ON GRAPHS ARISING FROM PROJECTIVE PLANES

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1. Introduction

We begin our discussion with finite projective planes and then proceed to discuss the properties of graphs associated with them.

Definition 1. A *design* is an ordered triple $(\mathcal{P}, \mathcal{I}, \mathcal{B})$ such that $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$. We say that $\mathcal{P}$ is the set of points and $\mathcal{B}$ is the set of blocks. We say that a point $p \in \mathcal{P}$ and a block $B \in \mathcal{B}$ are said to be incident if $(p, B) \in \mathcal{I}$.

Defitnition 1. A t-(v, k, b) design is a design such that ... A projective plane is a 1-$(v, k, 2)$ design, that is, every two points are contained in precisely one block and every two blocks intersect in precisely one point.

Remark 1. When referring to a finite projective plane, we will refer to points and blocks as points and lines. We will say that two points on the same line are colinear. Hence, in a finite projective plane any two points are colinear and any two lines intersect at exactly one point. There are no parallel lines in a finite projective plane.

Every known projective plane has $q^2 + q + 1$ points and lines. The number q is known as the order of the projective plane. All known nontrivial projective planes have order the power of a prime.

Definition 1. A graph Γ is an ordered pair $\Gamma = (V, E)$, where $E \subseteq V^{(2)}$, the set of all pairs in V. The set V is known as the vertex set and the set E is known as the edge set. If $(x, y) \in E$, then we write $x \sim y$. We say that two such vertices are adjacent and that y is a neighbor of x. The number of edges of a vertex x is known as its degree; a graph in which all vertices have the same degree is said to be regular.

Remark 1. Given a graph $\Gamma = (V, E)$, give an arbitrary ordering to its vertices $V = (v_1, v_2, \ldots, v_n)$. Define a matrx A where

$$A_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

The matrix A is known as the adjacency matrix of a graph Γ, and completely determines the graph, up to isomorphism. Let $M := D - A$, where D is the diagonal matrix $D = \text{diag}(d_1, \ldots, d_n)$, where d_i is the number of neighbors of the i^{th} vertex.

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Discusion 1. Let $(\mathcal{P},\mathcal{I},\mathcal{B})$ be a projective plane. Define a graph with vertex set $V=\mathcal{P}\bigcup\mathcal{B}$ and define a point p and a block B to be adjacent in this graph if p and B are incident in the projective plane. The resulting graph has $2(q^2+q+1)$ vertices and is $q+1$-regular. The matrix M is known as the Laplacian of the graph Γ.

Definition 1.

2. Background and Main Results

Here is out outline of what we need to do:
1. Introduce distance-regular graphs.
2. Establish nondegenerate projective plains as distance-regular.
3. Compute the eigenvalues and multiplicities of the Laplacian of a projective plane.

Definition 1. Given a graph $\Gamma=(V,E)$ and two vertices u and v in V, we define the distance between u and v to be the minimum path length between u and v. We denote this by $\partial(u,v)$. For each integer j and each vertex v, define

$$\Gamma_j(v)=\{w\in V(\Gamma)\mid \partial(v,w)=j\}$$

Now, for any vertices u and v in Γ, define the integer

$$\begin{aligned} s_{hi}(u,v) &= |\{w\in V(\Gamma)\mid \partial(u,w)=h \text{ and } \partial(v,h)=i\}| \\ &= |\Gamma_h(u)\bigcap\Gamma_i(v)| \end{aligned}$$

for all integers h and i.

Notice that $\Gamma_1(v)$ is the set of all neighbors of v in Γ.

Definition 1. A regular, connected graph Γ is said to be *distance-regular*, if there exist integers

$$b_0=k,\qquad b_1,\dots,b_{d-1},\qquad c_1=1,\qquad c-2,\dots,c_d,$$

such that for any pair of vertices u and v in Γ with $\partial(u,v)=j$, we have

$$c_j=|\Gamma_{j-1}(v)\cap\Gamma_1(u)| \text{ (for } 1\le j\le d\text{) and,}$$

$$b_j=|\Gamma_{j+1}(v)\cap\Gamma_1(u)| \text{ (for } 0\le j\le d-1\text{).}$$

The array $\{k,b_1,\dots,b_{d-1};1,c_2,\dots,c_d\}$ is the *intersection array* of Γ.

Notice that any projective plane of order k has intersection array $\{k,\ k-1,\ k-1;1,\ 1,\ k$ and so is distance-regular. In order to determine the critical groups of these graphs, we must first determine the eigenvalues of their Laplacian matrices.

