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Decomposition Numbers of Chevalley Groups

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INTRODUCTION

Consider a finite Chevalley group $G(p^n)$ defined over a field of p^n elements, p a prime. The projective indecomposable characters in characteristic p are indexed by the p^n-restricted weights. If a weight λ is p-restricted, it is also p^n-restricted. If $\Phi_{\lambda, n}$ is the corresponding projective indecomposable character of $G(p^n)$, we can ask how the decomposition of $\Phi_{\lambda, n}$ into ordinary irreducible characters varies with n.

The ordinary irreducible characters are obtained from the Deligne–Lusztig characters. In this paper we will show how to write the Deligne–Lusztig characters in terms of what we call a p-adic coding, so that we can answer the above question. We will show how to associate with each component of the p-adic coding a matrix, so that the trace of the product of the corresponding matrices gives the multiplicity of the corresponding character in $\Phi_{\lambda, n}$.

To avoid some technical complications we shall assume here that we are in the generic case, so that all the Deligne–Lusztig characters occurring in the decomposition of $\Phi_{\lambda, n}$ are irreducible. We shall also assume that $\Phi_{\lambda, n}$ can be expressed as a tensor product of $G(p)$-projective characters raised to a power of the Frobenius. However, the methods can be extended to non-generic cases; we shall deal with this elsewhere.

In fact, our result is a general one, and we apply it to decompose any projective character which can be expressed as a tensor product of $G(p)$-projective characters raised to a power of the Frobenius. The results apply to twisted groups as well, and illustrate the relationship between the decomposition numbers in the twisted and untwisted case.

We apply our method to determine the decomposition of $\Phi_{\lambda, n}$ for $SL_3(p^n)$ and $SU_3(p^n)$, λ a generic weight for $SL_3(p)$. We also show how generic Cartan invariants of $G(p^n)$ can be calculated using our methods, and again apply this to $SL_3(p^n)$ and $SU_3(p^n)$.

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Our method works when p is a large enough prime, although it can be extended to work for all primes.

1. PRELIMINARIES

Let G be a semi-simple simply connected algebraic group, defined over k, the algebraic closure of a finite field $\mathbb{F}_p$ of characteristic p. Denote the Frobenius automorphism by Fr. Let π be an automorphism of the Dynkin diagram, and let $F = Fr^n \circ \pi$ be the induced automorphism of G. Denote the finite group of fixed points of F on G by $G(p^n)_\pi$.

Let T be a maximal torus of G, split and defined over $\mathbb{F}_p$. Let $X = X(T)$ be the weight lattice. Let X_{p^n} be the subset of p^n-restricted weights, consisting of those elements of X whose coordinates in terms of the fundamental dominant weights lie between 0 and $p^n - 1$. Let ρ be half the sum of the positive roots.

The irreducible p-modular characters of $G(p^n)_\pi$ are indexed by X_{p^n}. For $\lambda \in X_{p^n}$, let $\chi_p(\lambda)$ be the corresponding p-modular irreducible, and let $\Phi_{\lambda, n}$ be the character of its projective cover. In the generic case, these characters have a nice tensor product property: (see [1, 2]). If λ is written p-adically, $\lambda = \sum p^k \lambda_k$, with each λ_k p-restricted, then $\Phi_\lambda = \Phi_{\lambda, n} = \prod_k \Phi_{\lambda k}^{Fr^k}$, where $\Phi_{\lambda_k} = \Phi_{\lambda_k, 1}$ is the projective cover of $\chi_p(\lambda_k)$ as a $kG(p)$ character. Moreover, each Φ_{λ_k} is divisible by the Steinberg character for $G(p)$, which we denote by St. Thus we can write $\Phi_{\lambda_k} = St \cdot \varphi(\lambda_k)$ and $\Phi_{\lambda, n} = St_n \cdot \prod_k \varphi(\lambda_k)^{Fr^k}$, where $St_n = \prod_k St^{Fr^k}$ is the Steinberg character for $G(p^n)_\pi$.

Let W be the Weyl group of G. For $\lambda \in X$, let $s(\lambda) = \sum_{\lambda'} e(\lambda')$ the sum over distinct λ' in the W-orbit of λ.

In [1], a formula is given for expressing $\varphi(\lambda)$ as a linear combination of the $s(\lambda)$, modulo a knowledge of the characters $\chi_p(\lambda)$. Thus in the generic case, we can write a projective indecomposable character of $G(p^n)_\pi$ as $\Phi = St_n \cdot \prod_{k=0}^{n-1} \sum_{\lambda \in X_p} a_k(\lambda) s(\lambda)^{Fr^k}$, where a_k is a function from the p-restricted weight lattice X_p to $\mathbb{Z}$.

The ordinary irreducible characters are obtained from the Deligne–Lusztig characters. We shall use Jantzen's notation for these characters (see [4]). Let $w \in W$ and let $n_w \in N_g(T)$ be a representative of the coset wT. There is an element $g_w \in G$ such that $g_w^{-1} F g_w = n_w$. Let $T_w = g_w T g_w^{-1}$. If $\lambda \in X$, we have a homomorphism from $T_w \cap G(p^n)_\pi \to k^*$ given by $t \to \lambda(g_w^{-1} t g_w)$, and consequently one also from $T_w \cap G(p^n)_\pi$ to $\mathbb{C}^*$, which we denote by θ. (We use the same embedding of roots of unity as we do in relating Brauer characters to ordinary characters.) This gives rise to a Deligne–Lusztig character $R_{T_w}(\theta)$ of $G(p^n)_\pi$, and we denote $\epsilon_G \epsilon_{T_w} R_{T_w}(\theta)$ by $R(w, \lambda)$. ($R(w, \lambda)$ depends on n and π but we shall suppress this in the notation.)

We have an action of $W \times X$ on itself given by $(w, \lambda)^{(v,\nu)} \to (vw\pi v^{-1}\pi^{-1}, \lambda + (p^n - vw\pi v^{-1})\nu)$. We have $R(w, \lambda) = R(y, \mu)$ if and only if (w, λ) and (y, μ) are conjugate under this action.

2. *P*-ADIC CODING OF DELIGNE–LUSZTIG CHARACTERS

To analyze the decomposition numbers, we will be considering Deligne–Lusztig characters $R(w, \lambda)$ with $\lambda = \sum_{k=0}^{n-1} p^k w_k \lambda_k$, $\lambda_k \in X_p$. It turns out that what is important for the analysis is how w_k is related to w_{k+1}. We therefore introduce an association between n-tuples $([\mu_0, z_{n-1}]; [\mu_{n-1}, z_{n-2}]; \cdots ; [\mu_1, z_0])$ with $\mu_i \in X^+$ and $z_i \in W$ and pairs (w, λ), with $w \in W$ and $\lambda \in X$, which we call a p-adic coding.

