Semibiplanes and Cell Complexes

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Contents

ABSTRACT. We will investigate a procedure in which a complex is constructed from certain graphs, called $(0,2)-graphs$. We will relate graph-theoretic properties to properties of the corresponding complex.

Introduction

A *graph* X consists of a *vertex* set $V(X)$ and an *edge* set $E(X)$, where an edge is an unordered pair of distinct vertices of X [**3**]. A $(0,2)-graph$ is a connected graph such that any two vertices have either 0 or 2 common neighbors [**1**]. Andries Brouwer provides a classification of small valency $(0,2)-graphs$ denoting the bipartite graph of valency k with serial number i by $\Delta_{k.i}$ [**1**]. While examining Brouwer's Lists, we noticed certain graphs crossed with K_2 contain three objects in the highest dimension. Our concern is to investigate what these graphs have in common and why certain patterns exist. In this paper we will give explicit descriptions of the graphs we studied and make conjectures based on our observations.

To construct the graphs, we use the computer algebra system GAP along with Brouwer's Lists of Adjacencies to compute the adjacency lists for each graph [**2, 1**]. We also associate a complex with each graph. An n-dimensional complex is a chain of sets $C_{-1}, C_0, \ldots, C_n$ together with an incidence relation between elements of C_i and C_{i+1} for $-1 \leq i \leq n-1$[1] such that if $a \in C_{i-1}$ is incident with $b \in C_i$ which is incident with c in C_{i+1}, then $\exists! d \in C_i, d \neq b$ such that $aIdIc$.[2] We take the point 1 to be our indexing point, making it the only element of dimension -1, which refers to the empty set. The entries of the adjacency list of 1 represent the elements of dimension zero (the points). We now look at the adjacency lists corresponding to each of these points, which give us the elements of dimension one (the lines). This process is repeated with successive dimensions by considering the adjacencies of the previous dimension.[3] (See construction example on next page). After constructing the complex, we consider the number of objects in each dimension, or the distribution. We begin by constructing the complexes of successive crosses of $\Delta_{4.1}$ with K_2. We examine patterns in the distributions of the Cartesian products of these graphs crossed with K_2. In this paper, we will often refer to the Cartesian product as taking the cross of two graphs. This leads us to our first definition.

DEFINITION 1. Given graphs Γ, Δ their Cartesian product $\Gamma \times \Delta$ is the graph with the vertex set $V(\Gamma) \times V(\Delta)$ and the adjacency $(a,b) \sim (c,d)$ if $a = c$ and $b \sim d$ or $a \sim c$ and $b = d$ for all $a, c \in V(\Gamma)$ and $b, d \in V(\Delta)$.

REMARK. The Cartesian product of two $(0,2)-graphs$ is a $(0,2)-graph$.

[1] If a in C_i is incident with b in C_{i+1} we think of a as a boundary of b.
[2] This is usually not a simplicial complex.
[3] This is the reverse procedure of a construction of a graph from a complex, sometimes called Barycentric Subdivision.

Construction of $\Delta_{4.1}$

Adjacency List for $\Delta_{4.1}$

[2, 3, 4, 5]
[1, 6, 7, 8]
[1, 6, 9, 10]
[1, 7, 9, 11]
[1, 8, 10, 11]
[2, 3, 12, 13]
[2, 4, 12, 14]
[2, 5, 13, 14]
[3, 4, 13, 14]
[3, 5, 12, 14]
[4, 5, 12, 13]
[6, 7, 10, 11]
[6, 8, 9, 11]
[7, 8, 9, 10]

FIGURE 0.0.1.

Brouwer's Lists gives the adjacencies for $\Delta_{4.1}$. See figure 0.0.1. We choose the vertex 1 as our indexing point making it the sole element of the -1 dimension. The elements adjacent to 1, specifically 2, 3, 4, and 5, make up the zeroth dimension and correspond to points in the complex. The adjacency list for the point 2 gives the elements of the first dimension which are incident with 2, namely 6, 7, and 8. We repeat this process for 3, 4, and 5 to obtain all the elements of the first dimension, corresponding to the lines in the complex. To determine which points make up the endpoints of the line 6, we look at the first two entries of its adjacency list. We repeat this process for the other lines. We return to the adjacency list of line 6, noting that the elements which have not yet been listed belong to the second dimension, in this instance 12 and 13. We repeat this process for the other lines to obtain all the elements of the second dimension. To determine which lines make up the boundaries of 12, we look at the one dimensional entries in its adjacency list, in particular 6, 7, 10, and 11. We then connect the constituent points of these lines to form the shape of 12, a quadrangle. We repeat this process for the other two dimensional elements, 13 and 14, to obtain their shapes. See Figure 0.0.2.

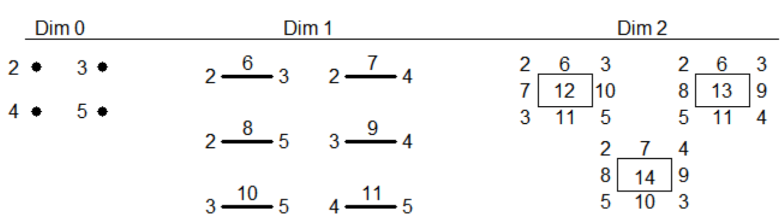

FIGURE 0.0.2.

