Constructing Vertices of (0,2)-graphs from Root Systems in A_n

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July 21, 2009

Abstract

A (0,2)-graph is a connected graph Γ where any pair of vertices $a, b \in V(\Gamma)$ have either 0 or 2 common neighbors. Known examples of (0,2)-graphs include hypercubes and incidence graphs of projective planes. Recently a construction of (0,2)-graphs from root systems associated with lie algebras of type A_n has been found. Given a target vector β, form a graph denoted by $\Gamma(A_n, \beta)$ consisting of the subsets of the set of positive roots of A_n such that the sum of the roots in that subset equals β. For example, $\Gamma(A_n, \alpha_0)$ where $\alpha_0 = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ is associated to the graph of the hypercube. It is natural to consider other cases with target $k\alpha_0$ for an integer $k > 1$. We have Weyl's dimension formula for counting the vertices in these graphs, but this formula is difficult to use. We show how tableaux can be used to represent the vertices of these graphs and show how this representation can be used to count the vertices of $\Gamma(A_n, k\alpha_0)$ using an inductive method. In particular, we have found a closed recursive formula for the vertices of $\Gamma(A_n, 2\alpha_0)$ and now we are trying to find a recursive formula that will work for $\Gamma(A_n, 3\alpha_0)$ with the intention of generalizing to $k\alpha_0$. I will discuss our progress on this construction.

1 Introduction

A (0,2)-graph is a connected graph Γ where any pair of vertices $a, b \in V(\Gamma)$ have either 0 or 2 common neighbors. These graphs arise in many different contexts. Examples of these graphs include hypercubes and the incidence graphs of projective planes. Recently a construction of (0,2)-graphs from root systems associated with lie algebras of type A_n has been found. We can count the number of vertices in these graphs by using Weyl's dimension formula, but computations can be difficult. We have tried to determine a closed formula that will allow for more straightforward computations. To this end, we have introduced tableau notation for representing the vertices of the (0,2)-graphs that come from our construction.

2 Target Vectors & Tableau Notation

Let Φ be the set of roots of a root system of type A_n and $\Phi^+ \subseteq \Phi$ denote the set of positive roots.

Definition 2.1. *The* **weight** *of a subset Ψ of Φ^+ is defined to be*

$$wt(\Psi) = \sum_{\alpha \in \Psi} \alpha$$

where α is a simple root or a sum of simple roots.

For a fixed target vector, we take all subsets Ψ with weight equal to our target vector and associate a vertex of the graph Γ with each subset. When we define adjacency between 2 subsets to be $\Psi_1 \sim \Psi_2$ when $\Psi_1 \oplus \Psi_2 = 3$, we get a (0,2)-graph.

Let $\Gamma(A_n, k\alpha_0)$ denote the (0,2)-graph associated with the root system of type A_n with target vector $k\alpha_0$ with $k \in \mathbb{Z}^+$. Our claim is that the number of vertices of $\Gamma(A_n, 2\alpha_0)$ denoted by $|\Gamma(A_n, 2\alpha_0)|$ is given by the following recursion

$$|\Gamma(A_n, 2\alpha_0)| = 3|\Gamma(A_{n-1}, 2\alpha_0)| + |\Gamma(A_{n-2}, 2\alpha_0)| + ... + |\Gamma(A_2, 2\alpha_0)| + 1.$$

To demonstrate that our claim holds we must first define some terms and describe our method for constructing the vertices.

For now, we are only concerned with the subsets Ψ of Φ^+ where $wt(\Psi) = 2\alpha_0$.

Lemma 2.2. *For a fixed weight of $2\alpha_0$, all subsets Ψ of Φ^+ must be of the form*

$$\begin{aligned} \{\ &\alpha_1 + ... + \alpha_{a_1},\ \alpha_{a_1+1} + ... + \alpha_{a_2}, ...,\ \alpha_{a_r+1} + ... + \alpha_n, \\ &\alpha_1 + ... + \alpha_{b_1},\ \alpha_{b_1+1} + ... + \alpha_{b_2}, ...,\ \alpha_{b_s+1} + ... + \alpha_n\}. \end{aligned}$$

