau) = (\sigma_1 + j - i, \sigma_2 + i)$, and so $\varphi_i(\sigma)$ is obtained by deleting j from $\varphi_i(\tau)$, *i.e.* $\varphi_i(\sigma) \sim \varphi_i(\tau)$. If $j = i$, then i must not occur in σ, so $\varphi_i(\sigma) = (\sigma_1, \sigma_2)$ and $\varphi_i(\tau) = (\sigma_1 + j - i, \sigma_2 + i) = (\sigma_1, \sigma_2 + i)$, and so σ is obtained by deleting i from τ in row 2. Therefore, φ_i preserves adjacency, and thus, is an automorphism.

(b) Now we show that φ_{12} is an automorphism, beginning with showing $\varphi_{12}^{-1} = \varphi_{12}$. We proceed by testing each case in the order that they appear in the definition of φ_{12}. Let $\tau = (\tau_1, \tau_2)$.

  (i) Suppose τ_1 contains $\begin{ytableau} 1 & 2 \end{ytableau}$ and τ_2 contains a 2. Then $\varphi_{12}(\tau) = (\tau_1 - 1 - 2, \tau_2 - 2)$. Now neither row of τ contains 1 or 2, and so applying φ_{12} again we obtain $\varphi_{12}^2(\tau) = (\tau_1 - 1 - 2 + 1 + 2, \tau_2 - 2 + 2) = \tau$. A similar argument applies if τ_1 contains $\begin{ytableau} 1 & 2 \end{ytableau}$ and τ_2 does not contain a 2. If τ_2 contains $\begin{ytableau} 1 & 2 \end{ytableau}$, then the same argument applies (the order of the rows is just a convention).

  (ii) Suppose that τ does not satisfy the above conditions and that τ_1 contains 1 and τ_2 contains 2. Then $\varphi_{12}^2(\tau) = \varphi_{12}((\tau_1, \tau_2 - 2)) = (\tau_1, \tau_2 - 2 + 2) = \tau$. The argument is similar if τ_2 does not contain a 2, as well as when τ_1 and τ_2 reverse roles.

  (iii) If τ does not meet above conditions and τ contains 2 in row 1, then $\varphi_{12}^2(\tau) = \varphi_{12}((\tau_1 + 1 + 2, \tau_2 - 2)) = (\tau_1 + 1 + 2 - 1 - 2, \tau_2 - 2 + 2) = \tau$, and similarly if 2 occurs in τ_2.

(iv) Finally, if τ contains no 1s or 2s, then $\varphi_{12}^2(\tau) = \varphi_{12}((\tau_1 + 1 + 2, \tau_2 + 2)) = (\tau_1 + 1 + 2 - 1 - 2, \tau_2 + 2 - 2) = \tau$.

Thus, φ_{12} is a bijection. Now suppose that $\tau = (\tau_1, \tau_2)$ is adjacent to $\sigma = (\sigma_1, \sigma_2)$ and that σ is obtained by deleting an integer $1 \leq j \leq n-1$ from $\tau = (\sigma_1 + j, \sigma_2)$, as before. We prove that $\varphi_{12}(\tau) \sim \varphi_{12}(\sigma)$ by cases in the same order as above.

(i) Suppose σ_1 contains $\boxed{1 \mid 2}$, σ_2 contains a 2, and note that $j \neq 1, 2$ since this would imply that $\boxed{1 \mid 2}$ is not in σ_1. Then $\varphi_{12}(\sigma) = (\sigma_1 - 1 - 2, \sigma - 2)$ and $\varphi_{12}(\tau) = (\sigma_1 - 1 - 2 + j, \sigma_2 - 2)$, and so $\varphi_{12}(\sigma)$ is obtained by deleting j from $\varphi_{12}(\tau)$, *i.e.* $\varphi_{12}(\tau) \sim \varphi_{12}(\sigma)$. A similar argument applies if σ_1 contains $\boxed{1 \mid 2}$ and σ_2 does not contain a 2.

If σ_2 contains $\boxed{1 \mid 2}$, then a similar argument works here as well, except that its possible for $j = 2$. If this is the case, then σ_1 does not contain 2, and so $\varphi_{12}(\sigma) = (\sigma_1 + 2, \sigma_2 - 1 - 2)$ and $\varphi_{12}(\tau) = (\sigma_1, \sigma_2 - 1 - 2)$. Thus, $\varphi_{12}(\sigma)$ is obtained by inserting $j = 2$ in $\varphi_{12}(\tau)$, and so $\varphi_{12}(\sigma)$ and $\varphi_{12}(\tau)$ are adjacent.

(ii) Suppose that σ does not satisfy either of the above two conditions and that σ_1 contains $\boxed{1}$ and σ_2 contains $\boxed{2}$. Then $\varphi_{12}(\sigma) = (\sigma_1, \sigma_2 - 2)$ and $\varphi_{12}(\tau) = (\sigma_1 + j, \sigma_2 - 2)$ if $j \neq 2$ and $\varphi_{12}(\tau) = (\sigma_1 - 1, \sigma_2 - 2)$ if $j = 2$. In either case, $\varphi_{12}(\tau) \sim \varphi_{12}(\sigma)$, and the argument is similar if σ_2 does not contain a 2.

If σ_1 and σ_2 reverse roles, so that σ_2 contains $\boxed{1}$ and σ_1 contains $\boxed{2}$, then $j \neq 1, 2$ since they would imply that τ has a 1 in both rows or that τ_1 has two 2s, both of which are not allowed. Thus, $\varphi_{12}(\sigma) = (\sigma_1 - 2, \sigma_2)$ and $\varphi_{12}(\tau) = (\sigma_1 + j - 2, \sigma_2)$, and so adjacency is preserved in this case as well. If σ_1 does not contain 2, then j can be 2, but the argument is still similar.

(iii) If σ does not meet above hypotheses and σ_1 contains $\boxed{2}$, then $j \neq 2$. Thus, $\varphi_{12}(\sigma) = (\sigma_1 - 2, \sigma_2 + 1 + 2)$ and $\varphi_{12}(\tau) = (\sigma_1 + j - 2, \sigma_2 + 1 + 2)$ if $j \neq 1$ and $\varphi_{12}(\tau) = (\sigma_1 - 2, \sigma_2 + 2)$ if $j = 1$. Adjacency is preserved in either case. If 2 occurs in σ_2, then $j \neq 2$ since both rows of τ can have a 2 only when there is a 1 in front of one of the 2s, which would force σ_1 to contain a 1, contrary to our assumption. Thus, $\varphi_{12}(\sigma) = (\sigma_1 + 1 + 2, \sigma_2 - 2)$ and $\varphi_{12}(\tau) = (\sigma_1 + j + 1 + 2, \sigma_2 - 2)$ if $j \neq 1$ and $\varphi_{12}(\tau) = (\sigma_1 + j, \sigma_1 - 2) = (\sigma_1 + 1, \sigma_2 - 2)$ if $j = 1$. In either case, adjacency is preserved.