Crosses with K_2

Graph	-1D	0D	1D	2D	3D	4D	5D	6D
K_2	1	1						
$\Delta_{4.1}$	1	4	6	3 3 quadrangles				
$\Delta_{5.3}$	1	5	10	9 3 quadrangles 6 triangles	3 3 square pyramids			
$\Delta_{6.12}$	1	6	15	19 3 quadrangles 16 triangles	12 6 square pyramids 6 tetrahedra	3 3 (2 square pyramids + 4 tetrahedra)		
$\Delta_{7.39}$	1	7	21	34 3 quadrangles 31 triangles	31 9 square pyramids 22 tetrahedra	15 9 (2 square pyramids + 4 tetrahedra) 6 (5 tetrahedra shape)	3	
$\Delta_{8.103}$	1	8	28	55 3 quadrangles 52 triangles	65 12 square pyramids 53 tetrahedra	46 18 (2 square pyramids + 4 tetrahedra) 28 (5 tetrahedra shape)	18	3

FIGURE 0.0.3.

Now we will give explicit descriptions of each of the graphs that we examined and their crosses with K_2, noting any patterns and similarities. We concentrate on dimensions zero through four where the patterns are most evident and comprehensible.[4] We now attempt to understand the relationships between the complexes of $\Delta_{4.1}$ with K_2 and their successive crosses. See figure 0.0.3. In figure 0.0.4, we use Mathematica to depict the complex of $\Delta_{4.1}$ as well as one of its individual quadrangles [**6**].

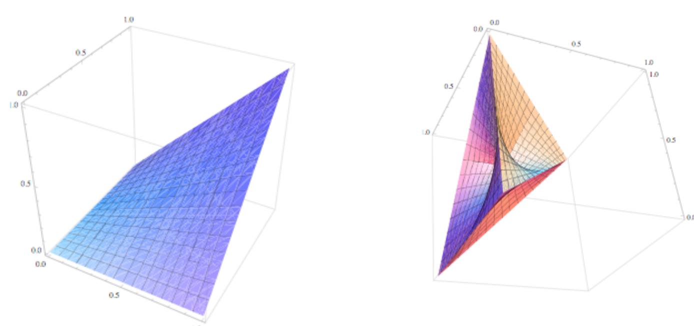

FIGURE 0.0.4.

The Cartesian product $\Delta_{4.1} \times K_2$ is the graph $\Delta_{5.3}$.

- $\Delta_{5.3}$
  - In dimension zero, the four points from $\Delta_{4.1}$ and the point from K_2 are added to obtain the five points in $\Delta_{5.3}$.
  - In dimension one, the original six lines from $\Delta_{4.1}$ remain. An additional four lines are formed between the point from K_2 and each point from $\Delta_{4.1}$, giving $\Delta_{5.3}$ ten total lines.

[4] The patterns do hold for dimensions above four.

- In dimension two, the three quadrangles from $\Delta_{4.1}$ remain. The six lines from $\Delta_{4.1}$ combined with the point from K_2 create six triangles in $\Delta_{5.3}$.
- In dimension three, three square-pyramids are formed in $\Delta_{5.3}$ by connecting the point of K_2 to every point on each of the three quadrangles in $\Delta_{4.1}$.

- $\Delta_{6.12}$
  - In dimensions zero through two, the construction follows the same pattern as in $\Delta_{5.3}$.
  - In dimension three, square-pyramids are formed in the same manner as above. In addition, the six triangles from $\Delta_{5.3}$ combined with the point from K_2 create six tetrahedra in $\Delta_{6.12}$.
  - In dimension four, every point on each of the three square-pyramids from $\Delta_{5.3}$ is connected to the point from K_2 to form three objects. Each object consists of two square-pyramids attached to four tetrahedra.
- $\Delta_{7.39}$
  - In dimensions zero through three, the construction follows the same patterns as in $\Delta_{6.12}$.
  - In dimension four, the objects consisting of two square-pyramids attached to four tetrahedra are formed as above. Additionally, every point on each of the six tetrahedra from $\Delta_{6.12}$ is connected to the point from K_2 to form six new objects consisting of five attached tetrahedra each.
- $\Delta_{8.103}$
  - In dimensions zero through four, the construction follows the same patterns as in the previous examples.

Thus, the pattern of the complexes can be explained by the complexes of successive crosses of $\Delta_{4.1}$ with K_2. While examining these particular Cartesian products, we mainly focused on the distributions, the highest dimensions, and the shapes. The similarities we observed are as follows:

- Each successive cross with K_2 raises the highest dimension by one.
- The number of quadrangles in the second dimension is exactly three.
- The number of elements in the highest dimension is exactly three.
- The number of square-pyramids in the third dimension increases by three with each successive cross.

Based on these specific cases, we conjecture that these observations hold true for any further crosses of $\Delta_{4.1}$ with K_2.