Proof. Assume we have a fixed weight of $2\alpha_0$ and we are interested in subsets of the positive roots Φ^+. Now consider $\beta_1, \beta_2, ..., \beta_k$ where

$$\sum_{i=1}^{k} \beta_i = 2\alpha_0.$$

So it must be the case that

$$\begin{aligned} \beta_1 &= \alpha_1 + ... + \alpha_r \\ \beta_2 &= \alpha_{r+1} + ... + \alpha_s \\ &\vdots \\ \beta_p &= \alpha_{s+t} + ... + \alpha_n \end{aligned}$$

for some p where $1 \leq p < k$. To see this suppose not, so that

$$\begin{aligned} \beta_1 &= \alpha_1 + ... + \alpha_r \\ \beta_2 &= \alpha_r + ... + \alpha_s \\ &\vdots \\ \beta_p &= \alpha_{s+t} + ... + \alpha_n \end{aligned}$$

where α_r shows up in both β_1 and β_2.

Then there must be some $\beta_{p+q} = \alpha_{r-l} + ... + \alpha_{r-1} + \alpha_{r+1} + ... + \alpha_{r+m}$ for some $q, l, m \in \mathbb{Z}^+$ so that the β_ks sum to $2\alpha_0$. However, because of the form of the roots of A_n it is not possible to have a root of this sort. Thus, for a fixed weight $2\alpha_0$, all subsets Ψ of Φ^+ must be of the form

$$\begin{aligned} \{\ &\alpha_1 + ... + \alpha_{a_1},\ \alpha_{a_1+1} + ... + \alpha_{a_2}, ...,\ \alpha_{a_r+1} + ... + \alpha_n, \\ &\alpha_1 + ... + \alpha_{b_1},\ \alpha_{b_1+1} + ... + \alpha_{b_2}, ...,\ \alpha_{b_s+1} + ... + \alpha_n\}. \end{aligned}$$

□

Thus, for a subset Ψ of Φ^+ we have a set containing simple roots and sums of simple roots of the form

$$\begin{aligned} \{\ &\alpha_1 + \ldots + \alpha_{a_1},\ \alpha_{a_1+1} + \ldots + \alpha_{a_2},\ \ldots,\ \alpha_{a_r+1} + \ldots + \alpha_n, \\ &\alpha_1 + \ldots + \alpha_{b_1},\ \alpha_{b_1+1} + \ldots + \alpha_{b_2},\ \ldots,\ \alpha_{b_s+1} + \ldots + \alpha_n\}. \end{aligned}$$

These subsets correspond to the vertices of $\Gamma(A_n, 2\alpha_0)$. However, working with these subsets is often inconvenient so introduce the following notation.

We can associate with Ψ a **tableau** with 2 rows. We look at $\phi_1 \subset \Psi$ where $wt(\phi_1) = \alpha_0$. In the first row of our tableau, we list in ascending order the largest subscript of each simple root or sum of simple roots in ϕ_1. Similarly, we consider the other subset ϕ_2 of Ψ which has $wt(\phi_2) = \alpha_0$. In the second row of our tableau, we list in ascending order the largest subscript of each simple root or sum of simple roots in ϕ_2. Thus the tableau associated with Ψ is follows:

$$\begin{array}{|c|c|c|c|}\hline a_1 & a_2 & \dots & a_r \\ \hline b_1 & b_2 & \dots & b_s \\ \hline \end{array}$$

where all a_i, $b_j \in \mathbb{Z}^+$.

Example Let Ψ be the set of positive roots $\{\alpha_1,\ \alpha_2,\ \alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_4\}$ from $\Gamma(A_4, 2\alpha_0)$. We associate with Ψ the tableau $\begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1}\end{array}$.

Example Consider the tableau $\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\ \hline 2&4 \\ \cline{1-2}\end{array}$ from $\Gamma(A_6, 2\alpha_0)$. This tableau can be written in the following ways

$$\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\ \hline 2&4 \\ \cline{1-2}\end{array} \sim \begin{array}{|c|c|c|c|}\hline 2&3&4&5\\ \hline 1&2&4 \\ \cline{1-3}\end{array} \sim \begin{array}{|c|c|c|c|}\hline 1&2&4&5\\ \hline 2&3&4 \\ \cline{1-3}\end{array} \sim \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 2&4&5 \\ \cline{1-3}\end{array}$$

Each of these tableaux represent the same subset of Φ^+. This defines an equivalence relation on tableaux. This characteristic of our tableau notation requires that we define a standard ordering for tableaux. Defining a standard order means choosing a representative for each equivalence class.