In the following definition, subscripts are to be read modulo n, so a subscript of -1 is understood to be the same as $n-1$.

Definition 2.1. Let $[\mu_k, z_{k-1}] \in X_p \times W$ for $k = 0, \ldots, n-1$. For $0 \le k \le n-2$, let $w_k = z_{n-2} z_{n-3} \cdots z_k$. Let $y = z_{n-2} z_{n-3} \cdots z_0 z_{n-1}$. Let $\lambda = p^{n-1}\mu_{n-1} + \sum_{k=0}^{n-2} p^k w_k \mu_k$. Define $\chi([\mu_0, z_{n-1}]; [\mu_{n-1}, z_{n-2}]; \cdots ; [\mu_1, z_0]) = R(y, \lambda)$. We call $\chi([\mu_0, z_{n-1}]; \cdots ; [\mu_1, z_0])$ a p-adic coding for $R(y, \lambda)$.

The following proposition shows how to get a p-adic coding, given a Deligne–Lusztig character.

Proposition 2.2. *Let $\lambda \in X$ and suppose we can write $\lambda = \sum_{k=0}^{n-1} p^k w_k \mu_k$ with $\mu_k \in X_p$. For $0 \le k \le n-2$, let $z_k = w_{k+1}^{-1} w_k$, and let $z_{n-1} = w_0^{-1} y \pi w_{n-1} \pi^{-1}$. Then $R(y, \lambda) = \chi([\mu_0, z_{n-1}]; [\mu_{n-1}, z_{n-2}], \cdots ; [\mu_1, z_0])$.*

Proof. By definition,

$$\chi([\mu_0, z_{n-1}]; \cdots ; [\mu_1, z_0]) = R\left(v, p^{n-1}\mu_{n-1} + \sum p^k u_k \mu_k\right),$$

where

$$\begin{aligned} u_k &= z_{n-2} \cdots z_k \\ &= \left(w_{n-1}^{-1} w_{n-2}\right)\left(w_{n-2}^{-1} w_{n-3}\right) \cdots \left(w_{k+1}^{-1} w_k\right) \\ &= w_{n-1}^{-1} w_k, \qquad \text{for } 0 \le k \le n-2, \end{aligned}$$

and

$$v = z_{n-2} \cdots z_0 z_{n-1} = w_{n-1}^{-1} y \pi w_{n-1} \pi^{-1}.$$

Now apply $(w_n, 0)$ to $(v, p^{n-1}\mu_{n-1} + \sum p^k u_k \mu_k)$, via the action described at the end of Section 1. We get $R(v, p^{n-1}\mu_{n-1} + \sum p^k u_k \mu_k) = R(w_{n-1} v \pi w_{n-1}^{-1} \pi^{-1}, p^{n-1} w_{n-1} \mu_{n-1} + \sum p^k w_{n-1} u_k \mu_k)$. But $w_{n-1} v \pi w_{n-1}^{-1} \pi^{-1} = y$ and $w_{n-1} u_k = w_k$ for $0 \le k \le n-2$, so the latter is equal to $R(y, \lambda)$, as claimed.

For purposes of analyzing how $R(w, \lambda)$ behaves in characteristic p, it is also useful to write λ as $\sum p^k w_k \mu_k$, with μ_k "small." Accordingly, we introduce the following notation.

Let $\tilde{W}_{a,p}$ be the extended affine Weyl group, i.e., the semi-direct product of W in its action on $\mathbb{R} \otimes X$ with translations by elements of pX. Let $\tilde{C}_p$ be the fundamental domain for $\tilde{W}_{a,p}$ in this action which is contained in X^+ and contains the origin. Then every element of $\lambda \in X$ can be written as $\lambda = p\lambda' + \tau\lambda''$ with $\lambda'' \in \tilde{C}_p$, $\lambda' \in X$, and $\tau \in W$.

PROPOSITION 2.3. *Let* $\lambda = \sum_{k=0}^{n-1} p^k y_k \lambda_k$, *with each* $\lambda_k \in X_p$, *and write* $\lambda_k = p\lambda'_k + \tau_k \lambda''_k$ *as above. Let* $y \in W$. *For* $0 \le k \le n-2$, *let* $w_k = \tau_{k+1}^{-1} y_{k+1}^{-1} y_k$, *and let* $w_{n-1} = \tau_0^{-1} y_0^{-1} y \pi y_{n-1} \pi^{-1}$. *Let* $\mu_0, \dots, \mu_{n-1} \in \tilde{C}_p$ *and* $z_0, \dots, z_{n-1} \in W$. *Assume* $\lambda''_{k+1} + w_k \lambda'_k$ *is conjugate under the Weyl group to an element of* $\tilde{C}_p$ *for all* k. *Then* $R(y, \lambda)$ *has p-adic coding* $\chi([\mu_{n-1}, z_{n-1}] \cdots ; [\mu_0, z_0])$ *if and only if there exist* $v_0, v_1, \dots, v_{n-1} \in W$ *such that for* $0 \le k \le n-2$, $z_k = v_k^{-1} w_k \tau_k v_{k-1}$ *and* $\mu_k = v_k^{-1}(\lambda''_{k+1} + w_k \lambda'_k)$, *while* $z_{n-1} = v_{n-1}^{-1} w_{n-1} \pi \tau_{n-1} v_{n-2} \pi^{-1}$ *and* $\mu_{n-1} = v_{n-1}^{-1}(\lambda''_0 + w_{n-1} \pi \lambda'_{n-1})$.

The subscripts, as usual, are to be read modulo n.