Graph	-1D	0D	1D	2D	3D	4D	5D	6D
$\Delta_{4.1}$	1	4	6	3 3 quadrangles				
$\Delta_{5.1}$	1	5	10	6 6 pentagons				
$\Delta_{5.2}$	1	5	10	7 5 quadrangles 2 pentagons	1 1 (5-quadrangle shape)			
$\Delta_{4.1}$ x $\Delta_{4.1}$	1	8	28	54 6 quadrangles 48 triangles	60 24 square pyramids 36 tetrahedra	36 36 (2 square pyramids + 4 tetrahedra)	9	
$\Delta_{4.1}$ x $\Delta_{5.1}$	1	9	36	79 3 quadrangles 6 pentagons 70 triangles	99 15 square pyramids 24 pentagonal pyramids 60 tetrahedra	66 30 (2 square pyramids + 4 tetrahedra) 36 (2 pentagonal pyramids + 5 tetrahedra)	18	
$\Delta_{4.1}$ x $\Delta_{5.2}$	1	9	36	80 8 quadrangles 2 pentagons 70 triangles	104 35 square pyramids 8 pentagonal pyramids 60 tetrahedra 1 (5-quadrangle shape)	76 60 (2 square pyramids + 4 tetrahedra) 12 (2 pentagonal pyramids + 5 tetrahedra) 4 (1 (5-quadrangle shape) + 5 square pyramids)	27	3

FIGURE 0.0.5.

Cartesian Products

After examining those graphs crossed with K_2, we were interested in seeing what patterns exist for crossing $\Delta_{4.1}$ with other graphs. Again, we focused on the distribution, the highest dimension, and the shapes. We now attempt to understand the relationships between the complexes of $\Delta_{4.1}$, $\Delta_{5.1}$, and $\Delta_{5.2}$ as well as their crosses with $\Delta_{4.1}$. See figure 0.0.5.

- $\Delta_{4.1} \times \Delta_{4.1}$
  - The construction of dimensions zero and one follows the same pattern as in the earlier crosses.
  - In dimension two, the three quadrangles from each copy of $\Delta_{4.1}$ remain. The six lines from each copy of $\Delta_{4.1}$ combined with the four points from the other copy create forty-eight triangles in $\Delta_{4.1} \times \Delta_{4.1}$.
  - In dimension three, twenty-four square-pyramids are formed by connecting the points on each copy of $\Delta_{4.1}$ to every point on each of the three quadrangles in the other copy. The six lines from one copy of $\Delta_{4.1}$ combined with the six lines from the other copy create thirty-six tetrahedra in $\Delta_{4.1} \times \Delta_{4.1}$.
  - In dimension four, the six lines from each copy of $\Delta_{4.1}$ combined with the three quadrangles from the other copy form thirty-six objects, each consisting of two square-pyramids attached to four tetrahedra.
- $\Delta_{4.1} \times \Delta_{5.1}$
  - The construction of dimensions zero and one follows the same pattern as in the earlier crosses.

  - In dimension two, the three quadrangles from $\Delta_{4.1}$ and the six pentagons from $\Delta_{5.1}$ remain. The six lines from $\Delta_{4.1}$ combined with the five points from $\Delta_{5.1}$ create thirty triangles. The ten lines from $\Delta_{5.1}$ combined with the four points from $\Delta_{4.1}$ create another forty triangles, giving seventy total triangles in $\Delta_{4.1} \times \Delta_{5.1}$.
  - In dimension three, fifteen square-pyramids are formed by connecting the five points of $\Delta_{5.1}$ to every point on each of the three quadrangles in $\Delta_{4.1}$. The six pentagons from $\Delta_{5.1}$ combined with the four points from $\Delta_{4.1}$ create twenty-four pentagonal-pyramids. The six lines from $\Delta_{4.1}$ together with the ten lines from $\Delta_{5.1}$ create sixty tetrahedra in $\Delta_{4.1} \times \Delta_{5.1}$.
  - In dimension four, the ten lines from $\Delta_{5.1}$ combined with the three quadrangles from $\Delta_{4.1}$ form thirty objects, each consisting of two square-pyramids attached to four tetrahedra. The six pentagons from $\Delta_{5.1}$ together with the six lines from $\Delta_{4.1}$ form thirty-six objects, each consisting of two pentagonal-pyramids and five tetrahedra.
- $\Delta_{4.1} \times \Delta_{5.2}$
  - The construction of dimensions zero and one follows the same pattern as in the earlier crosses.
  - In dimension two, the three quadrangles from $\Delta_{4.1}$ and the five quadrangles and two pentagons from $\Delta_{5.2}$ remain. The six lines from $\Delta_{4.1}$ combined with the five points from $\Delta_{5.2}$ create thirty triangles. The ten lines from $\Delta_{5.2}$ combined with the four points from $\Delta_{4.1}$ create another forty triangles, giving seventy total triangles in $\Delta_{4.1} \times \Delta_{5.2}$.
  - In dimension three, the 5-quadrangle shape remains from $\Delta_{5.2}$. Fifteen square-pyramids are formed by connecting the five points from $\Delta_{5.2}$ to every point on each of the three quadrangles in $\Delta_{4.1}$. An additional twenty square-pyramids are formed by connecting the four points from $\Delta_{4.1}$ to every point on each of the five quadrangles in $\Delta_{5.2}$. The two pentagons from $\Delta_{5.2}$ combined with the four points from $\Delta_{4.1}$ create eight pentagonal-pyramids. The six lines from $\Delta_{4.1}$ together with the ten lines from $\Delta_{5.2}$ create sixty tetrahedra in $\Delta_{4.1} \times \Delta_{5.2}$.
  - In dimension four, the six lines from $\Delta_{4.1}$ combined with the five quadrangles from $\Delta_{5.2}$ form thirty objects, each consisting of two square-pyramids attached to four tetrahedra. An additional thirty such objects are formed from the ten lines from $\Delta_{5.2}$ combined with the three quadrangles from $\Delta_{4.1}$. The 5-quadrangle shape from $\Delta_{5.2}$ combined with the four points from $\Delta_{4.1}$ form four objects, each consisting of one, 5-quadrangle shape and five square-pyramids. The two pentagons from $\Delta_{5.2}$ together with the six lines from $\Delta_{4.1}$ form twelve objects, each consisting of two pentagonal-pyramids and five tetrahedra.