We define our standard order in the following way. Here we can assume WLOG that $r \geq s$ and that if $r = s$ then $a_1 < b_1$ and also if for any i, j where $a_i = b_j$, then $a_{i+1} < b_{j+1}$. In terms of the above example, we would choose $\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\ \hline 2&4 \\ \cline{1-2}\end{array}$ for our representative of the equivalence class. We introduce this representation for weight $2\alpha_0$, but we will extend this notation later to $k\alpha_0$.

Definition 2.3. *A tableau associated with a subset $\Psi \subset \Phi^+$ with weight $2\alpha_0$ is called* **admissible** *if it satisfies the following properties:*

(1) The length of the first row is greater than or equal to the length of the second row.

(2) The first and last elements of the first row don't repeat in the second row as the first and last entries, respectively.

(3) 2 consecutive elements in the first row do not repeat in the second row.

Example The tableaux $\begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\ \hline 2&4 \\ \cline{1-2}\end{array}$ are admissible. The tableau $\begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 4 \\ \cline{1-1}\end{array}$ from $\Gamma(A_5, 2\alpha_0)$ is not admissible. Referring to the previous example, all of the ways of writing $\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\ \hline 2&4 \\ \cline{1-2}\end{array}$ are admissible, but when constructing tableaux we use a canonical representative following our standard order to describe a set. In the example $\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\ \hline 2&4 \\ \cline{1-2}\end{array}$ is our canonical representative.

3 Construction for $2\alpha_0$

First considering the target vector $2\alpha_0$, we build our tableaux so that the entries in each row are in increasing order and so that the longest rows are the top rows. When we construct the tableaux for $\Gamma(A_n, 2\alpha_0)$ we get several different types of tableaux. Here we assume that the tableaux that we build new tableaux from are ordered. We construct them in the following way.

Type 1 If $\Gamma(A_{n-1}, 2\alpha_0) \neq \emptyset$, then we add each of the tableaux from $\Gamma(A_{n-1}, 2\alpha_0)$ to the set of tableaux in $\Gamma(A_n, 2\alpha_0)$.

Type 2 If $\Gamma(A_{n-1}, 2\alpha_0) \neq \emptyset$, take the tableaux from $\Gamma(A_{n-1}, 2\alpha_0)$ and add $n-1$ to the top row of each tableau to get new tableaux of the form

a_1	...	a_r	$n-1$
b_1	...	b_s	

We include each of these new tableaux in the set of tableaux $\Gamma(A_n, 2\alpha_0)$.

Type 3 If $\Gamma(A_{n-1}, 2\alpha_0) \neq \emptyset$, to each of its tableaux we add $n-1$ to the bottom row to get the new tableaux of the form

a_1	...	a_r	
b_1	...	b_s	$n-1$

If adding $n-1$ here makes the bottom row longer than the top, we switch the rows so that the longer row is on top. We include each of these new tableaux in the set of tableaux for $\Gamma(A_n, 2\alpha_0)$.

Type 4 If $\Gamma(A_{n-t}, 2\alpha_0) \neq \emptyset$ where $t = 1, ..., n-2$, we take all of the tableaux from $\Gamma(A_{n-t}, 2\alpha_0)$ and add $n-t$ and then $n-1$ to the top row and then we add $n-t$ to the bottom to get tableaux of the form

a_1	...	a_r	$n-t$	$n-1$
b_1	...	b_s	$n-t$	

We include each of these new tableaux in the set of tableaux for $\Gamma(A_n, 2\alpha)$.

Type $*$ We also construct one additional tableau of the form

$n-1$
0

and include it in our set of tableaux for $\Gamma(A_n, 2\alpha_0)$. Using this construction, the tableaux from $\Gamma(A_{n-1}, 2\alpha_0)$ are each used 3 times, each tableau from $\Gamma(A_{n-t}, 2\alpha_0)$ where $t = 1, ..., n-2$ is used once, and one tableau of Type $*$ is included as well. Thus our construction follows the recursion

$$|\Gamma(A_n, 2\alpha_0)| = 3|\Gamma(A_{n-1}, 2\alpha_0)| + |\Gamma(A_{n-2}, 2\alpha_0)| + ... + |\Gamma(A_2, 2\alpha_0)| + 1$$

exactly.