Proof.

$$
\begin{aligned}
R\left(y, \sum_{k=0}^{n-1} p^k y_k \lambda_k\right)
&= R\left(y, \sum_{k=0}^{n-1} p^k y_k (p\lambda'_k + \tau_k \lambda''_k)\right) \\
&= R\left(y, p^n y_{n-1} \lambda'_{n-1} + \sum_{k=1}^{n-1} p^k y_k \tau_k \left(\lambda''_k + \tau_k^{-1} y_k^{-1} y_{k-1} \lambda'_{k-1}\right) + y_0 \tau_0 \lambda''_o\right) \\
&= R\Bigg(y, p^n y_{n-1} \lambda'_{n-1} + \sum_{k=1}^{n-1} p^k y_k \tau_k (\lambda''_k + w_{k-1} \lambda'_{k-1}) \\
&\qquad\qquad + y_0 \tau_0 \lambda''_0 + (p^n - y\pi)(-y_{n-1} \lambda'_{n-1})\Bigg) \\
&= R\Bigg(y, \sum_{k=1}^{n-1} p^k y_k \tau_k (\lambda''_k + w_{k-1} \lambda'_{k-1}) \\
&\qquad + y_0 \tau_0 \left(\lambda''_0 + \tau_0^{-1} y_0^{-1} y \pi y_{n-1} \lambda'_{n-1}\right)\Bigg) \\
&= R\left(y, \sum_{k=1}^{n-1} p^k y_k \tau_k (\lambda''_k + w_{k-1} \lambda'_{k-1}) + y_0 \tau_0 (\lambda''_0 + w_{n-1} \pi \lambda'_{n-1})\right).
\end{aligned}
$$

For $0 \leq k \leq n-2$, let μ_k be the element of X^+ conjugate to $\lambda''_{k+1} + w_k \lambda'_k$ and let v_k be an element of W such that $v_k \mu_k = \lambda''_{k+1} + w_k \lambda'_k$. Let μ_{n-1} be the element of X^+ conjugate to $\lambda''_0 + w_{n-1}\pi\lambda'_{n-1}$ and let v_{n-1} be an element of W such that $v_{n-1}\mu_{n-1} = \lambda''_0 + w_{n-1}\pi\lambda'_{n-1}$.

Then $R(y, \lambda) = R(y, \Sigma_{k=1}^{n-1} p^k y_k \tau_k v_{k-1} + y_0 \tau_0 v_{n-1} \mu_{n-1})$. By 2.2, this has p-adic coding

$$\chi([\mu_{n-1}, z_{n-1}]; [\mu_{n-2}, z_{n-2}]; \cdots ;[\mu_0, z_0]),$$

where $z_k = v_k^{-1}\tau_{k+1}^{-1}y_{k+1}^{-1}y_k\tau_k v_{k-1} = v_k^{-1}w_k\tau_k v_{k-1}$, for $1 \leq k \leq n-2$, while $z_0 = v_0^{-1}\tau_1^{-1}y_1^{-1}y_0\tau_0 v_{n-1} = v_0^{-1}w_0\tau_0 v_{n-1}$ and $z_{n-1} = v_{n-1}^{-1}\tau_0^{-1}y_0^{-1}y\pi y_{n-1}\tau_{n-1}v_{n-2}\pi^{-1} = v_{n-1}^{-1}w_{n-1}\pi\tau_{n-1}v_{n-2}\pi^{-1}$.

Thus $R(y, \lambda)$ has the p-adic coding given in the statement of the proposition, if and only if the values of μ_k and z_k are as stated there. This proves the proposition.

Remark. A p-adic coding for a given $R(y, \lambda)$ is certainly not unique. For example, if the W-stabilizer of μ_k is non-trivial, then there is a choice of v_k in the proof of 2.3, so there is a corresponding choice for z_k. Indeed, it is easy to see that if $w\mu_k = \mu_k$, then

$$\chi([\mu_{n-1}, z_{n-1}]; \cdots ;[\mu_k, z_k]; [\mu_{k-1}, z_{k-1}]; \cdots ;[\mu_0, z_0])$$

is equal to

$$\chi\big([\mu_{n-1}, z_{n-1}]; \cdots ;[\mu_k, z_k w]; \big[\mu_{k-1}, w^{-1}z_{k-1}\big]; \cdots ;[\mu_0, z_0]\big).$$

3. DECOMPOSITION OF PROJECTIVE CHARACTERS

Methods for determining the projective indecomposable characters of the groups we are considering are given by Jantzen [4], and the author [1]. The results there allow us to express a projective indecomposable character as a linear combination of characters of the form $St_n \cdot s(\lambda)$ (provided we know the decomposition of a Weyl module into irreducibles). These can be expressed in terms of Deligne–Lusztig characters using the following theorem of Humphreys [3]:

THEOREM 3.1 (Humphreys). *Let* $\lambda \in X$. *Then* $|Stab_W(\lambda)| St_n \cdot s(\lambda) = \Sigma_{w \in w} R(w, \lambda)$.

We now assume that we have a projective character Φ which can be written as $\Phi = St_n \cdot \prod_{k=0}^{n-1} \Sigma_{\lambda \in X_p} a_k(\lambda) s(\lambda)^{Fr^k}$, where a_k is a function from X_p to $\mathbb{Z}$. Humphreys' theorem allows us to decompose this character into Deligne–Lusztig characters. To determine the decomposition numbers, we need to determine the multiplicity with which a given Deligne–Lusztig character occurs.

We will now define the matrices, the product of whose traces will give the decomposition numbers.

DEFINITION 3.2. Let $\mu \in X^+$, $z \in W$. Suppose $a : X_p \to \mathbb{Z}$ is a function from X_p to the integers. Define a matrix $M_a(\mu, z)$ indexed by $W \times X^+$ as follows. The $(v, \lambda'') - (y, \nu'')$ entry of $M_a(\mu, z)$ is $\sum a(\lambda)$, where the sum is over $\lambda \in X_p$ such that $\lambda = p\lambda' + \tau\lambda''$ for some $\lambda' \in X^+$, $\tau \in W$, and there exists $w \in W$ with $\mu = y^{-1}(\nu'' + w\lambda')$ and $z = y^{-1}w\tau v$.

Our results will apply when for each λ for which $a_k(\lambda)$ is non-zero and $\lambda = p\lambda' + \tau\lambda''$ with $\lambda'' \in \tilde{C}_p$, λ'' is far enough away from the upper boundary of $\tilde{C}_p$. We state this in the following condition:

Condition 3.3. $\Phi = St_n \cdot \prod_k \sum_{\lambda \in X_p} a_k(\lambda)s(\lambda)^{Fr^k}$ such that if $a_k(\lambda)$ and $a_{k-1}(\mu)$ are both non-zero (subscript modulo n), and we write $\lambda = p\lambda' + \tau_1\lambda''$ and $\mu = p\mu' + \tau_2\mu''$ as in Section 2 then $\lambda'' + w\mu'$ is conjugate under W to an element of $\tilde{C}_p$, not lying on the upper boundary of $\tilde{C}_p$.

We can now state our main results. We first assume that we are in the untwisted case.