Proposition 1. *The adjacency algebra $\mathcal{A}(\Gamma)$ of a distance-regular graph Γ with diameter d can be faithfully reprsented by an algebra of matrices with $d+1$ rows and columns. A basiss for this representation is the set $\{B_0,B_1,\dots,B_d\}$, where $(B_h)_{ij}$ is the intersection number s_{hij} for $h,i,j\in\{0,1,\dots,d\}$.*

Proof. See [1], Proposition 21.1. ∎

Let Γ be a distance-regular graph with degree K and diameter d. The Γ has $d+1$ distinct eigenvalues $k = \lambda_0, \lambda_1, \ldots, \lambda_d$ which are the eigenvalues of the intersection matrix B.

Definition 1. If B is an $n \times n$ matrix, we define left eigenvector of B to be a row vector u_i such that

$$u_i B = \lambda_i u_i.$$

A right eigenvector is defined analogously and is denoted v_i. We normalize all left and right eigenvectors to have first entry one.

Proposition 1. *With the notation above, the multiplicity of the eigenvalue λ_i of a distance-regular graph with n vertices is*

$$m(\lambda_i) = \frac{n}{u_i, v_i} \qquad (0 \le i \le d) \tag{1}$$

Remark 1. By our above discussion, the graph of any finite projective plane has associated matrix

$$\mathrm{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ k & 0 & 1 & 0 \\ 0 & k-1 & 0 & k \\ 0 & 0 & k-1 & 0 \end{pmatrix}$$

Proposition 1. *The eigenvalues of the adjacency matrix of a projective plane of order $k+1$ are $\pm k$ and $\pm\sqrt{k-1}$ with multiplicities $1, 1, k^2-k$, and k^2-k, respectively.*

Proof. A simple computation shows that the matrix **B** above has left and right eigenvalues of $\pm k$ and $\pm\sqrt{k-1}$ with associated left and right eigenvectors:

$$k: \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ k \\ k(k-1) \\ 0 \end{bmatrix} \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ k \\ 1 \\ 1 \end{bmatrix}$$

$$-k: \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -k \\ k(k-1) \\ -(k-1)^2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\sqrt{k-1}: \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ -\sqrt{k-1} \\ -1 \\ \sqrt{k-1} \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ \frac{(k-1)^2}{(k^2-k)\sqrt{k-1}} \\ \frac{-1}{k^2-k} \\ \frac{-1}{(k-1)\sqrt{k-1}} \end{bmatrix}$$

$$-\sqrt{k-1}: \quad \mathbf{u}_4 = \begin{bmatrix} 1 \\ \sqrt{k-1} \\ -1 \\ -\sqrt{k-1} \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ -\frac{(k-1)^2}{(k^2-k)\sqrt{k-1}} \\ \frac{-1}{k^2-k} \\ \frac{1}{(k-1)\sqrt{k-1}} \end{bmatrix}$$

Plugging these eigenvectors into 1, we have that $(\mathbf{u}_1, \mathbf{v}_1) = (\mathbf{u}_2, \mathbf{v}_2) = 2(k^2 - k + 1)$ and $(\mathbf{u}_3, \mathbf{v}_3) = (\mathbf{u}_4, \mathbf{v}_4) = 2(k^2 - k + 1)/(k^2 - k)$ □

An immediate corollary to this is that the Laplacian of a projective plane of order $k+1$ has eigenvectors $0, 2k, k \pm \sqrt{k-1}$ with multiplicities $1, 1, k^2 - k$, and $k^2 - k$, respectively. Hence, we have that:

Definition 1. Given an $n \times n$ integer matrix M, we denote by $\Phi(M)$ the group $(\mathbb{Z}^n/\mathrm{Im}(M))_{\mathrm{tor}}$. When M is the Laplacian matrix of a graph, $\Phi(M)$ is known as the critical group of the associated graph.

Corollary 1. *The order of the critical group of a finite projective plane of order* $k+1$ *is* $k(k^2 - k + 1)^{k^2-k-1}$.

Proof. It is well-known (see [1] 6.5) that the order of the critical group of a graph is equal to the product of the nonzero eigenvalues of its Laplacian divided by the number of vertices. Hence, we have that the order is

$$2\mathrm{k}(\mathrm{k}+\sqrt{k-1})^{k^2-k}(k - \sqrt{k-1})^{k^2-k}/(2(k^2 - k + 1) = k(k^2 - k + 1)^{k^2-k-1}$$ □

We require several results on the general form of a critical group of a graph, which we state here.

To begin, let M be a symmetric, integer $n \times n$-matrix of rank $n - 1$. Let R denote the $n \times 1$ column vector generating the kernel of M. Define the integer r by $r = R \cdot R$. Note that if M is the Laplacian of a graph, $R = (1, \dots, 1)^\intercal$, and $r = n$, the number of vertices of the graph.