In order to show that the recursion holds, we must prove 2 propositions. We will do so inductively by assuming that the recursion holds for $\Gamma(A_n, 2\alpha_0)$ and show that this holds for $\Gamma(A_{n+1}, 2\alpha)$. Our propositions are:

Proposition 3.1. *Given our construction for* $2\alpha_0$

1. *Each tableau created by our construction above is admissible.*
2. *Every admissible tableau in* $\Gamma(A_{n+1}, 2\alpha)$ *arises in this way.*

Proof. Let $r, s \in \mathbb{Z}^+$ where r is the length of the first row and s is the length of the second row. Here we'll denote the entries in the first and second rows by $a_1, ..., a_r$ and $b_1, ..., b_s$, respectively.

1. The Type 1 tableaux are all admissible because they were admissible for $\Gamma(A_n, 2\alpha_0)$. All tableaux of Type $*$ are trivially admissible.

Now consider the Type 2 tableaux. They were previously admissible which implies that $r \geq s$ and thus by simply adding an entry to the top row we know that $r + 1 \geq s$ is true $\implies$ (1). Because the

Type 2 tableaux were previously admissible, we know that $a_1 \neq b_1$ and that $a_r \neq b_s$ and since there is only one n entry in the tableaux, we know (2) holds. The tableaux were previously admissible so there would not be 2 consecutive elements in the first row that repeat in the second. Furthermore, by only adding n to the top row we would not cause there to be 2 consecutive elements in the first row that repeat in the second $\Longrightarrow$ (3). Thus all tableaux of Type 2 are admissible.

Now we consider the Type 3 tableaux. All Type 3 tableaux were previously admissible so we know $r \geq s$. If $r = s$, then by construction n would appear in the top row to keep the longer rows on top. So $r \geq s+1 \implies$ (1). Because these tableaux were previously admissible we know $a_1 \neq b_1$ and $a_r \neq b_s$. Since n only appears once in these tableaux, we know $a_r \neq n \implies$ (2). These tableaux were previously admissible so there are no instance in these tableaux where 2 consecutive elements in the first row repeat in the second. So it follows that since n only appears once in these tableaux there will be no instances created by adding n where (3) is violated. Thus all tableaux of Type 3 are admissible.

Now consider the Type 4 tableaux. These tableaux were previously admissible so $r \geq s$ and thus $r+2 \geq s+1$ is true $\implies$ (1). Because these tableaux were previously admissible, $a_1 \neq b_1$ and $a_r \neq b_s$. For Type 4 tableaux the last entries in the tableaux are n and $n-t \neq n \implies$ (2). Lastly, since these tableaux were previously admissible and since $a_r \neq b_s$, there are no instances in these tableaux where 2 consecutive elements in the first row repeat in the second row. That is, (3) holds. Thus all tableaux of Type 4 are admissible. In all, each tableau created by our construction above is admissible.

2. Choose any admissible tableau from $\Gamma(A_{n+1}, 2\alpha_0)$. This gives us a few different cases.

Case 1: Suppose n does not appear at the end of any row in the tableau, then by our construction n does not appear anywhere in the tableau. Because the tableau is admissible and because n does not appear in the tableau, it must have come from $\Gamma(A_n, 2\alpha_0)$. That is, it must be of Type 1.

Case 2: Suppose $a_r = n$ and $a_{r-1} \neq b_s$. If $r-1 > s$, then the vertex was created by simply adding n to the top row; it is of Type 2. If $r-1 = s$, then n was either added to the top row (it is Type 2) or n was added to the second row (it is Type 3) and, following our construction, the rows were switched. Thus these admissible tableaux are either Type 2 or Type 3.

Case 3: Suppose $b_s = n$ and $a_r \neq b_{s-1}$. It follows that $r > s-1$ or else by following our construction we would switch the rows to have the longer row on top. Thus, the admissible tableaux of this form must be of Type 3.

Case 4: Suppose $a_r = n$ and $a_{r-1} = b_s$. Because these tableaux are admissible we know that $a_{r-2} \neq b_{s-1}$ so it follows that all tableaux of this form are of Type 4.

Case 5: Suppose n is the only nonzero entry. Then these tableaux must be of Type $*$. In all, every admissible tableau in $\Gamma(A_{n+1}, 2\alpha_0)$ is created following our construction.