The similarities we observed are as follows:

- The highest dimension of the Cartesian product is one more than the sum of the highest dimensions of the two graphs.
- The number of elements in the highest dimension of the Cartesian product is the product of the numbers of elements in the highest dimensions of the two graphs.

Cartesian Product and Join

In all of these cases, we are able to explain the patterns in the distributions and representative shapes by considering combinations of the cells of the complexes of the constituent graphs. These patterns prompted us to consider how the complex of the Cartesian product of any two graphs could be created. Thus, we entertain the topological concept of a join as our next definition.

Definition 2. The join of two topological spaces A and B, often denoted by $A \star B$, is defined to be the quotient space $A \times B \times I/R$, where I is the interval $[0,1]$ and R is the relation defined by $(a, b_1, 0) \sim (a, b_2, 0)$ for all $a \in A$ and $b_1, b_2 \in B$, $(a_1, b, 1) \sim (a_2, b, 1)$ for all $a_1, a_2 \in A$ and $b \in B$. In effect, one is collapsing $A \times B \times \{0\}$ to A and $A \times B \times \{1\}$ to B. Intuitively, $A \star B$ is formed by taking the disjoint union of two spaces and attaching a line segment joining every point in A to every point in B [**5, 4**].

This standard definition is not enough to fully explain our examples. We thus extend this notion to an abstract definition which applies to the entire complex of a graph.

Definition 3. The join of two complexes X and Y consists of the cells of the complex of X, the cells of the complex of Y, and the set of all joins of the cells $x \star y$ such that $x \in X$ and $y \in Y$. The join, $x \star y$, is as described in Definition 2. Thus, we obtain new cells of dimension n from the join of a cell in X of dimension p and a cell in Y of dimension q, where $n = p + q + 1$.

We now state our theorem.

Theorem 4. *Let Γ and Δ be graphs. Then the complex of their Cartesian product, $\Gamma \times \Delta$, is the join of their individual complexes.*

Proof. Let Γ and Δ be graphs.

Take the Cartesian product of Γ and Δ.

The complex is then constructed in the usual way. (We omit the details.)

We first claim that the existing cells from the complex of Γ and that of Δ remain in the complex of $\Gamma \times \Delta$.

Take any point (x, y) in the product. The point (x, y) is adjacent to the points (z, y) where $z \sim x$ and the points (x, m) where $m \sim y$.

By fixing y, we obtain a subgraph isomorphic to Γ (i.e. a copy). Similarly, by fixing x, we obtain a subgraph isomorphic to Δ (i.e. a copy).

The point (x, y) is thus a component of a copy of Γ and a copy of Δ.

Making (x, y) the indexing point, it follows that the cells of the complexes of Γ and Δ are included in the complex of $\Gamma \times \Delta$.

We now consider the vertices adjacent to the zero dimensional cells from the copy of Γ and the copy of Δ incident with (x, y).

These vertices are the one dimensional cells (the lines); the (z, m), where $z \neq x$ and $m \neq y$, retaining the adjacencies described above.

This procedure follows similarly in the higher dimensions.

Since we have a copy of the complex of Γ and that of Δ, as well as additional elements created from combinations of the elements of the smaller complexes, then by our definition of join, the complex of the Cartesian product of $\Gamma \times \Delta$ is clearly the join of the individual complexes. □

Distribution Patterns

With this notion of join, we now present the following equations to give the elements in each dimension, where d represents the dimension.