(iv) Finally, if σ does not contain any 1s or 2s, then $\varphi_{12}(\sigma) = (\sigma_1 + 1 + 2, \sigma_2 + 2) = (\sigma_1 + 2, \sigma_2 + 1 + 2)$ and $\varphi_{12}(\tau) = (\sigma_1 + j + 1 + 2, \sigma_2 + 2)$ if $j \neq 1, 2$, $\varphi_{12}(\tau) = (\sigma_1 + j, \sigma_2 + 2)$ if $j = 1$, or $\varphi_{12}(\tau) = (\sigma_1, \sigma_2 + 1 + 2)$ if $j = 2$. In each case, adjacency is preserved.

Therefore, we conclude that φ_{12} is an automorphism of Γ_n.

(c) Showing that change-$(n-2, n-1)$ is an automorphism is analogous to the above case, with $n - 1$ in the place of 1 and $n - 2$ in the place of 2.

(d) Now we show that the palindromic function ρ_n is an automorphism. Once more, we first show that $\rho_n^{-1} = \rho_n$. This is apparent since applying ρ_n to a tableau τ once replaces every nonzero integer i with $n - i$. Applying ρ_n again replaces $n - i$ with $n - (n - i) = i$, and so $\rho_n^2(\tau) = \tau$ for any tableau τ. To show that ρ_n preserves adjacency, suppose that $\tau \sim \sigma$. Then we may assume that $\tau = (\sigma_1 + j, \sigma)$ for some $1 \leq j \leq n-1$. If we write the image of σ under ρ_n as $\rho_n(\sigma) = (\overline{\sigma_1}, \overline{\sigma_2})$, then $\rho_n(\tau) = (\overline{\sigma_1} + (n-j), \overline{\sigma_2})$. Thus, $\rho_n(\sigma)$ is obtained by deleting $n-j$ from $\rho_n(\tau)$, and so $\rho_n(\sigma) \sim \rho_n(\tau)$. Therefore, ρ_n is a automorphism.

$\square$

Lemma 3. *The automorphism obtained by conjugating slide-*1 *by change-*$(1,2)$ *has the following image given a tableau* τ*: Replace* 1 *with* 2 *and* 2 *with* 1 *whenever they occur in* τ*, except when* $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$ *occurs in one row and* $\begin{array}{|c|}\hline 2 \\ \hline\end{array}$ *occurs in the opposite row, in which case* τ *is left fixed. The analogous description holds for the automorphism obtained by conjugating slide-*$n-1$ *by change-*$(n-2, n-1)$ *(with* $n-1$ *in place of* 1 *and* $n-2$ *in place of* 2*). We refer to these as* switch-$(1,2)$ *and* switch-$(n-2, n-1)$ *and denote them by* ς_{12} *and* $\varsigma_{n-2,n-1}$*, respectively.*

Proof. First note that in the proof that change-(1,2) is an automorphism, we showed that $\varphi_{12}^{-1} = \varphi_{12}$. Hence, the claim is that $\varsigma_{12} = \varphi_{12}\varphi_1\varphi_{12}$ (and analogously for $\varsigma_{n-2,n-1}$). Let $\tau = (\tau_1, \tau_2)$ be a tableau in Γ_n. We proceed by cases for change-(1,2), in the same order as we have above. Keep in mind that replacing 1 with 2 in row τ_i is equivalent to writing $\tau_i - 1 + 2$ and replacing 2 with 1 is equivalent to $\tau_i + 1 - 2$.

(i) Suppose τ_1 contains $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$ and τ_2 contains a 2. Then

$$\begin{aligned}\varphi_{12}\varphi_1\varphi_{12}(\tau) &= \varphi_{12}\varphi_1(\tau_1 - 1 - 2, \tau_2 - 2) \\ &= \varphi_{12}(\tau_1 - 1 - 2, \tau_2 - 2) \\ &= (\tau_1, \tau_2) = \tau,\end{aligned}$$

as expected (this was the exception in the definition of ς_{12}). Now if τ_1 contains $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$ and τ_2 does not contain a 2, then

$$\begin{aligned}\varphi_{12}\varphi_1\varphi_{12}(\tau) &= \varphi_{12}\varphi_1(\tau_1 - 1 - 2, \tau_2 + 2) \\ &= \varphi_{12}(\tau_1 - 1 - 2, \tau_2 + 2) \\ &= (\tau_1, \tau_2) = \tau,\end{aligned}$$

which agrees with the definition of ς_{12} since replacing 1 with 2 and 2 with 1 in the same row yields an equivalent row.

(ii) If τ_2 contains $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$, then the argument from part (i) works here as well since the order of the rows is only a convention.

(iii) Suppose that τ does not satisfy either of the above two conditions and that τ_1 contains $\begin{array}{|c|}\hline 1 \\ \hline\end{array}$ and τ_2 contains $\begin{array}{|c|}\hline 2 \\ \hline\end{array}$. Then

$$\begin{aligned}\varphi_{12}\varphi_1\varphi_{12}(\tau) &= \varphi_{12}\varphi_1(\tau_1, \tau_2 - 2) \\ &= \varphi_{12}(\tau_1 - 1, \tau_2 + 1 - 2) \\ &= (\tau_1 - 1 + 2, \tau_2 + 1 - 2) = \varsigma_{12}(\tau).\end{aligned}$$

Similarly, if τ_1 contains $\boxed{1}$ and τ_2 does not contain 2, then

$$\begin{aligned}\varphi_{12}\varphi_1\varphi_{12}(\tau) &= \varphi_{12}\varphi_1(\tau_1, \tau_2 + 2)\\ &= \varphi_{12}(\tau_1 - 1, \tau_2 + 1 + 2)\\ &= (\tau_1 - 1 + 2, \tau_2) = \varsigma_{12}(\tau),\end{aligned}$$

The argument is similar if τ_1 and τ_2 reverse roles.