THEOREM 3.4. *Assume* $\pi = 1$. *Suppose* Φ *is a projective character, with* $\Phi = St_n \cdot \prod_{k=0}^{n-1} \sum_{\lambda \in X_p} a_k(\lambda)s(\lambda)^{Fr^k}$ *satisfying Condition* 3.3. *Also assume that every Deligne–Lusztig character occurring in the decomposition of* Φ *is irreducible. Let* $\mu_0, \dots, \mu_{n-1} \in \tilde{C}_p$ *and* $z_0, \dots, z_{n-1} \in W$. *Then the multiplicity of the Deligne–Lusztig character with p-adic coding* $\chi([\mu_{n-1}, z_{n-1}]; \cdots; [\mu_0, z_0])$ *in* Φ *is equal to* $\operatorname{Tr}(\prod_{k=0}^{n-1} M_{a_k}(\mu_k, z_k))$.

Proof. Expanding the product $\prod_{k=0}^{n-1} \sum_{\lambda \in X^+} a_k(\lambda)s(\lambda)^{Fr^k}$ gives

$$\sum_{(\lambda_0, \dots, \lambda_{n-1}) \in (X_p)^n} \prod_{k=0}^{n-1} a_k(\lambda_k)s(p^k\lambda_k),$$

which is equal to

$$\sum_{(\lambda_0, \dots, \lambda_{n-1}) \in (X_p)^n} \prod_{k=0}^{n-1} a_k(\lambda_k)s\left(p^{n-1}\lambda_{n-1} + \sum_{k=0}^{n-2} \sum_{y_k \in W} p^k y_k \lambda_k\right).$$

We decompose this using Humphreys' theorem 3.1. Since we are assuming that the Deligne–Lusztig characters occurring are irreducible, all the weights occurring have trivial stabilizer. We get

$$\begin{aligned}\Phi &= \sum_{(\lambda_0, \dots, \lambda_{n-1}) \in (X_p)^n} \prod_{k=0}^{n-1} a_k(\lambda_k)St_n s\left(p^{n-1}\lambda_{n-1} + \sum_{k=0}^{n-2} \sum_{y_k \in W} p^k y_k \lambda_k\right) \\ &= \sum_{\substack{(\lambda_0, \dots, \lambda_{n-1}) \in (X_p)^n \\ (y, y_0, \dots, y_{n-2}) \in W^n}} \prod_{k=0}^{n-1} a_k(\lambda_k)R\left(y, p^{n-1}\lambda_{n-1} + \sum_{k=0}^{n-2} p^k y_k \lambda_k\right).\end{aligned}$$

Now by 2.3, $\chi([\mu_{n-1}, z_{n-1}]; \cdots ;[\mu_0, z_0])$ is a p-adic coding for $R(y, p^{n-1}\lambda_{n-1} + \sum p^k y_k \lambda_k)$ if and only if there exist $v_{-1}, v_0, v_1, \ldots, v_{n-1} \in W$, with $v_{-1} = v_{n-1}$ and

$$z_k = v_k^{-1} w_k \tau_k v_{k-1} \qquad \text{for all } k,$$
$$\mu_k = v_k^{-1}(\lambda''_{k+1} + w_k \lambda'_k) \qquad \text{for } 0 \le k \le n-2,$$

and

$$\mu_{n-1} = v_{n-1}^{-1}(\lambda''_0 + w_{n-1}\lambda'_{n-1}),$$

where $w_k = \tau_{k+1}^{-1} y_{k+1}^{-1} y_k$ for $0 \le k \le n-2$ and $w_{n-1} = \tau_0^{-1} y_0^{-1} y$. Now for a given sequence $\lambda_0, \ldots, \lambda_{n-1}$, as $(y_1, y_0, \ldots, y_{n-2})$ ranges over W^n, so does $(w_0, w_1, \ldots, w_{n-1})$. Thus the multiplicity of $\chi([\mu_{n-1}, z_{n-1}]; \cdots ;[\mu_0, z_0])$ in Φ is equal to $\sum \prod_k a_k(\lambda_k)$, the sum over all sequences $v_{-1}, v_0, v_1, \ldots, v_{n-1} \in W$ with $v_{-1} = v_{n-1}$ and sequences $\lambda''_0, \lambda''_1, \ldots, \lambda''_n$ with $\lambda''_n = \lambda''_0$, such that there exist $\lambda'_0, \lambda'_1, \ldots, \lambda'_{n-1} \in X^+$; $\tau_0, \tau_1, \ldots, \tau_{n-1}$; $w_0, w_1, \ldots, w_{n-1} \in W$, with $\lambda_k = p\lambda'_k + \tau_k \lambda''_k$ and $\mu_k = v_k^{-1}(\lambda''_{k+1} + w_k \lambda'_k)$; and $z_k = v_k^{-1} w_k \tau_k v_{k-1}$ for $0 \le k \le n-1$. Thus by definition of $M_{a_k}(\mu_k, z_k)$, the multiplicity is equal to

$$\sum m^{(0)}_{\alpha_0, \alpha_1} m^{(1)}_{\alpha_1, \alpha_2} \cdots m^{(n-1)}_{\alpha_{n-1}, \alpha_0},$$

where the sum is over $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in W \times X^+$, and where $m^{(k)}_{\alpha, \beta}$ is the $\alpha - \beta$ element of $M_{a_k}(\mu_k, z_k)$. But this is precisely equal to the trace of the matrices as stated.

We now generalize the theorem so that it applies to twisted groups as well. We need to define another matrix to account for the twisting by π.

DEFINITION 3.5. Let M_π be the permutation matrix indexed by $W \times X^+$ whose $(w, \nu_1) - (y, \nu_2)$ entry is 1 if $y = \pi w \pi^{-1}$ and $\nu_2 = \pi\nu_1$, and 0 otherwise.

THEOREM 3.6. *Suppose* $\Phi = St_n \prod_{k=0}^{n-1} \sum_{\lambda \in X_p} a_k(\lambda) s(\lambda)^{Fr^k}$ *is a projective character of* $G(p^n)_\pi$ *satisfying the same conditions as in* 3.4. *Let* $\mu_0, \ldots, \mu_{n-1} \in \tilde{C}_p$ *and* $z_0, \ldots, z_{n-1} \in W$. *The multiplicity of the Deligne–Lusztig character with p-adic coding* $\chi([\mu_{n-1}, z_{n-1}]; \cdots ;[\mu_0, z_0])$ *is* $\mathrm{Tr}(\prod_{k=0}^{n-2} M_{a_k}(\mu_k, z_k) M_\pi M_{a_{n-1}}(\mu_{n-1}, z_{n-1}))$.