If $n \in \mathbb{Z}$ is an integer, then for any prime p, we denote by $\mathrm{ord}_p(n)$ the largest integer such that $p^{\mathrm{ord}_p(n)}$ divides n.

Proposition 1. [2] *Let M be any symmetric, $(n \times n)$ integer matrix of rank $n-1$. Let $\lambda \neq \pm 1$ be an integer eigenvalue of M and $m(\lambda)$ its multiplicity. Then*

1. *If there exists a prime p such that $p \mid \lambda$ but $p \nmid r$, then $\Phi(M)$ contains a subgroup isomorphic to $\left(\mathbb{Z}/p^{ord_p(\lambda)}\mathbb{Z}\right)^{m(\lambda)}$*
2. *If the vector R has one entry r_i with $r_i = \pm 1$, then $\Phi(M)$ contains a subgroup isomorphic to $(\mathbb{Z}/\lambda\mathbb{Z})^{m(\lambda)-1}$.*

Proof. See [2], Proposition 2.3. □

Notation 1. Let M be a symmetric integer matrix and λ is an eigenvalue of M with minimal polynomial $f(x) = a_d x^d + \cdots + a_0 \in \mathbb{Z}[x]$. Denote by $L(\lambda)$ the least common multiple of the roots of $f(x)$ (when computed in the ring of algebraic integers).

Proposition 1. *Let M be an integer $(n \times n)$-matrix of rank $n-1$. Let r be defined as above, with $r > 0$. Suppose λ is an eigenvalue of M. Then $\Phi(M)$ contains an element of order $L(\lambda)/gcd(L(\lambda), r)$.*

Proof. See [2], Proposition 2.11. □

Let λ and $f(x)$ be as above. Define the polynomial $g(x)$ by $f(x) = xg(x) + a_0$.

Proposition 1. *With the notation above and assuming $a_1 \neq 0$, the kernel of the map $\mathbb{Z}^n/Im(Mg(M)) \to \mathbb{Z}^n/M \times \mathbb{Z}/Im(g(M))$ is killed by a_1, and the natural map*

$$q : \Phi(Mg(M)) \longrightarrow \Phi(M) \times \Phi(g(M))$$

induces an isomorphism on the prime-to-a_1 part of these groups.

Theorem 1. *The critical group of the incidence graph of a nondegenerate projective plane of order $k-1 = p^n$, p a prime, is of the form $\mathbb{Z}_k \oplus (\mathbb{Z}_{k^2-k+1})^{k^2-k-1}$.*

Proof. Let M be the Laplacian of the incidence graph of a finite projective plane of order $k-1$.

Recall that if M is the Laplacian matrix of a projective plane graph, then M has eigenvalues $0,\ 2k,\ k \pm \sqrt{k-1}$ with multiplicities $1,\ 1,\ k^2-k,\ k^2-k$. Now, as $2k$ is an integer, its minimal polynomial of $\mathbb{Q}$ is $f(x) = x - 2k$. By 1, $\Phi(M)$ contains an element of order $2k/\gcd(2k, 2(k^2-k+1)) = k$. Hence, we have that $\Phi(M)$ contains a subgroup isomorphic to $\mathbb{Z}/k\mathbb{Z}$.

Suppose $k-1 = p^{2n}$, p a prime. Notice, then, that if $q | p^{2n} - p^n + 1$ and $q | p^{2n} + p^n + 1$, q a prime, then $q | 2p^n$, whence $q = 2$ or $q = p$, which is impossible. Hence, $q = 1$ and $(p^{2n} + p^n + 1, p^{2n} - p^n + 1) = 1$. Then, by **??**, we have that $\Phi(M)$ contains subgroups isomorphic to $(\mathbb{Z}_{k-\sqrt{k-1}})^{k^2-k-1}$ and $(\mathbb{Z}_{k+\sqrt{k-1}})^{k^2-k-1}$, whence $\Phi(M)$ contains a subgroup isomorphic to $\mathbb{Z}_{k^2-k+1}^{k^2-k-1}$. Hence, $\Phi(M) \cong \mathbb{Z}_k \oplus \left(\mathbb{Z}/(k^2-k+1)\mathbb{Z}\right)^{k^2-k-1}$.

Now supposed $k-1=p^{2n+1}$. The minimal polynomial of both $k+\sqrt{k-1}$ and $k-\sqrt{k-1}$ is $f(x)=x^2-2kx+k^2-k+1$. Let $g(x)$ be defined by $f(x)=xg(x)+k^2-k+1$. Then, by 5.20 in 1, there is an isomorphism

$$\Phi(M(M-2kI)) \cong \Phi(M) \times \Phi(M-2kI) \tag{2}$$

on the prime-to-$2k$ parts of these groups.