(iv) If τ does not meet above hypotheses and τ has a $\boxed{2}$ in row 1, then

$$\begin{aligned}\varphi_{12}\varphi_1\varphi_{12}(\tau) &= \varphi_{12}\varphi_1(\tau_1 - 2, \tau_2 + 1 + 2)\\ &= \varphi_{12}(\tau_1 + 1 - 2, \tau_2 - 2)\\ &= (\tau_1 + 1 - 2, \tau_2) = \varsigma_{12}(\tau),\end{aligned}$$

and similarly if $\boxed{2}$ is in row 2.

(v) Finally, if τ contains no 1s or 2s, then

$$\begin{aligned}\varphi_{12}\varphi_1\varphi_{12}(\tau) &= \varphi_{12}\varphi_1(\tau_1 + 1 + 2, \tau_2 + 2)\\ &= \varphi_{12}(\tau_1 + 2, \tau_2 + 1 + 2)\\ &= (\tau_1, \tau_2) = \varsigma_{12}(\tau).\end{aligned}$$

Thus, the claim holds in all cases. The proof for switch-$(n-2, n-1)$ is analogous. ☐

5 Subgroups of $\mathrm{Aut}(\Gamma_n)$

In this section, we assume $n \geq 6$ throughout.

Proposition 2. *The slide-*1 (φ_1) *and change-*(1, 2) (φ_{12}) *automorphisms generate a subgroup of* $\mathrm{Aut}(\Gamma_n)$ *isomorphic to* D_8 *(the dihedral group of order* 16*). Similarly, slide-*1 (φ_{n-1}) *and change-*$(n-1, n-2)$ ($\varphi_{n-1,n-2}$) *generate a subgroup isomorphic to* D_8*. Together, these four automorphisms generate a subgroup isomorphic to* $D_8 \times D_8$*.*

Proof. It suffices to show that φ_1 and φ_{12} satisfy the relations for the presentation of $D_8 = \langle r, f \mid r^2 = f^2 = (rf)^8 = e\rangle$. In the proof that φ_1 and φ_{12} were actually automorphisms it was shown that they both have order 2, and so it remains to show that $(\varphi_1\varphi_{12})^8 = 1$, where 1 is the identity automorphism. Let $\tau = (\tau_1, \tau_2)$ be a tableau in Γ_n. By Lemma 3, we have that $(\varphi_1\varphi_{12})^8 = (\varphi_1\varsigma_{12})^4$, which is easily checked by expanding the product. We will denote the image of τ under the ith application of $\varphi = \varphi_1\varsigma_{12}$ by $\tau^i = (\tau_1^i, \tau_2^i)$. We proceed by cases for change-(1,2) in the same order as we have above.

(i) If τ_1 contains $\boxed{1}\boxed{2}$ and τ_2 contains $\boxed{2}$, then

$$\begin{aligned}\varphi^4(\tau) &= \varphi^3\varphi_1(\tau_1, \tau_2) \quad \text{(note that } \varsigma_{12}(\tau) = \tau \text{ in this case by definition)}\\ &= \varphi^3(\tau_1 - 1, \tau_2 + 1) \quad \text{(now } \tau_1^1 \text{ contains } \boxed{2} \text{ and } \tau_2^1 \text{ contains } \boxed{1}\boxed{2}\text{)}\\ &= \varphi^2\varphi_1(\tau_1 - 1, \tau_2 + 1)\\ &= \varphi^2(\tau_1, \tau_2) = \tau \quad \text{(after repeating the same steps as above)}\\ &\Rightarrow \varphi^4(\tau) = \varphi^2(\tau) = \tau.\end{aligned}$$

However, if τ_1 contains $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$ but τ_2 does not contain $\begin{array}{|c|}\hline 2 \\ \hline\end{array}$, then

$$\begin{aligned}
\varphi^4(\tau) &= \varphi^3\varphi_1(\tau_1, \tau_2) \\
&= \varphi^3(\tau_1 - 1, \tau_2 + 1) \quad \text{(now } \tau_1^1 \text{ contains } \begin{array}{|c|}\hline 2 \\ \hline\end{array} \text{ and } \tau_2^1 \text{ contains } \begin{array}{|c|}\hline 1 \\ \hline\end{array}\text{)} \\
&= \varphi^2\varphi_1(\tau_1 - 2, \tau_2 + 2) \\
&= \varphi^2(\tau_1 - 1 - 2, \tau_2 + 1 + 2) \\
&= \varphi\varphi_1(\tau_1 - 1 - 2, \tau_2 + 1 + 2) \\
&= \varphi_1\varsigma_{12}(\tau_1 - 2, \tau_2 + 2) \\
&= \varphi_1(\tau_1 - 1, \tau_2 + 1) \\
&= (\tau_1, \tau_2) = \tau,
\end{aligned}$$

and so $\varphi^4(\tau) = \tau$. The same argument works if τ_2 contains $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$.

(ii) Suppose that τ does not satisfy either of the above two conditions, τ_1 contains $\begin{array}{|c|}\hline 1 \\ \hline\end{array}$, and τ_2 contains $\begin{array}{|c|}\hline 2 \\ \hline\end{array}$. Then

$$\begin{aligned}
\varphi^4(\tau) &= \varphi^3\varphi_1(\tau_1 - 1 + 2, \tau_2 + 1 - 2) \\
&= \varphi^3(\tau_1 + 2, \tau_2 - 2) \quad \text{(now } \tau_1^1 \text{ contains 1 and 2, while } \tau_2^1 \text{ contain neither)} \\
&= \varphi^2\varphi_1(\tau_1 + 2, \tau_2 - 2) \\
&= \varphi^2(\tau_1 - 1 + 2, \tau_2 + 1 - 2) \\
&= \varphi\varphi_1(\tau_1, \tau_2) \\
&= \varphi_1\varsigma_{12}(\tau_1 - 1, \tau_2 + 1) \quad \text{(now } \tau_2^3 \text{ contains 1 and 2, while and } \tau_1^3 \text{ contains neither)} \\
&= \varphi_1(\tau_1 - 1, \tau_2 + 1) \\
&= (\tau_1, \tau_2) = \tau.
\end{aligned}$$

Now if τ_1 contains $\begin{array}{|c|}\hline 1 \\ \hline\end{array}$ but τ_2 does not contain $\begin{array}{|c|}\hline 2 \\ \hline\end{array}$, then

$$\begin{aligned}
\varphi^4(\tau) &= \varphi^3\varphi_1(\tau_1 - 1 + 2, \tau_2) \\
&= \varphi^3(\tau_1 - 1 + 2, \tau_2) \\
&= \varphi^2\varphi_1(\tau_1, \tau_2) \\
&= \varphi^2(\tau_1 - 1, \tau_2 + 1) \quad \text{(now } \tau_2^2 \text{ contains 1, while and } \tau_1^2 \text{ contains neither 1 nor 2)} \\
&= \varphi\varphi_1(\tau_1 - 1, \tau_2 + 2) \\
&= \varphi_1\varsigma_{12}(\tau_1 - 1, \tau_2 + 2) \\
&= \varphi_1(\tau_1 - 1, \tau_2 + 1) \\
&= (\tau_1, \tau_2) = \tau.
\end{aligned}$$

The argument is the same when τ_1 and τ_2 reverse roles.