Proof. As in the proof of 3.4, we have

$$\Phi = \sum_{\substack{(\lambda_0, \ldots, \lambda_{n-1}) \in X^n \\ (y, y_0, \ldots, y_{n-2}) \in W^N}} \prod_{k=0}^{n-1} a_k(\lambda_k) R\left(y, p^{n-1}\lambda_{n-1} + \sum_{k=0}^{n-2} p^k y_k \lambda_k \right).$$

By 2.3, $\chi([\mu_{n-1}, z_{n-1}]; \cdots ;[\mu_0, z_0])$ is a p-adic coding for $R(y, p^{n-1}\lambda_{n-1} + \Sigma p^k y_k \lambda_k)$ if and only if there exist $v_{-1}, v_0, \ldots, v_{n-1} \in W$ with $v_{-1} = v_{n-1}$ and for $0 \leq k \leq n-2$

$$z_k = v_k^{-1} w_k \tau_k v_{k-1}$$

and $\mu_k = v_k^{-1}(\lambda''_{k+1} + w_k \lambda'_k)$, while

$$z_{n-1} = v_{n-1}^{-1} w_{n-1} \pi \tau_{n-1} v_{n-2} \pi^{-1}$$

and $\mu_{n-1} = \lambda''_0 + w_{n-1}\pi\lambda'_{n-1}$, where $w_k = \tau_{k+1}^{-1} y_{k+1}^{-1} y_k$ for $0 \leq k \leq n-2$ and $w_{n-1} = \tau_0^{-1} y_0^{-1} y$. Thus the multiplicity of $\chi([\mu_{n-1}, z_{n-1}]; \cdots ;[\mu_0, z_0])$ in Φ is equal to $\Sigma \prod_k a_k(\lambda_k)$ where the sum is over sequences $v_{-1}, v_0, \ldots, v_{n-1}, v_\pi \in W$ with $v_{-1} = v_{n-1}$, and sequences λ''_0, $\lambda''_1, \ldots, \lambda''_{n-1}, \lambda''_\pi \in X^+$ with $\lambda''_n = \lambda''_0$, such that there exist $\lambda'_0, \ldots, \lambda'_{n-1}$, $\lambda'_\pi \in X^+$, $\tau_0, \ldots, \tau_{n-1}, \tau_\pi, w_0, \ldots, w_{n-1} \in W$ with $\lambda_k = p\lambda'_k + \tau_k \lambda''_k$ for $0 \leq k \leq n-1$ and

$$\mu_k = v_k^{-1}(\lambda''_{k+1} + w_k \lambda'_k),$$

$$z_k = v_k^{-1} w_k \tau_k v_{k+1} \qquad \text{for } 0 \leq k \leq n-2;$$

$$\mu_{n-1} = v_{n-1}^{-1}(\lambda''_n + w_{n-1}\lambda'_\pi), \qquad z_{n-1} = v_{n-1}^{-1} w_{n-1} \tau_\pi v_\pi,$$

$$\tau_\pi = \pi \tau_{n-1} \pi^{-1}, \qquad v_\pi = \pi v_{n-2} \pi^{-1}, \qquad \lambda'_\pi = \pi \lambda_{n-1}, \qquad \lambda''_\pi = \pi \lambda''_{n-1}.$$

Thus by definition of $M_{a_k}(\mu_k, z_k)$, the multiplicity is equal to

$$\sum m^{(0)}_{\alpha_0, \alpha_1} m^{(1)}_{\alpha_1, \alpha_2} \cdots m^{(n-2)}_{\alpha_{n-2}, \alpha_\pi} m^{\pi}_{\alpha_\pi, \alpha_{n-1}} m^{(n-1)}_{\alpha_{n-1}, \alpha_0},$$

where the sum is over $\alpha_0, \alpha_1, \ldots, \alpha_{n-2} \alpha_\pi \alpha_{n-1} \in W \times X^+$, and where $m^{(k)}_{\alpha, \beta}$ is the $\alpha - \beta$ element of the matrix $M_{a_k}(\mu_k, z_k)$ and $m^{\pi}_{\alpha, \beta}$ is the $\alpha - \beta$ element of the matrix M_π. But this is equal to the trace of the matrices as stated.

Remark. We need the condition of smallness on the λ''_k (Condition 3.3) because a Deligne–Lusztig character may be represented by more than one p-adic coding. If $\mu_k = v_k^{-1}(\lambda''_k + w_k \lambda'_k)$ lies in an alcove above $\tilde{C}_p$, the corresponding Deligne–Lusztig character may be equal to one whose p-adic coding has the corresponding component inside $\tilde{C}_p$. The multiplicity would not be given correctly. If μ_k lies on a lower boundary, we can also have different codings representing the same character. (See the remark at the end of Section 2.) However, each coding is accounted for, since v_k is allowed to range over all of $Stab_w(\mu_k)$ in the proof of the theorem. Thus the multiplicity is correctly given in this case. The only problem which could arise would be if the stabilizer of every μ_k were non-trivial, when there would be a choice of v_k for every component. However, the corresponding Deligne–Lusztig character would not be irreducible, and we are assuming it is. We will discuss this situation elsewhere when we consider non-generic cases.

Note that the theorem applies to the situation described in the Introduction.

Suppose p is large enough and ν is deep enough in the interior of a $\tilde{W}_{a,p}$ alcove of X_p, and let Φ_ν be the projective cover of the corresponding irreducible $G(p^n)_\pi$ character. Then by the results of [1, 2] we can write $\Phi_\nu = St_n \prod_{k=0}^{n-1} \sum_{\lambda \in X_p} a_k(\lambda) s(\lambda)^{Fr^k}$, where $a_k(\lambda) = a(\lambda)$ is the same for all $k > 0$. Here $St \sum a_0(\lambda) s(\lambda)$ is the restriction to $G(p)$ of the projective cover of the $u_n - T$ module with highest weight λ while $St \sum a_k(\lambda) s(\lambda)$ for $k > 1$ is the restriction to $G(p)$ of the projective cover of the trivial $u_n - T$ module. Also, λ'' is small whenever $a_k(\lambda)$ is non-zero. For $k = 0$ we can ensure this by taking λ deep enough in the interior of a $\tilde{W}_{a,p}$ alcove. For $k > 1$, this follows from the fact that λ'' is either conjugate under the affine Weyl group $W_{a,p}$ to ρ, or to $\rho + w\zeta$ where $w \in w$ and ζ has all components in terms of the fundamental weights equal to or less than one (see [1]).