Now, as M is symmetric, it is diagonalizable, whence $M = SDS^{-1}$, for some invertible matrix S. Moreover, $M-2kI = S(D-2kI)S^{-1}$, and so $M-2kI$ has eigenvalues $-2k,\ 0,\ -k\pm\sqrt{k-1}$, with multiplicities $1,\ 1, k^2-k,\ k^2-k$. Hence $M(M-2kI) =$S(D$^2-$ $2kD)S^{-1}$ has corresponding eigenvalues $0,\ -k^2+k-1$ with multiplicities $2,\ 2k^2-2k$. Now,

$$\begin{aligned} M^2-2kM &= (\Delta-A)^2-2k+2kA \\ &= \Delta^2-2kA+A^2-2\Delta^2+2kA \\ &= A^2-\Delta^2 \\ &= A^2-k^2I, \end{aligned}$$

where A is the adjacency matrix of the graph and Δ is the diagonal matrix with entries corresponding to vertex degree, which in this case is kI. Now, it is well-known that if A is the adjacency matrix (see [1] 2.5), $(A^k)_{ij}$ represents the number of paths of length k from vertex i to vertex j. Now, as our graph is a projective plane, there is exactly one path of length two between any pair of points and lines. There are no paths of length two between a point and a line, and between any point or line and itself, there are precisely k paths of length 2. Hence,

$$M^2-2kM = \begin{pmatrix} J-(k^2-k+1)I & 0 \\ 0 & J-(k^2-k+1)I \end{pmatrix} \tag{3}$$

where J is a $(k^2-k+1)\times(k^2-k+1)$ matrix of ones and I is the corresponding identity. Now, by addding rows and columns 2 through k^2-k+1 to the first row and column and by adding rows and columns 2 through k^2-k+2 through $2k^2-2k+1$ to the last row and column, we obtain the matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & J-(k^2-k+1)I & 0 & \vdots \\ \vdots & 0 & J-(k^2-k+1)I & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

where J is a $(k^2-k)\times(k^2-k)$ matrix of ones and I is the corresponding identity. Notice that this matrix has rank $2k^2-2k$, and that the nonzero part of it is a principal submatrix of M^2-2kM. Denote this matrix by M'. Notice that $\Phi(M^2-2kM)\cong\Phi(M')$. Now, the $(k^2-k)\times(k^2-k)$-matrix J has eigenvalues 0 and k^2-k, with multiplicities k^2-k-1 and 1,respectively. Hence, the eigenvalues of M' are -1 and $-k^2+k-1$, with multiplicities 2 and $2(k^2-k-1)$.

Hence, as M' is nonsingular, we may apply [3], Theorem 4, to see that $\Phi(M')$ contains a subgroup isomorphic to $\left(\mathbb{Z}/(k^2-k+1)\mathbb{Z}\right)^{2(k^2-k-1)}$. Hence, as k^2-k+1 is prime to $2k$, the group $\Phi(M)\times\Phi(g(M))$ contains a subgroup isomorphic to $\left(\mathbb{Z}/(k^2-k+1)\mathbb{Z}\right)^{2(k^2-k-1)}$.

But, by Corollary 3 in [3], if s_n is the largest integer in the Smith Normal Form of N, then $s_n|k^2-k+1$, and so $\Phi(M^2-2kM) \lhd \left(\mathbb{Z}/(k^2-k+1)\mathbb{Z}\right)^{2(k^2-k-1)}$. Hence, $\Phi(M^2-2kM)$ is isomorphic to $\left(\mathbb{Z}/(k^2-k+1)\mathbb{Z}\right)^{2(k^2-k-1)}$. Now, by our above proposition, as the order or $\Phi(M)$ is $k(k^2-k+1)^{k^2-k-1}$, the prime-to-$2k$ part is precisely $\left(\mathbb{Z}/(k^2-k+1)\mathbb{Z}\right)^{(k^2-k-1)}$. Hence, $\Phi(M) \cong (\mathbb{Z}/k\mathbb{Z}) \oplus \left(\mathbb{Z}/(k^2-k+1)\mathbb{Z}\right)^{k^2-k-1}$. $\square$

References

[1] Norman Biggs. *Algebraic Graph Theory.* Cambridge University Press, 1993.

[2] Dino Lorenzini. Smith normal form and laplacians. *J. Combin. Theory B*, 95:1271–1300, 2008.

[3] Joseph J. Rushanan. Eigenvalues and the smith normal form. *Linear Algebra Appl.*, 216:177–184, 1995.