(iii) If τ does not meet above hypotheses and τ_1 contains $\boxed{2}$, then

$$\begin{aligned}\varphi^4(\tau) &= \varphi^3\varphi_1(\tau_1 + 1 - 2, \tau_2)\\ &= \varphi^3(\tau_1 - 2, \tau_2 + 1)\\ &= \varphi^2\varphi_1(\tau_1 - 2, \tau_2 + 2)\\ &= \varphi^2(\tau_1 - 2, \tau_2 + 2)\\ &= \varphi\varphi_1(\tau_1 - 2, \tau_2 + 1)\\ &= \varphi_1\varsigma_{12}(\tau_1 + 1 - 2, \tau_2)\\ &= \varphi_1(\tau_1, \tau_2)\\ &= (\tau_1, \tau_2) = \tau,\end{aligned}$$

and similarly if τ_2 contains $\boxed{2}$.

(iv) Finally, if τ contains no 1s or 2s, then $\varphi_1(\tau) = \varsigma_{12}(\tau) = \tau$, and so $\varphi^4(\tau) = \tau$.

Finally, we can conclude that $(\varphi_1\varphi_{12})^8 = (\varphi_1\varsigma_{12})^4 = 1$, and so $H_1 = \langle\varphi_1, \varphi_{12}\rangle \cong D_8$. An analogous proof would show that $H_2 = \langle\varphi_{n-1}, \varphi_{n-1,n-2}\rangle \cong D_8$.

In order show that $\langle\varphi_1, \varphi_{12}, \varphi_{n-1}, \varphi_{n-1,n-2}\rangle \cong D_8 \times D_8$, it is necessary to show that $H_1 \cap H_2 = \{1\}$ and φ_1 and φ_{12} both commute with φ_{n-1} and $\varphi_{n-2,n-1}$.

H_1 and H_2 intersect trivially for $n \geq 6$ since automorphism in H_1 only affect 1 and 2, while automorphisms in H_2 only affect $n-2$ and $n-1$. It is necessary here that $n \geq 6$ because having configurations of $3, \ldots, n-3$ in tableaux remain unchanged under the changes and slides under consideration. This prevents automorphism from having a description in terms of φ_1 and φ_{12} and in terms of φ_{n-1} and $\varphi_{n-2,n-1}$, and thus occurring in H_1 and H_2. For example, in Γ_5 $\varphi_1\varphi_2 = \varphi_{n-2}\varphi_{n-1}$ (it can be shown that $\varphi_2 = \varphi_{12}\varphi_1\varsigma_{12}\varphi_{12}\varsigma_{12} = \varphi_{12}(\varphi_1\varphi_{12})^3 \in H_1$ and analogously that $\varphi_{n-2} \in H_2$).

Finally, since all four elements are involutions, to show commutativity its suffices to show that $(\varphi_1\varphi_{n-1})^2 = (\varphi_1\varphi_{n-1,n-2})^2 = (\varphi_{n-1}\varphi_{12})^2 = (\varphi_{1,2}\varphi_{n-1,n-2})^2 = 1$. To avoid the tedious computations that are similar to the ones already written above, we remark that these equations follow from the fact that the definitions of slide-i and change-(j, k) only depend on and affect the occurrence and position of i and j, k, respectively. Thus, they act independent of each other when $i \neq j, k$, *i.e.* the order in which they are applied does not matter. □

Proposition 3. *The group* $G = \langle\varphi_1, \varphi_3, \ldots, \varphi_{n-4}, \varphi_{n-1}, \varphi_{12}, \varphi_{n-2,n-1}\rangle$ *is isomorphic to*

$$\mathbb{Z}_2^{n-6} \times D_8 \times D_8.$$

Proof. The previous proposition shows that $H = \langle\varphi_1, \varphi_{n-1}, \varphi_{12}, \varphi_{n-2,n-1}\rangle \cong D_8 \times D_8$. It is clear that $K = \langle\varphi_3, \ldots, \varphi_{n-4}\rangle \cong \mathbb{Z}_2^{n-6}$ since all slides commute with each other and are involutions (note that $|\{3, \ldots, n-4\}| = n-6$). To show that $G \cong \mathbb{Z}_2^{n-6} \times D_8 \times D_8$, we must show that $H \cap K = \{1\}$ since the commutativity of the slides $\varphi_3, \ldots, \varphi_{n-4}$ with elements of H is implied by the remarks at the end of the proof of proposition 2. Note that for $n = 6$, $\varphi_{n-4} = \varphi_2 \in H$ and that $\varphi_3 = \varphi_1\varphi_2\varphi_4\varphi_5 \in H$, so $G = H \cong D_8 \times D_8 \times \cong \mathbb{Z}_2^{6-6} \times D_8 \times D_8$ by proposition 2.