Zero Dimensional: $0d = 0d\,(\Gamma) + 0d\,(\Delta)$

One Dimensional: $1d = 1d\,(\Gamma) + 1d\,(\Delta) + (0d\,(\Gamma) \star 0d\,(\Delta))$

Two Dimensional: $2d = 2d\,(\Gamma) + 2d\,(\Delta) + ((0d\,(\Gamma) \star 1d\,(\Delta)) + (1d\,(\Gamma) \star 0d\,(\Delta)))$

Three Dimensional: $3d = 3d\,(\Gamma) + 3d\,(\Delta) + ((0d\,(\Gamma) \star 2d\,(\Delta)) + (2d\,(\Gamma) \star 0d\,(\Delta)) + (1d\,(\Gamma) \star 1d\,(\Delta)))$

Four Dimensional: $4d = 4d\,(\Gamma) + 4d\,(\Delta) + ((0d\,(\Gamma) \star 3d\,(\Delta)) + (3d\,(\Gamma) \star 0d\,(\Delta)) + (1d\,(\Gamma) \star 2d\,(\Delta)) + (2d\,(\Gamma) \star 1d\,(\Delta)))$

The pattern of these equations holds for a general case.

n-dimensional (n odd):

$$nd = nd\,(\Gamma) + nd\,(\Delta) + \left(\frac{n-1}{2}d\,(\Gamma) \star \frac{n-1}{2}d\,(\Delta)\right) + \sum_{i=0}^{(n-3)/2} (id\,(\Gamma) \star (n-i-1)\,d\,(\Delta) + (n-i-1)\,d\,(\Gamma) \star id\,(\Delta))$$

n-dimensional (n even):

$$nd = nd\,(\Gamma) + nd\,(\Delta) + \sum_{i=0}^{(n-2)/2} (id\,(\Gamma) \star (n-i-1)\,d\,(\Delta) + (n-i-1)\,d\,(\Gamma) \star id\,(\Delta))$$

While studying these graphs we were also intrigued by patterns in the distribution which we believe can be explained by the above equations as a corollary of our theorem. We first observed that the distribution of each cross with K_2 can be obtained by some sum of the distribution of $\Delta_{4.1}$, offsetting it by one dimension in some succession. Explicitly, add the distribution of $\Delta_{4.1}$ until the sum of that dimensional column is complete. Then, shift over one dimension until the sum in that dimensional column is complete. Continue adding and shifting until the original distribution has been fulfilled. We observe through this method that the pattern of the sum corresponds to the distribution of whatever is being crossed with $\Delta_{4.1}$. This method also works for the opposite direction. See the appendix for the distribution charts.

The offsetting as we take the sum corresponds to the combinations of the elements of different dimensions given in the above equations. The column of the zeroth dimension has the number of elements in the zeroth dimension of the distribution plus the same number of 1's as the number of elements of the zeroth dimension of the graph being crossed. This corresponds to $0d\,(\Gamma) + 0d\,(\Delta)$. The column of the first dimension has the similar numbers for the first dimension of Γ and Δ, as well as the same number of $\mid 0d\,(\Gamma) \mid$ as the $\mid 0d\,(\Delta) \mid$. The pattern follows similarly for the remaining equations.

Until this point, we have only focused on graphs with a single distribution; however, we encounter multiple distributions as we consider more related graphs on Brouwer's Lists.

Introducing Multiple Orbits

According to Brouwer's Lists, $\Delta_{4.1}$ is a subgraph of $\Delta_{6.10}$ which, in turn, is a subgraph of $\Delta_{8.98}$. We examine the complexes of these graphs to see if this relationship is evident. For ease of comprehension, we focus on the second and third dimensions only. In considering these graphs, we encounter graphs with more than one distribution. Different distributions arise when graphs have multiple orbits. When a graph is non-transitive, not all of the vertices are in the same orbit. Therefore, there can be a different distribution for each different orbit resulting in distinct complexes. See figure 0.0.6.

Graph	Orbit	-1D	0D	1D	2D	3D	4D	5D
$\Delta_{4.1}$	1	1	4	6	3 3 quadrangles			
$\Delta_{6.10}$	1	1	6	15	18 6 quadrangles 12 triangles	8 6 (2 quadrangles + 4 triangles) 2 (6 triangle shape)		
$\Delta_{6.10}$	2	1	6	15	17 9 quadrangles 8 triangles	8 2 (2 quadrangles + 4 triangles) 4 (triangular prisms) 2 (square pyramids)	1	
$\Delta_{8.98}$	1	1	8	28	53 9 quadrangles 44 triangles	53 12 (2 quadrangles + 4 triangles) 12 (square pyramids) 4 (6 triangle shape) 25 (tetrahedra)	21	
$\Delta_{8.98}$	4	1	8	28	51 15 quadrangles 36 triangles	49 4 (2 quadrangles + 4 triangles) 8 (triangular prisms) 28 (square pyramids) 9 (tetrahedra)	23	4
$\Delta_{8.98}$	5	1	8	28	52 12 quadrangles 40 triangles	50 14 (2 quadrangles + 4 triangles) 4 (triangular prisms) 8 (square pyramids) 4 (6 triangle shape) 20 (tetrahedra)	22	3
$\Delta_{8.98}$	9	1	8	28	53 9 quadrangles 44 triangles	52 18 (2 quadrangles + 4 triangles) 6 (6 triangle shape) 28 (tetrahedra)	21	1
$\Delta_{8.98}$	30	1	8	28	50 18 quadrangles 32 triangles	48 8 (2 quadrangles + 4 triangles) 16 (triangular prisms) 8 (square pyramids) 16 (tetrahedra)	24	5
$\Delta_{8.98}$	31	1	8	28	50 18 quadrangles 32 triangles	48 7 (2 quadrangles + 4 triangles) 16 (triangular prisms) 8 (square pyramids) 17 (tetrahedra)	24	5

Figure 0.0.6.