COROLLARY 3.7. *Assume p is large enough and that ν is deep enough in the interior of a $\tilde{W}_{a,p}$ alcove contained in X_p. Let $\Phi_{\nu,n}$ be the character of the projective cover of the corresponding irreducible $G(p^n)_\pi$ module. Write $\Phi_{\nu,n} = St_n \varphi_\nu \prod_{k=1}^{n-1} \varphi_0^{Fr^k}$, where $\varphi_\nu = \sum_{\lambda \in X_p} a_0(\lambda) s(\lambda)$ and $\varphi_0 = \sum_{\lambda \in X_p} a(\lambda) s(\lambda)$. Then the multiplicity of $\chi([\mu_0, z_0]; \cdots ;[\mu_{n-1}, z_{n-1}])$ in $\Phi_{\nu,n}$ is equal to*

$$\operatorname{Tr}\left(M_{a_0}(\mu_0, z_0) \prod_{k=1}^{n-2} M_a(\mu_k, z_k) M_\pi M_a(\mu_{n-1}, z_{n-1}) \right).$$

4. DECOMPOSITION NUMBERS OF $SL_3(p^n)$ AND $SU_3(p^n)$

We will apply the results of the previous section to the groups $SL_3(p^n)$ and $SU_3(p^n)$ for large enough p. We will consider a weight sufficiently deep in the interior of the $\tilde{W}_{a,p}$ alcove $\tilde{C}_p$ of $SL_3(p)$ and determine the decomposition of the projective indecomposable Φ_ν of $SL_3(p^n)$ and $SU_3(p^n)$. Similar computations apply to weights in other $\tilde{W}_{a,p}$ alcoves.

The weight lattice X is isomorphic to $\mathbb{Z}^+ \times \mathbb{Z}^+$. The Weyl group W is isomorphic to S_3 and is generated by σ_1 and σ_2 where these elements act on X via $\sigma_1(r, s) = (-r, r + s)$ and $\sigma_2(r, s) = (r + s, -s)$.

Let $W_{a,p}$ be the ordinary affine Weyl group, that is, the semidirect product of W with translations by elements of p times the root lattice.

There are two $W_{a,p}$-alcoves in X_p. The top alcove consists of (r, s) in X_p with $r + s \geq p - 1$, while the bottom alcove consists of those with $r + s < p - 1$.

If ν is in the bottom $W_{a,p}$ alcove of X_p, then $\varphi_\nu = (s(\nu + \rho) + s(p\rho + \sigma_0(\nu + \rho))\prod_{k=1}^{n-1}(s(\rho) + s(p\rho + \sigma_0 \rho))^{Fr^k}$. Thus we have

$$\varphi_\nu = \prod_k \sum_\lambda a_k(\lambda) s(\lambda)^{Fr^k}$$

where $a_0(\lambda) = 1$ if $\lambda = \nu + \rho$ or $p\rho + \sigma_0(\nu + \rho)$, and is equal to 0 otherwise, and for $k > 1$, $a_k(\lambda) = 1$ if $\lambda = \sigma$ or $p\rho + \sigma_0 \rho$, and is equal to 0 otherwise.

Thus for $k > 1$, if $a_k(\lambda_k)$ is non-zero then $\lambda_k = \lambda'_k + \tau_k \lambda''_k$ with $\lambda'_k = 0$, $\tau_k = 1$, $\lambda''_k = \rho$ or $\lambda'_k = \rho$, $\tau_k = \sigma_0$, $\lambda''_k = \rho$. If $a_0(\lambda_0)$ is non zero then $\lambda_0 = \lambda'_0 + \tau_0 \lambda''_0$ with $\lambda'_0 = 0$, $\tau_0 = 1$, $\lambda'' = \nu + \rho$ or $\lambda'_0 = \rho$, $\tau_0 = \sigma_0$, $\lambda''_0 = \nu + \rho$.

We will now determine the matrices $M_{a_k}(\mu, z)$. We will see that the only entries which contribute to a non-zero trace are those with indices whose W component corresponds to 1, σ_1, or σ_2. The only non-zero rows correspond to indices with X^+ component ρ for $k > 0$ and $\nu + \rho$ for $k = 0$. The only non-zero columns correspond to indices with X^+ component ρ for $0 \leq k < n - 2$ and $\nu + \rho$ for $k = n - 1$. We will only give the portion of $M_{a_k}(\mu, z)$ corresponding to these rows and columns. The order will correspond to 1, σ_1, σ_2.

We first consider $M_{a_0}(\mu, z)$.

Proposition 4.1. *With the above notation,*

$$M_{a_0}(\nu + \rho, w) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$M_{a_0}(\nu + \rho + w\rho, w\sigma_0) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$M_{a_0}(\nu + \rho + w\rho, w\sigma_1\sigma_2) \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$M_{a_0}(\nu + \rho + w\rho, w\sigma_2\sigma_1) \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \text{for all } w \in W.$$

$M_{a_0}(\mu, z)$ *is* 0 *for all other* $\mu \in X^+$, $z \in W$.

Proof. If $M_{a_0}(\mu, z)$ is non-zero then $\mu = y^{-1}(\lambda'' + w\lambda')$ and $z = y^{-1}w\tau v$ for some $w \in W$, where $\lambda'' = \nu + \rho$ and either $\lambda' = 0$, $\tau = 1$, or $\lambda' = \rho$, $\tau = \sigma_0$. We then get an entry in the row corresponding to v and column corresponding to y. Since we are assuming that $\nu + \rho$ is deep enough in $\tilde{C}_p$, y must be the identity, so all columns bu the first are 0. If $\lambda' = 0$ and $\tau = 1$, then $\mu = \nu + \rho$ and $z = wv$. For any v, as w ranges over W, so does z. This gives the values for $M_{a_0}(\nu + \rho, z)$. If $\lambda' = \rho$ and $\tau = \sigma_0$, we have $\mu = \nu + \rho + w\rho$ and $z = w\sigma_0\nu$. Taking $v = 1$, σ_1 and σ_2 give the remaining matrices.

We now consider $M_{a_k}(\mu, z)$ for $k > 0$.

PROPOSITION 4.2. *With the above notation, for $k > 0$ we have*

$$M_{a_k}(2\rho, \sigma_0) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_{a_k}(2\rho, \sigma_1\sigma_2) \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_{a_k}(2\rho, \sigma_2\sigma_1) \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_{a_k}((0,3), \sigma_2\sigma_1) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$M_{a_k}((0,3), \sigma_1\sigma_2) \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_{a_k}((0,3), \sigma_0) \leftrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_{a_k}((0,3), \sigma_2) \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_{a_k}((3,0), \sigma_1\sigma_2) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_{a_k}((3,0),\sigma_2\sigma_1) \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{a_k}((3,0),\sigma_0) \leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_{a_k}((3,0),\sigma_1) \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_{a_k}(\rho,\sigma_1\sigma_2) \leftrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_{a_k}(\rho,\sigma_2\sigma_1) \leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_{a_k}(\rho,\sigma_0) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$M_{a_k}(\rho,\sigma_1) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$M_{a_k}(\rho,\sigma_2) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_{a_k}(\rho,1) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_{a_k}(0,1) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$M_{a_k}(\mu, z)$ is 0 for all other non-zero $\mu \in X^+$, $z \in W$. There are also non-zero $M_{a_k}(0, w)$ for other $w \in W$, whose values we omit.