For $n \geq 7$, let $\tau = \begin{array}{|c|c|c|}\hline 3 & \cdots & n-4 \\ \hline 0 \\ \cline{1-1}\end{array}$. Any nonidentity automorphism $\varphi \in K$ can be written as string of the generators $\varphi_3, \ldots, \varphi_{n-4}$ in the form $\varphi_{i_1} \cdots \varphi_{i_m}$, where $m \leq n-6$, $i_j \in \{3, \ldots, n-4\}$ for $j = 1, \cdots, m$, and $j \neq k$ implies $i_j \neq i_k$. This follows from the facts that $\varphi_3, \ldots, \varphi_{n-4}$ commute with each other and each is an involution. Hence, if $\varphi = \varphi_{i_1} \cdots \varphi_{i_m}$ and $m < n-6$ (*i.e.* φ does not include all of $\varphi_3 \ldots, \varphi_{n-4}$), then

$$\varphi(\tau) = (\tau_1 - i_1 - \cdots - i_m, \tau_2 + i_1 + \cdots + i_m) \neq \tau.$$

However, τ is left fixed by $\varphi_1, \varphi_{n-1}, \varphi_{12}, \varphi_{n-2,n-1}$, and thus, left fixed by H. If $n = m$ (all of $\varphi_3 \ldots, \varphi_{n-4}$ occur in φ), then

$$\varphi(\tau_1 + (n-3), \tau_2) = \varphi\left(\begin{array}{|c|c|c|c|}\hline 1 & \cdots & n-4 & n-3 \\ \hline 0 \\ \cline{1-1}\end{array}\right) = \begin{array}{|c|c|c|}\hline 3 & \cdots & n-4 \\ \hline n-3 \\ \cline{1-1}\end{array},$$

but $(\tau_1 + (n-3), \tau_2)$ is left fixed by H as well. It follows that no element in K is generated by elements if H, and so H and K intersect trivially for $n \geq 7$. This ends the proof. ☐

Notice that $\varphi_{n-3} = \varphi_1 \varphi_2 \cdots \varphi_{n-4} \varphi_{n-2} \varphi_{n-1} \in G$, where G is defined as in the above proposition, and so G is the group generated by all changes and slides.

Proposition 4. *Let* $G_n = \langle \varphi_1, \varphi_3, \ldots, \varphi_{n-4}, \varphi_{n-1}, \varphi_{12}, \varphi_{n-2,n-1}, \rho_n \rangle$. *Then* G_n *is isomorphic to*

$$(\mathbb{Z}_2^{n-6} \times D_8 \times D_8) \rtimes \mathbb{Z}_2,$$

and in particular, $|G_n| = 2^{n+3}$.

Proof. Let $N = \langle \varphi_1, \varphi_3, \ldots, \varphi_{n-4}, \varphi_{n-1}, \varphi_{12}, \varphi_{n-2,n-1} \rangle \cong \mathbb{Z}_2^{n-6} \times D_8 \times D_8$ (by previous proposition), and $H = \langle \rho_n \rangle \cong \mathbb{Z}_2$. To show that $G_n = N \rtimes H$, we must show that N is normal in G_n, $G_n = NH$, and $H \cap N = \{1\}$.

It is clear that ρ_n is not generated by $\varphi_1, \varphi_{12}, \varphi_{n-1}$, and $\varphi_{n-2,n-1}$, e.g. $\rho_n\left(\begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline\end{array}\right) = \begin{array}{|c|}\hline n-1 \\ \hline 0 \\ \hline\end{array}$, but any combination of the given changes and slides will never map $\begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline\end{array}$ to $\begin{array}{|c|}\hline n-1 \\ \hline 0 \\ \hline\end{array}$. Hence N and H intersect trivially.

To prove that $G_n = NH$ and N is normal in G_n, we use the following identities hold:

$$\rho_n \varphi_i \rho_n = \varphi_{n-i}, \text{ for all } i \in \{1, \ldots, n-1\} \quad \text{and} \quad \rho_n \varphi_{12} \rho_n = \varphi_{n-2,n-1}.$$

These identities imply that $G_n = NH$ since given any string of generators of G_n, every instance of ρ_n can be moved to the left end of the string by replacing $\varphi_i \rho_n$ with $\rho_n \varphi_i$, $\varphi_{12} \rho_n$ with $\rho_n \varphi_{n-2,n-1}$, or $\varphi_{n-2,n-1} \rho_n$ with $\rho_n \varphi_{12}$. Similarly, these identities imply that N is normal. Let $h \in N$ and $g \in G_n$, and suppose g contains k instances of ρ_n when written in terms of the generators. Since $\rho_n^2 = 1 \Leftrightarrow \rho_n^{-1} = 1$, there will be $2k$ occurrences of ρ_n in ghg^{-1}. After moving all $2k$ instances of ρ_n to the left, they cancel to 1, leaving a string of generators of N. Thus, N is normal in G_n.

It remains to show that the above identities hold. Let $\tau = (\tau_1, \tau_2)$ be a tableau in Γ_n, and write $\rho_n(\tau) = \overline{(}\tau) = (\overline{\tau_1}, \overline{\tau_2})$. First assume $n - i$ occurs in τ_1.

$$\begin{aligned} \rho_n \varphi_i \rho_n(\tau) &= \rho_n \varphi_i(\overline{\tau_1}, \overline{\tau_2}) \quad (n - i \text{ is changed to } i) \\ &= \rho_n(\overline{\tau_1} - i, \overline{\tau_2} + i) \\ &= (\tau_1 - (n - i), \tau_2 + (n - i)) = \varphi_{n-i}(\tau). \end{aligned}$$

If $n - i$ does not occur in τ or it occurs in both rows, then there will be no is or two is in $\overline{\tau}$, and so $\varphi_i(\overline{\tau}) = \overline{\tau}$ while $\varphi_{n-i}(\tau) = \tau$. Thus,

$$\rho_n \varphi_i \rho_n(\tau) = \rho_n \varphi_i(\overline{\tau}) = \rho_n(\overline{\tau}) = \tau = \varphi_{n-i}(\tau).$$

This proves the first identity. For the second identity, we will avoid going the tedious cases explicitly. However, the principle is the same as for the first identity. The configuration of $n-2$ and $n-1$ in τ will turn into the analogous configuration of 1 and 2 in $\overline{\tau}$. Then $\varphi 12(\overline{\tau})$ will affect the 1 and 2 analogously to how $\varphi_{n-2,n-1}(\tau)$ affects the original $n - 2$ and $n - 1$. Finally, applying the second ρ_n turns the configuration of 1 and 2 back into the analogous configuration with $n - 2$ in place of 2 and $n - 1$ in place of 1, while every other number is returned unchanged. ☐

6 O_n Orbit

The goal of this section we be to characterize an orbit and the stabilizer of an element in this orbit under the action of $\mathrm{Aut}(\Gamma_n)$, and use this information to show that $|\mathrm{Aut}(\Gamma_n)| = 2^{n+3}$.

Consider the set of tableaux for Γ_n $(n > 5)$ that contain

(i) no repeats,

(ii) at least one of 1 and 2, and at least one of $n - 1$ and $n - 2$,

(iii) and all of $3, \ldots, n - 3$.

Call this subset O_n.