□

4 Tableaux for Larger k

In moving to larger k, we need a more general description of tableaux.

Definition 4.1. *For fixed target vector $k\alpha_0$ with $k \in \mathbb{Z}^+$ and a subset Ψ of positive roots of A_n we associate with Ψ the* **tableau** *with k rows of the form*

a_1	$\ldots$	a_r
b_1	$\ldots$	b_s
$\vdots$	$\vdots$	$\vdots$
p_1	$\ldots$	p_t

with integer entries and $r, s, t \in \mathbb{Z}^+$.

The next definition identifies a special type of tableau that will be important in our construction for $3\alpha_0$.

Definition 4.2. *A tableau from* $\Gamma(A_n, k\alpha_0)$ *where* $k > 1$ *is called* **skinny** *if it is of the form*

a_1	...	a_r
b_1		
$\vdots$		
p_1		

where $b_1 \neq 0$, $\ldots$, $p_1 \neq 0$.

As k gets bigger we have instances where repeat entries in tableaux are more common. Recall that repeat entries in a tableau indicate that the tableau can be written in more than one way.

Definition 4.3. *Given a tableau from* $\Gamma(A_n, k\alpha_0)$ *where* $k > 1$ *containing entries that repeat in different rows, the entries in a row following a repeat entry are called the* **tail** *of the row.*

Defining the tail of a tableau provides an easier way to discuss larger tableaux with repeat entries.

Now we provide a general description of admissible tableaux. Note that we follow the same standard order that we used for $2\alpha_0$.

Definition 4.4. *For rows* a, b *where* $1 \leq a < b \leq k$ *in a tableau with* k *rows associated with a subset* $\Psi \subseteq \Phi$ *with weight* $k\alpha_0$ *where* $k \in \mathbb{Z}^+$ *is called* **admissible** *if it satisfies the following properties:*

(1) length(a) $\geq$ *length(b)*
(2) $a_1 \neq b_1$
(3) $a_{length(a)} \neq b_{length(b)}$
(4) if $a_l = b_m$ *for some* l, m, *then* $a_{l+1} \neq b_{m+1}$.

5 Construction for $3\alpha_0$

Here we describe how we may construct the tableaux for $\Gamma(A_n, 3\alpha_0)$. As in the case of $\Gamma(A_n, 2\alpha_0)$, we create ordered tableaux. If rows have the same length, we order the rows so that

$$a_1 < b_1 < c_1$$

If repeat entries occur in a tableau we arrange the tails of the tableau by length with the longer tails closer to the top row. In the case where the tails have equal length, we order them in the same fashion that we order rows of equal length.

Unfortunately, we were not able to describe how all tableaux arise in $\Gamma(A_n, 3\alpha_0)$. We found that trying to determine the rules for building these tableaux becomes much more difficult. Because of the additional row in each tableau, we need to define many more rules for building these.

Type 1 If $\Gamma(A_{n-1}, 3\alpha_0) \neq \emptyset$, we add all the tableaux from $\Gamma(A_{n-1}, 3\alpha_0)$ to the set of tableaux for $\Gamma(A_n, 3\alpha_0)$. Thus $\Gamma(A_n, 3\alpha_0)$ contains $|\Gamma(A_{n-1}, 3\alpha_0)|$ tableaux.

Type 2 If $\Gamma(A_{n-1}, 3\alpha_0) \neq \emptyset$, we construct 3 distinct tableaux for $\Gamma(A_n, 3\alpha_0)$ from each tableau in $\Gamma(A_{n-1}, 3\alpha_0)$ in by adding $n-1$ to each row. The new tableaux are of the form

a_1	$\ldots$	a_r	$n-1$
b_1	$\ldots$	b_s	
c_1	$\ldots$	c_t	

,

a_1	$\ldots$	a_r	
b_1	$\ldots$	b_s	$n-1$
c_1	$\ldots$	c_t	

,

a_1	$\ldots$	a_r	
b_1	$\ldots$	b_s	
c_1	$\ldots$	c_t	$n-1$

If by adding $n-1$ we do not get an ordered tableau, then we order each tableau. After building these tableaux, $\Gamma(A_n, 3\alpha_0)$ gets $3|\Gamma(A_{n-1}, 3\alpha_0)|$ more tableaux.