Clearly, the complex of $\Delta_{4.1}$ is present in each complex of $\Delta_{6.10}$. In the second dimension, it is evident that the complex in each orbit of $\Delta_{6.10}$ is contained in the second dimension of the complexes of the orbits of $\Delta_{8.98}$. In the third dimension, the first orbit of $\Delta_{6.10}$ is contained in the complex of orbits 1, 5, and 9 of $\Delta_{8.98}$. The second orbit of $\Delta_{6.10}$ is contained in the complex of orbits 4, 5, 30, and 31 of $\Delta_{8.98}$. Both of the orbits of $\Delta_{6.10}$ do not have to be present in every orbit of $\Delta_{8.98}$ as long as one of the two is present in each.

Observations on Quotients

In our construction of the complexes of many graphs on Brouwer's List, we encountered graphs containing cells with identified interiors but distinct boundaries. We attempt to understand these graphs through the following examples. We observe the usual relationship between the graphs of $\Delta_{6.11}, \Delta_{7.38}, \Delta_{8.100}$ which are successive crosses of $\Delta_{5.2}$ with K_2. In addition, we note that there are correspondences between these complexes and those of $\Delta_{7.2}$ and $\Delta_{8.34}$. See figure 0.0.7.

Graph	-1D	0D	1D	2D	3D	4D	5D	6D
$\Delta_{6.11}$	1	6	15	17	18	1		
				10 triangles 5 quadrangles 2 pentagons	2 pentagonal pyramids 5 square pyramids 1 (5 quadrangle shape)			
$\Delta_{7.2}$	1	7	21	17	2			
				2 pentagons 10 double triangles 5 (quadrangle + triangle)	2 (2 pentagons + 5 double triangles)			
$\Delta_{7.38}$	1	7	21	32	25	9	1	
				25 triangles 5 quadrangles 2 pentagons	3 pentagonal pyramids 12 square pyramids 10 tetrahedra			
$\Delta_{8.34}$	1	8	28	37	19	3		
				15 triangles 5 quadrangles 1 pentagon 15 double triangles 1 (pentagon + triangle)	3 pentagonal pyramids 15 (square pyramid + tetrahedron) 1 (5 quadrangle shape)			
$\Delta_{8.100}$	1	8	28	53	57	34	10	1
				46 triangles 5 quadrangles 2 pentagons	5 pentagonal pyramids 17 square pyramids 35 tetrahedra			

FIGURE 0.0.7.

In the second dimension we note that the complexes of $\Delta_{7.2}$ and $\Delta_{7.38}$ contain the same number of triangles, quadrangles, and pentagons; however, in $\Delta_{7.2}$ some of these shapes have their interiors identified so that one cell contains the exterior of two shapes. A similar identification occurs in the graphs below. Assuming σ is an involution of Γ with no fixed points, then it appears that the complex of the quotient Γ/σ is a quotient of the complex of Γ. Another example of quotients occurs below.

Graph	Orbit	-1D	0D	1D	2D	3D	4D	5D	6D
$\Delta_{7.40}$	1	1	7	21	35	35	21	7	1
					35 triangles	35 tetrahedra			
$\Delta_{8.75}$	1	1	8	28	46	35	10		
					36 triangles 10 double triangles	15 tetrahedra 20 double tetrahedra			
$\Delta_{8.75}$	8	1	8	28	56	35			
					56 triangles	35 double tetrahedra			

FIGURE 0.0.8.

In our exploration of various graphs on Brouwer's Lists, we observed many patterns in the areas of distribution, shapes, etc. Our research is certainly not extensive, for, other related graphs might give rise to different patterns in these or other areas.

Bibliography

[1] Brouwer, Andries E. Classification of small (0, 2)-graphs. *J. Comb. Theory Ser. A*, 113(8):1636–1645, 2006.

[2] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.4.12*, 2008.

[3] Chris Godsil and Gordon Royle. *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[4] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, UK, 2002.

[5] Wikipedia. Join (topology) — Wikipedia, The Free Encyclopedia, 2006. [Online, accessed 24-June-2009].

[6] Inc. Wolfram Research. *Mathematica Edition: Version 7.0*. Champaign, Illinois, 2008.