Lemma 4. *(i) The size of O_n is 2^n.*

(ii) O_n is mapped to itself by changes, slides, and switches.

(iii) Tableaux in O_n have either $n - 3$, $n - 2$, or $n - 1$ positive entries.

(iv) The maximum distance from a vertex in O_n to any other vertex in Γ_n is $n - 1$.

(v) If w is any vertex not in O_n, then there is a vertex in Γ_n whose distance from w is at least n.

Proof. (i) Given a tableau in O_n, either one of 1 and 2 occur, they both occur in the same row, or they both occur in opposite rows, and similarly with $n-2$ and $n-1$. There are also two row orientations for each of the four possibilities for the 1 and 2 and the $n-2$ and $n-1$. Thus, there are $(4 \cdot 2)^2 = 64$ ways to place the 1, 2, $n-2$ and $n-1$. Now fix placement of 1, 2, $n-2$, and $n-1$. Notice that $|\{3, \ldots, n-3\}| = n-5$. There are 2^{n-6} ways to place $n-3$ in the top row among the fixed 1, 2, $n-2$, and $n-1$ and to place the remaining $n-6$ numbers $3, \ldots, n-4$ in either of the two rows. Fixing the $n-3$ in the top row does not force us to undercount $|O_n|$ since we are concerned with the relative positions of 1, 2, $n-2$, $n-1$ and $3, \ldots, n-3$, and not the absolute position of the rows with respect to the conventions (e.g. the longer row is written on top). Therefore, there is a total of $2^{n-6} \cdot 64 = 2^n$ tableaux in O_n.

(ii) It is clear that O_n is mapped to itself under slides since slides do not change which numbers appear in a tableau, only where they appear, and the definition of O_n only includes rules about which numbers appear. Change-(1,2) automorphisms only affect the 1s and 2s that appear in a tableau. The possible configurations (relative row positions) of 1 and 2 for a tableau in O_n is $\boxed{1}$, $\boxed{2}$, $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$, and $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$, so we must check that applying φ_{12} to these valid configuration will produce another valid configuration. Abusing notation, we see that this is the case since $\varphi_{12}\left(\boxed{1}\right) = \boxed{2}$, $\varphi_{12}\left(\boxed{2}\right) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$, $\varphi_{12}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right) = \boxed{2}$, and $\varphi_{12}\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right) = \boxed{1}$. The argument that O_n is invariant under change-$(n-2, n-1)$ is analogous. Finally, switches are just slide automorphisms conjugated by changes, so it follows automatically that O_n is mapped to itself by switches.

(iii) It is clear from (ii) and (iii) of then definition of O_n that there are at least $n-3$ positive entries in each tableau τ in O. There can be $n-2$ entries if either 1 and 2 both occur in τ or $n-2$ and $n-1$ both occur, and there can be $n-1$ entries in τ if all of $1, 2, n-2$, and $n-1$ are in τ. This covers all possibilities since the only way to get more than $n-1$ positive entries is to include repeats.

(iv) We proceed by strong induction on $n \geq 6$. The claim is easily verified by computer for $n = 6, 7$. Suppose the claim holds for some $n \geq 8$, and let τ be a tableau in O_{n+1} and let σ any other tableau in Γ_n. From Alexander Garver's inductive construction of the tableaux in [3], we have that

(1) $\sigma = \begin{array}{|c|}\hline \mathrm{n} \\ \hline 0 \\ \hline \end{array}$

(2) σ is a vertex in Γ_n,

(3) σ is a vertex from Γ_n with n added to a row,

(4) or $\sigma = \sigma' + \begin{array}{|c|c|}\hline \mathrm{t} & \mathrm{n} \\ \hline \mathrm{t} \\ \cline{1-1} \end{array}$, where σ' is a vertex from $\Gamma(2\alpha_0, A_t)$ for $t = 2, \ldots, n-1$.

We proceed by cases,

(1) Suppose σ satisfies 1. Recall that τ has at most $n-1$ positive entries by (iii). If n does not occur in τ, then we can create a path from τ to σ by adding n to τ and deleting every other positive entry in τ one at a time. Thus, $\partial(\tau, \sigma) \leq n$. If n does occur in τ, then deleting every other positive integer results in a path of length at most $n-2$.

(2) Suppose that σ satisfies 2. Let τ' by the tableau obtain by deleting n from τ if n occurs and $\tau' = \tau$ if n does not occur. Since $\tau \in O_{n+1}$, we have that $\tau' \in O_n$ and so $\partial(\tau', \sigma) \leq n-1$ by the inductive hypothesis. If n does not occur in τ, then we are done, and if n does occur, then $\partial(\tau, \sigma) \leq n$ since $\tau \sim \tau'$.

(3) Suppose σ satisfies 3. Let τ' be as above and let σ' be the tableau obtained by deleting n from σ. Then $\partial(\tau', \sigma') \leq n-1$ by the inductive hypothesis. If n does not occur in τ, then $\partial(\tau, \sigma) \leq n$ since $\sigma \sim \sigma'$. If n does occur in σ, then inserting n to each vertex in the path from τ' to σ' in Γ_n gives a path of length at most $n-1$ in Γ_{n+1} from τ to σ.

(4) Suppose σ satisfies 4. For $t \geq 6$, deleting every vertex in τ greater that or equal to t yields a tableau $\tau' \in O_t$. If σ' is the tableau obtained by deleting both ts and the n from σ, then $\partial(\tau', \sigma') \leq t-1$ by the inductive hypothesis. If n does not occur in τ, then we would have deleted at most $n-t-1$ entries from τ to get τ', and we can also be certain that t is in τ since $3, \ldots, n-1$ must all occur in τ. Adding t to both rows and n to a row of every vertex in the path from τ' to σ' in Γ_t gives a path from $\tau' + \begin{array}{|c|c|}\hline \mathrm{t} & \mathrm{n} \\ \hline \mathrm{t} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ to σ in Γ_{n+1} of length at most $t-1$. Since $\partial\left(\tau, \tau' + \begin{array}{|c|c|}\hline \mathrm{t} & \mathrm{n} \\ \hline \mathrm{t} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) \leq n-t+1$, there is path of length at most $(t-1)+(n-t+1) = n$ from τ to σ.

If n does occur in τ, then we would have deleted at most $n-t$ entries from τ to get τ' and we can still be certain that t occurs once in τ. Thus, $\partial\left(\tau, \tau' + t\right) \leq n-t+1$, and so we have a path of length at most $(n-t+1)+(t-1) = n$ from τ to σ.