Type 3 For all $\Gamma(A_{n-l}, 3\alpha_0) \neq \emptyset$ where $l = 3, \dots, n-3$, we build tableaux for $\Gamma(A_n, 3\alpha_0)$ from each tableau in each of the $\Gamma(A_{n-l}, 3\alpha_0)$s in the following way.

(1) Add $n-l$ followed by $n-1$ to the end of row a . Next, add $n-l$ to the end of row b. Include this tableau in the set of tableaux for $\Gamma(A_n, 3\alpha_0)$.

(2) Make a new tableau by adding $n-l$ followed by $n-1$ to the end of row a. Next, add $n-l$ to the end of row c. Include this tableau in the set of tableaux for $\Gamma(A_n, 3\alpha_0)$.

(3) Make a new tableau by adding $n-l$ followed by $n-1$ to the end of row b. Next, add $n-l$ to the end of row c. Include this tableau in the set of tableaux for $\Gamma(A_n, 3\alpha_0)$.

To illustrate, we build 3 tableaux from each tableau in each of the $\Gamma(A_{n-l}, 3\alpha_0)$s that are of the form

a_1	$\dots$	a_r	$n-l$	$n-1$
b_1	$\dots$	b_s	$n-l$	
c_1	$\dots$	c_t		

,

a_1	$\dots$	a_r	$n-l$	$n-1$
b_1	$\dots$	b_s		
c_1	$\dots$	c_t	$n-l$	

,

a_1	$\dots$	a_r		
b_1	$\dots$	b_s	$n-l$	$n-1$
c_1	$\dots$	c_t	$n-l$	

Once we build these tableaux we order them and then add them to the set tableaux for $\Gamma(A_n, 3\alpha_0)$. In doing so, we add $3(\Gamma(A_{n-2}, 3\alpha_0) + \Gamma(A_{n-3}, 3\alpha_0) + \dots + \Gamma(A_3, 3\alpha))$ tableaux to the set of tableaux for $\Gamma(A_n, 3\alpha_0)$.

Type 4 If $\Gamma(A_{n-1}, 2\alpha_0) \neq \emptyset$, take all of its tableaux with a row whose only entry is 0. To these tableaux, create a third row whose only entry in $n-1$. Once we create the new row we order each tableau. This give tableaus of the form

a_1	$\dots$	a_r
$n-1$		
0		

This gives us $2^{n-2} - 1$ new tableaux that we include in the set of tableaux for $\Gamma(A_n, 3\alpha_0)$.

Type 5 For all $\Gamma(A_{n-l}, 2\alpha_0)$s where $l = 2, \dots, n-2$, take all the tableaux that have a row whose only entry is 0 and add $n-l$ followed by $n-1$ to the top row. Then, create a third row whose only entry is $n-l$. After creating these tableaux we need to order them. These tableaux are of the form

a_1	$\dots$	$n-l$	$n-1$
$n-l$			
0			

For each $\Gamma(A_{n-l}, 2\alpha_0)$, we get $l-1$ new tableaux. Altogether this part of the construction gives us the following number of tableaux that we include in the set of tableaux for $\Gamma(A_n, 3\alpha_0)$

$$(2^1 - 1) + (2^2 - 1) + \dots + (2^{n-3} - 1).$$

Type 6 Take a tableau in $\Gamma(A_{n-1}, \alpha_0)$, say $a_1, \dots, a_r$ where $r \geq 2$ and if $r = 2$, $a_r \neq n-2$ and construct a new tableau that for $\Gamma(A_n, 3\alpha_0)$ in the following way.

(1) Add $n-1$ to the end of the row.

(2) Create a new row containing only a_r.

(3) Create a third row that contains either a_{r-1} or $s \in \mathbb{Z}^+$ where $a_r < s < n-1$.

This part of our construction gives us all the skinny tableaux for $\Gamma(A_n, 3\alpha_0)$.

Type 7 Take a tableau in $\Gamma(A_{n-1}, 2\alpha_0)$ of the form

a_1	...	a_r
b_1		

with $a_1 \neq \max(a_i < n-2)$, $b_1 \neq a_{r-2}$, $b_1 \neq \max(a_i < n-2)$, and $b_1 \neq n-2$ and construct tableaux for $\Gamma(A_n, 3\alpha_0)$ in the following way.

(1) Add $n-1$ to position a_{r+1}.

(2) Add $n-2$ to position b_2.