Proof. If $M_{a_k}(\mu, z)$ is non-zero then

$$\mu = y^{-1}(\lambda'' + w\lambda')$$

and $z = y^{-1}w\tau v$ for some $w \in W$, where $\lambda'' = \rho$ and either $\lambda' = 0$, $\tau = 1$ or $\lambda' = \rho$, $\tau = \sigma_0$. We then get an entry in the column corresponding to y and row corresponding to v.

If $\lambda' = 0$, $\tau = 1$, then $y = 1$, $\mu = \rho$, and $z = wv$. For any v, we get all $z \in W$ as w ranges over W. This gives an entry of 1 in the first column of $M_{a_k}(\rho, z)$ for each $z \in W$.

If $\lambda' = \rho$, $\tau = \sigma_0$ then $\mu = y^{-1}(\rho + w\rho)$ and $z = y^{-1} w \sigma_0 v$. The following table gives the possibilities (blank entries signify the same value as in the previous row);

w	$\rho + w\rho$	y	μ	v	z
1	2ρ	1	2ρ	1	σ_0
				σ_1	$\sigma_1\sigma_2$
				σ_2	$\sigma_2\sigma_1$
σ_1	$(0,3)$	1	$(0,3)$	1	$\sigma_2\sigma_1$
				σ_1	σ_2
				σ_2	σ_0
		σ_1	$(0,3)$	1	σ_0
				σ_1	$\sigma_1\sigma_2$
				σ_2	$\sigma_2\sigma_1$
σ_2	$(3,0)$	1	$(3,0)$	1	$\sigma_1\sigma_2$
				σ_1	σ_0
				σ_2	σ_1
		σ_2	$(3,0)$	1	σ_0
				σ_1	$\sigma_1\sigma_2$
				σ_2	$\sigma_2\sigma_1$
$\sigma_1\sigma_2$	$(-1,2)$	σ_1	ρ	1	$\sigma_1\sigma_2$
				σ_1	σ_0
				σ_2	σ_1
$\sigma_2\sigma_1$	$(2,-1)$	σ_2	ρ	1	$\sigma_2\sigma_1$
				σ_1	σ_2
				σ_2	σ_0
σ_0	0	v^{-1}	0	v	1

In the last row, y can be any element of W, but we only choose to give the one that is the same as v^{-1}, so that z is always the identity.

Putting together all of the above gives the values for $M_{a_k}(\mu, z)$ stated in the proposition.

From the computations in the proof of 4.2, we see that there is a non-zero entry in $M_{a_k}(\mu_k, z_k)$ only in a row corresponding to 1, σ_1, σ_2, except when $(\mu_k, z_k) = (0, 1)$. Since not all components can be of this type, we are indeed justified in ignoring all rows except those corresponding to these three elements.

Proposition 4.3. *If* $\pi \neq 1$, *then*

$$M_\pi \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Proof. We have $1^\pi = 1$. If $(r, s) \in X$,

$$\begin{aligned}\sigma_1^\pi(r,s) &= \pi\sigma_1\pi^{-1}(r,s) = \pi\sigma_1(s,r) = \pi(-s, r+s)\\ &= (r+s, -s) = \sigma_2(r,s),\end{aligned}$$

so $\sigma_1^\pi = \sigma_2$

Similarly $\sigma_2^\pi = \sigma_1$.

We now have by 3.6

THEOREM 4.4. *With notation as above, the multiplicity of* $\chi([\mu_0, z_0], \cdots ;[\mu_{n-1}, z_{n-1}])$ *in* Φ_ν *is equal to*

$$\operatorname{Tr}\left(\prod_{k=0}^{n-2} M_{a_k}(\mu_k, z_k) M_\pi M_{a_0}(\mu_{n-1} z_{n-1})\right),$$

where the M_{a_k} *are given in* 4.1 *and* 4.2 *and*

$$M_\pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{if } \pi \neq 1,$$

while $M_\pi =$ *identity matrix if* $\pi = 1$.

Remark 1. When a Deligne–Lusztig character has a p-adic coding with component $\mu_k = (0,3)$ or $(3,0)$, it will have more than one such coding. This corresponds to the fact that there is a choice of $y = 1$ or σ_1 (resp. $y = 1$ or σ_2) in the proof of 4.2, in these cases. Thus when making an inventory of the characters appearing in Φ_ν, we must take this into account. We can choose a component of type $[(0,3), \sigma_2\sigma_1]$ or $[(0,3), \sigma_0]$, but not both; of type $[(0,3), \sigma_2]$ or $[(0,3), \sigma_1\sigma_2]$, but not both; of type $[(3,0), \sigma_1\sigma_2]$ or $[(3,0), \sigma_0]$, but not both; of type $[(3,0), \sigma_1]$ or $[(3,0), \sigma_2\sigma_1]$, but not both.

Similarly, we get multiple representations of a Deligne–Lusztig character with a component of type $[0, w]$. As noted above, we will always represent such a character with a component of $[0, 1]$.

Remark 2. It is not hard to see that 4.3 holds, not only for $\nu \in \tilde{C}_p$, but for any ν deep enough in the interior of any $\tilde{W}_{a,p}$ alcove contained in the bottom $W_{a,p}$-alcove of X_p. In the definition of the decomposition of $\lambda_0 = \lambda_0' + \tau_0 \lambda_0''$, we can relax the requirement that $\lambda_0'' \in \tilde{C}_p$ and allow it to be in any particular $\tilde{W}_{a,p}$ alcove contained in the bottom $\tilde{W}_{a,p}$-alcove, and all the computations above will be valid.

Similar computations apply to ν in the upper $W_{a,p}$-alcove.

THEOREM 4.5. *Suppose* ν *is in the top* $W_{a,p}$*-alcove of* X_p*, and deep enough in the interior of a* $\tilde{W}_{a,p}$*-alcove. Then the multiplicity of* $\chi([\mu_0, z_0]; \cdots ;[\mu_{n-1}, z_{n-1}])$ *in* Φ_ν *considered as a character of* $G(p^n)_\pi$*, is* $\mathrm{Tr}(\prod_{k=0}^{n-2} M_{a_k}(\mu_k, z_k) M_\pi M_{a_{n-1}}(\mu_{n-1}, z_{n-1}))$*, where* $M_{a_k}(\mu_k, z_k)$ *are as in* 4.1 *and* 4.2 *for* $1 \le k \le n-1$*, while*

$$M_{a_0}(\mu_0, z_0) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

if $\mu_0 = \nu + \rho$ *and z is any element of W, and is* 0 *otherwise.*

5. CARTAN INVARIANTS

We will show that our methods can also be used to calculate Cartan invariants. To do so we need to assume that we have resolved any ambiguities due to the fact that p-adic codings of Deligne–Lusztig characters may not be unique. We shall assume that we have a complete listing of all the irreducible ordinary characters occurring in a projective indecomposable character Φ, and each such character is represented exactly once in the p-adic coding.