The remainder of this argument is incomplete If τ has $n-2$ positive entries, then there are two or three matches between τ and σ. If there are two matches, then deleting the $n-4$ non matching entries, adding another t to τ, and up to 3 entries still missing yields a path of length at most $n-4+1+3 = n$ from τ to σ. The cases when $t=4$ and $t=5$ are similar.

In each case, we find that the distance from τ to any other vertex is at most n, which proves the induction step.

(v) Let τ be a vertex tableau in Γ_n that is not in O_n. Then either τ contains repeats, does not contain 1 and 2 or $n-2$ and $n-1$, or does not contain all of $3, \ldots, n-3$. Let $k \leq n-1$ be the number of distinct positive integers in τ and let $m = |\tau|$ be the total number of positive integers in τ.

If τ contains repeats, then $m > k$. If $k = n-1$ and $m = n$ (*i.e.* τ has a single repeated entry), then there exist consecutive integers $i, j < n$, neither of which are repeated in τ. Deleting each of the $n-1$ entries besides i and then inserting j into the row opposite to where it appears with respect to i in τ produces a path of length $(n-1)+1=n$. If $k=n-1$ and there are multiple repeats, then $m>n$ and deleting all but one integer i from τ will produce a path of length $m-1\geq n$ from τ to $\begin{array}{|c|}\hline \text{i}\\\hline 0\\\hline\end{array}$.

If $k<n-1$, then inserting the $n-1-k$ integers that do not appear in τ and deleting the m original entries produces a path of length $n-1-k+m\geq n$ (since $m>k$).

For the rest of the cases, assume that τ does not have any repeats, *i.e.* $k=m$. For each case, insert the $n-1-m$ integers which do not appear in τ. Choose one of the $n-1-m$ integers, say i not appearing τ to be inserted in both rows, and only choose 2 or $n-2$ if 1 or $n-1$ do not appear τ, respectively. Now the original m entries are "separated" by the two is, that is the original entries less than i and those greater than i. Pair the original m entries less than i as much as possible (if there are an odd number of these, there will be one leftover), and for each of these pairs move one of the number to the opposite row, and do the same action for the original entries greater than i. Finally, delete any unpaired original entries. Here is an example for clarity where $i=4$, the pairs are (1,3) and (5,6), and the 2 is unpaired:

$$\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline 4&6\\\cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1&4&5&6\\\hline 3&4\\\cline{1-2}\end{array}$$

Notice that moving an entry to the opposite row in this manner takes two steps (deleting and inserting). Thus, if m is even, there are $m/2$ pairs, each of which contribute two steps to the path for a total of $2\cdot(m/2)=m$ steps, and if m is odd, there are $(m-1)/2$ pairs and 1 unpaired entry for a total of $2\cdot(m-1)/2+1=m$ steps. Therefore, the process described for this case has a total of $(n-1-m)+1+m=n$ steps, and it is left to show that this is the shortest possible path between the endpoints. **(Not Sure How To Prove This Is the Shortest Path).**

□

Proposition 5. *O_n forms a single orbit under the action of* $\mathrm{Aut}(\Gamma_n)$.

Proof. A vertex in $\tau\in O_n$ cannot be mapped by an automorphism φ to a vertex $\varphi(\tau)$ outside the O_n because the maximum distance from τ to another vertex would have to be the same as the maximum distance from $\varphi(\tau)$ to another vertex and by (iv) and (v) in the previous lemma, this is not the case. By (ii) above, $\mathrm{Aut}(\Gamma_n)$ is transitive on O_n, and so O_n is an orbit (a more explicit description of how these automorphisms act on O_n will become apparent below). □

Lemma 5. *There are 2^{n-6} connected components in the subgraph of Γ_n induced by O_n. Each connected component is isomorphic to the graph obtained by taking eight octagons and joining corresponding corners (to make a sort of toroidal shape).*

Proof. We can see that each connected component of O_n is 4-regular. Given a vertex $\tau\in O_n$, the entries $3,\ldots,n-3$ have a certain configuration relative to each other, e.g. they can all be in the same row, all but fixed integer are in the same row, and so on. A vertex with a certain configuration of $3,\ldots,n-3$ cannot be adjacent to a vertex with a difference configuration since this would require that at least one of $3, \ldots, n-3$ switch rows, which requires at least two steps of deleting and inserting numbers (since there are no repeats in τ). Thus, the vertices adjacent to τ depend only on the configuration of 1, 2, $n-2$, and $n-1$ in τ. If only one of 1 and 2 occur, then we obtain two vertices adjacent to τ by inserting the other to either of the two rows in τ. If both 1 and 2 occur in τ, then we obtain two vertices adjacent to τ by deleting 1 or 2. Similarly, there are two ways to obtain vertices adjacent to τ if one of $n-2$ and $n-1$ occur or if both of $n-2$ and $n-1$ occur. Therefore, there are 4 total ways to obtain vertices in O_n adjacent to a given vertex.

The above observation implies that there are 2^{n-6} connected components in the subgraph induced by O_n since there are 2^{n-5} configurations of $3, \ldots, n-3$, which over counts the by a factor of two since the orientation of the rows does not matter in terms of connectivity. Thus, the number of components is $2^{n-5}/2 = 2^{n-6}$. As in the proof that $|O_n| = 2^n$, there are 64 ways of placing 1, 2, $n-2$, and $n-1$ for a given configuration of $3, \ldots, n-3$, and so there are 64 vertices in each connected component.