(3) Add $\max(a_i < n-2)$ to a new row. Now add this tableau to the set of tableaux for $\Gamma(A_n, 3\alpha_0)$.

(4) Make an additional tableau by switching $n-2$ and $\max(a_i < n-2)$ if the following conditions hold:

(a) $a_2 \neq b_1$

(b) $\max(a_i < n-2) > b_1$ (these conditions are necessary but I don't know if they are sufficient; they do appear to hold for all observed cases and to be violated for all cases that don't produce a tableau with $n-2$ and $\max(a_i < n-2)$ switched if the above conditions are violated).

Type 8 Take all skinny tableaux from $\Gamma(A_{n-2}, 2\alpha_0)$ of the form

a_1	...	a_r
b_1		

where $a_1 \neq a_\gamma$, $b_1 \neq a_{\gamma-1}$, $b_1 \neq a_\gamma$, and $b_1 \neq n-3$ where $a_\gamma = \max(a_i < n-3)$ and $a_{\gamma-1}$ denotes the entry before a_γ. For each of these tableaux we construct tableaux for $\Gamma(A_n, 3\alpha_0)$ in the following way.

(1) Add $n-1$ to position a_{r+1}.

(2) Add $n-3$ to position b_2.

(3) Add a_γ to a new row. Now add this tableau to the set of tableaux for $\Gamma(A_n, 3\alpha_0)$ in α_0.

(4) Make an additional tableau by switching $n-3$ and a_γ if the following conditions hold:

(a) $b_1 \neq a_\gamma$

(b) $\max(a_i < n-2) > b_1$.

Type 9 Take all tableaux from $\Gamma(A_{n-3}, 2\alpha_0)$ without any 0 entries. To each tableau of this form we construct tableaux for $\Gamma(A_n, 3\alpha_0)$ in the following way.

(1) Add $n-3$ to the end of both rows.

(2) Create a new row containing only $n-3$.

(3) Add $n-1$ to the end of the top row.

(4) Add $n-2$ to the end of the second row.

This method gives tableaux of the form

a_1	...	a_r	$n-3$	$n-1$
b_1	...	b_s	$n-3$	$n-2$
$n-3$				

We include all $|\Gamma(A_{n-3}, 2\alpha_0)| - (2^{n-4} - 1)$ of these tableaux in our set of tableaux for $\Gamma(A_n, 3\alpha_0)$.

Type 10 Take all tableaux from $\Gamma(A_{n-3}, 3\alpha_0)$ that have no rows where 0 is the only entry. To each tableau of this form we construct tableaux for $\Gamma(A_n, 3\alpha_0)$ in the following way.

(1) Add $n-3$ to the end of each row.

(2) Add $n-1$ to the end of the top row.

(3) Add $n-2$ to the end of the second row.

This method gives tableaux of the form

a_1	$\ldots$	a_r	$n-3$	$n-1$
b_1	$\ldots$	b_s	$n-3$	$n-2$
c_1	$\ldots$	c_t	$n-3$	

We include all these tableaux in our set of tableaux for $\Gamma(A_n, 3\alpha_0)$.

The above construction gives the following recursion

$$\begin{aligned} |\Gamma(A_n, 3\alpha_0)| &= 4|\Gamma(A_{n-1}, 3\alpha_0)| + 3(|\Gamma(A_{n-2}, 3\alpha_0)| + \ldots + |\Gamma(A_3, 3\alpha_0)|) \\ &\quad + (2^{n-2} - 1) + (2^{n-3} - 1) + \ldots + (2^1 - 1) \\ &\quad + |\Gamma(A_{n-3}, 2\alpha_0)| - (2^{n-4} - 1) \end{aligned}$$

From this construction we can account for the tableaux in the graphs up to $\Gamma(A_5, 3\alpha_0)$, but for larger graphs more rules for building the tableaux become necessary.

6 Conclusion

We were unable to determine all of these rules because the tableaux in larger graphs have many entries which causes the necessary rules to be very subtle. Consequently, we have not found a solution to the problem of finding a closed formula for the vertices of $\Gamma(A_n, 3\alpha_0)$ nor have we found one for $\Gamma(A_n, k\alpha_0)$. Future research could again look at these tableaux in $3\alpha_0$ and try to define the remaining rules. Discovering these rules could allow for a generalization to $k\alpha_0$.

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