Appendix

$\Delta_{4.1}$	**1**	**4**	**6**	**3**

$\Delta_{5.3}$	**1**	**5**	**10**	**9**	**3**
	1	4	6	3	
		1	4	6	3

$\Delta_{6.12}$	**1**	**6**	**15**	**19**	**12**	**3**
	1	4	6	3		
		1	4	6	3	
		1	4	6	3	
			1	4	6	3

$\Delta_{7.39}$	**1**	**7**	**21**	**34**	**31**	**15**	**3**
	1	4	6	3			
		1	4	6	3		
		1	4	6	3		
		1	4	6	3		
			1	4	6	3	
			1	4	6	3	
			1	4	6	3	
				1	4	6	3

$\Delta_{8.103}$	**1**	**8**	**28**	**55**	**65**	**46**	**18**	**3**
	1	4	6	3				
		1	4	6	3			
		1	4	6	3			
		1	4	6	3			
		1	4	6	3			
			1	4	6	3		
			1	4	6	3		
			1	4	6	3		
			1	4	6	3		
			1	4	6	3		
			1	4	6	3		
				1	4	6	3	
				1	4	6	3	
				1	4	6	3	
				1	4	6	3	
					1	4	6	3

FIGURE 0.0.9.

$\Delta_{4.1}$ x $\Delta_{4.1}$	**1**	**8**	**28**	**54**	**60**	**36**	**9**
	1	4	6	3			
		1	4	6	3		
		1	4	6	3		
		1	4	6	3		
		1	4	6	3		
			1	4	6	3	
			1	4	6	3	
			1	4	6	3	
			1	4	6	3	
			1	4	6	3	
			1	4	6	3	
				1	4	6	3
				1	4	6	3
				1	4	6	3

FIGURE 0.0.10.

$\Delta_{4.1}$ x $\Delta_{5.1}$	**1**	**9**	**36**	**79**	**99**	**66**	**18**
	1	4	6	3			
		1	4	6	3		
		1	4	6	3		
		1	4	6	3		
		1	4	6	3		
		1	4	6	3		
			1	4	6	3	
			1	4	6	3	
			1	4	6	3	
			1	4	6	3	
			1	4	6	3	
			1	4	6	3	
			1	4	6	3	
			1	4	6	3	
			1	4	6	3	
			1	4	6	3	
				1	4	6	3
				1	4	6	3
				1	4	6	3
				1	4	6	3
				1	4	6	3
				1	4	6	3

$\Delta_{5.1}$ x $\Delta_{4.1}$	**1**	**9**	**36**	**79**	**99**	**66**	**18**
	1	5	10	6			
		1	5	10	6		
		1	5	10	6		
		1	5	10	6		
		1	5	10	6		
			1	5	10	6	
			1	5	10	6	
			1	5	10	6	
			1	5	10	6	
			1	5	10	6	
			1	5	10	6	
				1	5	10	6
				1	5	10	6
				1	5	10	6

$\Delta_{4.1}$ x $\Delta_{5.2}$	**1**	**9**	**36**	**80**	**104**	**76**	**27**	**3**
	1	4	6	3				
		1	4	6	3			
		1	4	6	3			
		1	4	6	3			
		1	4	6	3			
		1	4	6	3			
			1	4	6	3		
			1	4	6	3		
			1	4	6	3		
			1	4	6	3		
			1	4	6	3		
			1	4	6	3		
			1	4	6	3		
			1	4	6	3		
			1	4	6	3		
			1	4	6	3		
				1	4	6	3	
				1	4	6	3	
				1	4	6	3	
				1	4	6	3	
				1	4	6	3	
				1	4	6	3	
				1	4	6	3	
					1	4	6	3

$\Delta_{5.2}$ x $\Delta_{4.1}$	**1**	**9**	**36**	**80**	**104**	**76**	**27**	**3**
	1	5	10	7	1			
		1	5	10	7	1		
		1	5	10	7	1		
		1	5	10	7	1		
		1	5	10	7	1		
			1	5	10	7	1	
			1	5	10	7	1	
			1	5	10	7	1	
			1	5	10	7	1	
			1	5	10	7	1	
			1	5	10	7	1	
				1	5	10	7	1
				1	5	10	7	1
				1	5	10	7	1

FIGURE 0.0.11.

Graph	Orbit	-1D	0D	1D	2D	3D	4D	5D	6D
$\Delta_{4.1}$	1	1	4	6	3 3 quadrangles				
$\Delta_{4.2}$	1	1	4	6	4 4 triangles	1 1 tetrahedron			
$\Delta_{5.1}$	1	1	5	10	6 6 pentagons				
$\Delta_{5.2}$	1	1	5	10	7 5 quadrangles 2 pentagons	1 1 (5-quadrangle shape)			
$\Delta_{5.3}$	1	1	5	10	9 3 quadrangles 6 triangles	3 3 quadrangle pyramids			
$\Delta_{5.4}$	1	1	5	10	10 10 quadrangles	5 5 tetrahedra	1 1 (5-tetrahedra shape)		
$\Delta_{6.1}$	1	1	6	15	10 4 hexagons 6 double triangles				
$\Delta_{6.2}$	1	1	6	15	10 6 hexagons 4 double triangles				
$\Delta_{6.3}$	1	1	6	15	10 10 double triangles				
$\Delta_{6.4}$	1	1	6	15	12 12 pentagons	2 2 (6-pentagon shape)			
$\Delta_{6.7}$	1	1	6	15	14 5 quadrangles 2 triangles 6 pentagons	4			
$\Delta_{6.9}$	1	1	6	15	16 10 triangles 6 pentagons	6 6 pentagonal pyramids			
$\Delta_{6.10}$	1	1	6	15	18 6 quadrangles 12 triangles	8 6 (2 quadrangles + 4 triangles) 2 (6 triangle shape)			
$\Delta_{6.10}$	2	1	6	15	17 9 quadrangles 8 triangles	8 2 (2 quadrangles + 4 triangles) 4 (triangular prisms) 2 (square pyramids)	1		
$\Delta_{6.11}$	1	1	6	15	17 10 triangles 5 quadrangles 2 pentagons	18 2 pentagonal pyramids 5 square pyramids 1 (5 quadrangle shape)	1		

FIGURE 0.0.12.