Now we show that each component if isomorphic to the graph obtained by taking eight octagons and joining corresponding corners (imagine stacking eight octagons so that the corners are aligned, then connecting the corners to make an octagonal cylinder, and then connecting the ends of the cylinder to make torus). Fix a configuration of $3, \ldots, n-3$ for the connected component. As described above in showing that $|O_n| = 2^n$, there are eight ways to place $n-2$ and $n-1$ with respect to the configuration of $3, \ldots, n-3$. Fix a placement of $n-2$ and $n-1$. There are also eight ways of placing 1 and 2, giving us the following template for an octagon: (Notation: (placed in first row, placed in second row) and 0 means insert nothing)

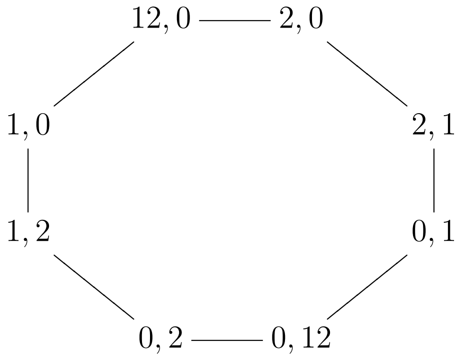

Figure 1: (1,2)-Octagon Template

Attaching the fixed configuration of $3, \ldots, n-3$ and $n-2$ and $n-1$ to each vertex above gives the full octagon. Notice that since $3, \ldots, n-1$ are fixed, adjacency only depends on 1 and 2, and it is easily checked that the vertices in the above figure satisfy the rule for adjacency given by lemma 2. There are eight octagons with the above template determined by each of the eight configurations of $n-2$ and $n-1$. Take these eight octagons and match corners with the same placement of $n-2$ and $n-1$, and then arrange the octagons in the order specified by the second template shown below:

As above, adjacency above is easily confirmed after attaching a fixed configuration of

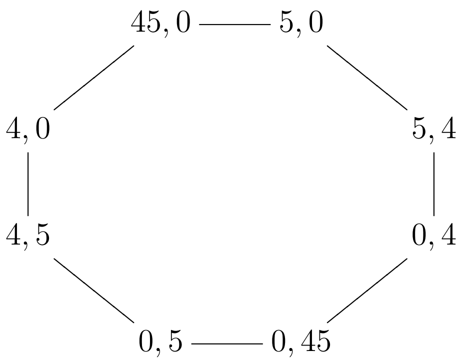

Figure 2: (4,5)-Octagon Template

$1, \ldots, n-3$. This completes the construction (each vertex has four neighbors, two in each template above). In short, each component can be thought of as the intersection points of eight latitudinal circles and eight longitudinal circles on a torus. If one picks a tableau τ in a connected component of O_n, one obtains one octagon by fixing the configuration of $1, \ldots, n-3$ in τ and varying the placement of $n-2$ and $n-1$, and another octagon by fixing the configuration of $3, \ldots, n-1$ and varying the placement of 1 and 2. □

Let C_0 be the component of O_n where $3, \ldots, n-3$ are all in the same row. Adjacency in C_0 is indicated by the two octagonal templates shown above. Thus, an automorphism φ of C must must be an automorphism of the templates, which clearly have automorphism groups isomorphic to D_8. Furthermore, there is an automorphism that "switches" these templates, namely the palindromic automorphism ρ_n. For a more geometric description, the eight latitudinal octagons all have the same template, while the eight longitudinal octagons have the other template. Thus, the automorphisms of these templates correspond to the symmetries of the octagonal torus, *i.e.* rotating about an axis through the "hole" of the torus and twisting the torus latitudinally. One can imagine the palindromic automorphism which maps between templates as puncturing a hole in the torus and then turning it inside-out to form a new torus. These automorphisms must generate all of $\mathrm{Aut}(C_0)$. The following gives a more technical description of $\mathrm{Aut}(C_0)$ in terms of known automorphisms.

Proposition 6. *Let C_0 be the component of O_n where $3, \ldots, n-3$ are all in the same row. Then*

$$\mathrm{Aut}(C_0) = \langle \varphi_1, \varphi_{12}, \varphi_{n-1}, \varphi_{n-2,n-1}, \rho_n \rangle$$

and is isomorphic to $(D_8 \times D_8) \rtimes \mathbb{Z}_2$. In particular, $|\mathrm{Aut}(C_0)| = 2^9$.

Proof. It is clear that the change and switch automorphisms map C_0 onto itself since they do not the configuration of $3, \ldots, n-3$. Similarly, since $3, \ldots, n-3$ occur in the same in C_0, ρ_n does not change their configuration, and thus maps C_0 onto itself. The $\langle \varphi_1, \varphi_{12} \rangle \cong D_8$ correspond to the automorphisms of the $(1,2)$-octagons from figure 1, and similarly $\langle \varphi_{n-1}, \varphi_{n-2,n-1} \rangle \cong D_8$ correspond to the automorphisms of the $(n-2, n-1)$-octagons from figure 2.

Finally, proposition 3 implies that $\mathrm{Aut}(C_0) \cong (D_8 \times D_8) \rtimes \mathbb{Z}_2$. □

$\mathrm{Aut}(\Gamma_n)$ acts transitively on the set of connected components of O_n. If a vertex in a component C_1 is mapped to a vertex from another component C_2, then all of C_1 must be mapped to C_2 since automorphisms must preserve adjacency. Transitivity follows because applying an appropriate combination of $\varphi_3, \ldots, \varphi_{n-3}$ will map between the various possible configurations of $3, \ldots, n-3$ that correspond to the different components. Moreover, the stabilizer of C_0 under this action is $\mathrm{Aut}(C_0)$ since these automorphisms necessarily map C_0 to C_0. Since there are 2^{n-6} components, the orbit-stabilizer theorem implies that

$$|\mathrm{Aut}(\Gamma_n)| = 2^9 \cdot 2^{n-6} = 2^{n+3}.$$

Since this is the size of the subgroup of $G_n \leq \mathrm{Aut}(\Gamma_n)$ from proposition 4, we have proven the following theorem:

Theorem 1. *For $n \geq 6$*

$$\mathrm{Aut}(\Gamma_n) = \langle \varphi_1, \varphi_3, \ldots, \varphi_{n-3}, \varphi_{n-1}, \varphi_{12}, \varphi_{n-1,n-2}, \rho_n \rangle \cong (\mathbb{Z}_2^{n-6} \times D_8 \times D_8) \rtimes \mathbb{Z}_2.$$

Note that the generators in the theorem is not the smallest generating set, but is easy relatively easy to describe. For $n < 6$, the group structure returned by GAP ([2]) is summarized in table 1, where $PSL(3,2)$ is the projective special linear group over a vector space of dimension 3 over $\mathbb{F}_2$.

Table 1: Structure of $\mathrm{Aut}(\Gamma_n)$ for $n < 6$

n	$\mathrm{Aut}(\Gamma_n)$	$\lvert\mathrm{Aut}(\Gamma_n)\rvert$
2	1	1
3	D_4	8
4	$PSL(3,2) \rtimes \mathbb{Z}_2$	336
5	$((D_4 \times D_4) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	256

References

[1] Leonard Chastkofsky. (0,2)-graphs and root systems. 2009.

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[4] James Humphreys. *Reflection Groups and Coxeter Groups.* Cambridge University Press, 1992.