$\Delta_{6.12}$	1	1	6	15	19 3 quadrangles 16 triangles	12 6 square pyramids 6 tetrahedra	3 3 (2 square pyramids + 4 tetrahedra)		
$\Delta_{7.2}$	1	1	7	21	17 2 pentagons 10 double triangles 5 (quadrangle + triangle)	2 2 (2 pentagons + 5 double triangles)			
$\Delta_{7.38}$		1	7	21	32 25 triangles 5 quadrangles 2 pentagons	25 3 pentagonal pyramids 12 square pyramids 10 tetrahedra	9	1	
$\Delta_{7.39}$	1	1	7	21	34 3 quadrangles 31 triangles	31 9 square pyramids 22 tetrahedra	15 9 (2 square pyramids + 4 tetrahedra) 6 (5-tetrahedra shape)	3	
$\Delta_{7.40}$	1	1	7	21	35 35 triangles	35 35 tetrahedra	21	7	1
$\Delta_{8.34}$	1	1	8	28	37 15 triangles 5 quadrangles 1 pentagon 15 double triangles 1 (pentagon + triangle)	19 3 pentagonal pyramids 15 (square pyramid + tetrahedron) 1 (5 quadrangle shape)	3		
$\Delta_{8.75}$	1	1	8	28	46 36 triangles 10 double triangles	35 15 tetrahedra 20 double tetrahedra	10		
$\Delta_{8.75}$	8	1	8	28	56 56 triangles	35 35 double tetrahedra			
$\Delta_{8.98}$	1	1	8	28	53 9 quadrangles 44 triangles	53 12 (2 quadrangles + 4 triangles) 12 (square pyramids) 4 (6 triangle shape) 25 (tetrahedra)	21		
$\Delta_{8.98}$	4	1	8	28	51 15 quadrangles 36 triangles	49 4 (2 quadrangles + 4 triangles) 8 (triangular prisms) 28 (square pyramids) 9 (tetrahedra)	23	4	
$\Delta_{8.98}$	5	1	8	28	52 12 quadrangles 40 triangles	50 14 (2 quadrangles + 4 triangles) 4 (triangular prisms) 8 (square pyramids) 4 (6 triangle shape) 20 (tetrahedra)	22	3	

FIGURE 0.0.13.

$\Delta_{8.98}$	9	1	8	28	53 9 quadrangles 44 triangles	52 18 (2 quadrangles + 4 triangles) 6 (6 triangle shape) 28 (tetrahedra)	21	1	
$\Delta_{8.98}$	30	1	8	28	50 18 quadrangles 32 triangles	48 8 (2 quadrangles + 4 triangles) 16 (triangular prisms) 8 (square pyramids) 16 (tetrahedra)	24	5	
$\Delta_{8.98}$	31	1	8	28	50 18 quadrangles 32 triangles	48 7 (2 quadrangles + 4 triangles) 16 (triangular prisms) 8 (square pyramids) 17 (tetrahedra)	24	5	
$\Delta_{8.100}$	1	1	8	28	53 46 triangles 5 quadrangles 2 pentagons	57 5 pentagonal pyramids 17 square pyramids 35 tetrahedra	34	10	1
$\Delta_{8.103}$	1	1	8	28	55 3 quadrangles 52 triangles	65 12 square pyramids 53 tetrahedra	46 18 (2 square pyramids + 4 tetrahedra) 28 (5 tetrahedra shape)	18	3
$\Delta_{4.1} \times \Delta_{4.1}$	1	1	8	28	54 6 quadrangles 48 triangles	60 24 square pyramids 36 tetrahedra	36 36 (2 square pyramids + 4 tetrahedra)	9	
$\Delta_{4.1} \times \Delta_{5.1}$	1	1	9	36	79 3 quadrangles 6 pentagons 70 triangles	99 15 square pyramids 24 pentagonal pyramids 60 tetrahedra	66 30 (2 square pyramids + 4 tetrahedra) 36 (2 pentagonal pyramids + 5 tetrahedra)	18	
$\Delta_{4.1} \times \Delta_{5.2}$	1	1	9	36	80 8 quadrangles 2 pentagons 70 triangles	104 35 square pyramids 8 pentagonal pyramids 60 tetrahedra 1 (5-quadrangle shape)	76 60 (2 square pyramids + 4 tetrahedra) 12 (2 pentagonal pyramids + 5 tetrahedra) 4 (1 (5-quadrangle shape) + 5 square pyramids)	27	3

FIGURE 0.0.14.