We now assume that we have 2 projective indecomposable characters Φ_1 and Φ_2, and that for each k we have a subset $\mathscr{S}_k$ of $X^+ \times W$ such that $\{\chi([\mu_0, z_0], \dots, [\mu_{n-1}, z_{n-1}]) : [\mu_k, z_k] \in \mathscr{S}_k\}$ gives a complete simultaneous listing, without repetition, of the Deligne–Lusztig characters appearing in both Φ_1 and Φ_2. With the assumptions on Φ_1 and Φ_2 in Section 3, the multiplicity of the character $\chi([\mu_0, z_0]; \cdots ;[\mu_{n-1} z_{n-1}])$ in Φ_i for $i = 1, 2$, can be written as $\mathrm{Tr}(\prod_{k=0}^{n-2} M_{a_i, k}(\mu_k, z_k) M_\pi M_{a_{i,n-1}}(\mu_{n-1}, z_{n-1}))$, where $\Phi_i = St_n \prod \Sigma a_{i,k}(\lambda) s(\lambda)^{Fr^k}$. Write $M_{i,k}(\mu, z) = M_{a_{i,k}}(\mu, z)$ for $k = 0, \dots, n-2$ and $M_{i,n-1}(\mu, z) = M_\pi M_{a_{i,n-1}}(\mu, z)$.

Let $c(\Phi_1, \Phi_2)$ be the Cartan invariant corresponding to Φ_1 and Φ_2, i.e., the number of times the irreducible $kG(p^n)_\pi$ character corresponding to Φ_1 occurs in Φ_2. We have the following formula for $c(\Phi_1, \Phi_2)$:

THEOREM 5.1. *With the above notation,* $c(\Phi_1, \Phi_2) = \mathrm{Tr}(\prod_{k=0}^{n-1} \sum_{(\mu, z) \in \mathscr{S}_k} M_{1,k}(\mu, z) \otimes M_{2,k}(\mu, z))$.

Proof. The Cartan invariant is equal to the inner product (Φ_1, Φ_2), so it is given by the sum of products of decomposition numbers. We have

$$\Phi_i = \sum_{\substack{(\mu_0, z_0), \dots, (\mu_{n-1}, z_{n-1}) \\ (\mu_k, z_k) \in \mathscr{S}_k}} \mathrm{Tr}\Big(\prod M_{i,k}(\mu_k, z_k)\Big) \times \chi([\mu_0, z_0]; \cdots ;[\mu_{n-1}, z_{n-1}]).$$

Thus

$$\begin{aligned}
c(\Phi_1, \Phi_2) &= \sum_{\substack{(\mu_0, z_0); \cdots ; (\mu_{n-1}, z_{n-1}) \\ (\mu_k, z_k) \in \mathcal{S}_k}} \operatorname{Tr}\Bigl(\prod_k M_{1,k}(\mu_k, z_k) \Bigr) \\
&\qquad \times \operatorname{Tr}\Bigl(\prod_k M_{2,k}(\mu_k, z_k) \Bigr) \\
&= \operatorname{Tr}\Biggl(\prod_k \sum_{\substack{(\mu, z) \\ \in \mathcal{S}_k}} M_{1,k}(\mu, z) \Biggr) \operatorname{Tr}\Biggl(\prod_k \sum_{\substack{(\mu, z) \\ \in \mathcal{S}_k}} M_{2,k}(\mu, z) \Biggr) \\
&= \operatorname{Tr}\Bigl(\prod_k \sum_{(\mu, z) \in \mathcal{S}_k} M_{1,k}(\mu, z) \otimes M_{2,k}(\mu, z) \Bigr) \qquad \text{as required.}
\end{aligned}$$

We apply this result to compute the generic Cartan invariants $c(\Phi_\nu, \Phi_\nu)$ for $\nu \in X_p$, for $SL_3(p^n)$ and $SU_3(p^n)$.

THEOREM 5.2. *Suppose ν is in the bottom alcove of $SL_3(p)$ and sufficiently deep in the interior of a $\tilde{W}_{a,p}$-alcove. Then the Cartan invariant $c(\Phi_{\nu,n}, \Phi_{\nu,n})$ of $SL_3(p^n)$ or $SU_3(p^n)$ is equal to the coefficient of x^n in the series expansion of $12x(1-4x)/(48x^2 - 18x + 1)$.*

Proof. By 4.1, the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

each occur as $M_{a_0}(\mu, z)$ for 6 pairs (μ, z). Thus

$$\sum_{(\mu, z) \in \mathcal{S}_{n-1}} M_{a_0}(\mu, z) \otimes M_{a_0}(\mu, z) = \begin{pmatrix} 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 6 & & & & & & & & \\ 6 & & & & & & & & \\ 6 & & & & & & & & \\ 12 & & & & & & & & \\ 6 & & & & & & & & \\ 6 & & & & & & & & \\ 6 & & & & & & & & \\ 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

where omitted entries in the 9×9 matrix are 0. Call this matrix $M^{(2)}_{n-1}$. By 4.2, the following 12 matrices occur as $M_{a_k}(\mu, z)$ for $1 \leq k \leq n-1$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The first 10 occur once each while the last 2 each occur for two $M(\mu, z)$. As explained before, when $\mu = (0, 3)$ or $(3, 0)$ we must make a choice for z. Here we are choosing the pairs $[(0,3), \sigma_2\sigma_1]$, $[(0,3), \sigma_2]$, $[(3,0), \sigma_1\sigma_2]$, $[(3,0), \sigma_1]$, and omitting $[(0,3), \sigma_0]$, $[(0,3), \sigma_1\sigma_2]$, $[(3,0), \sigma_0]$, and $[(3,0), \sigma_2\sigma_1]$.

We also take $[0,1]$ but omit $[0, w]$ for $w \neq 1$. This gives for $1 \leq k \leq n-1$,

$$\sum_{(\mu, z) \in \mathscr{S}_k} M_{a_k}(\mu, z) \otimes M_{a_k}(\mu, z) = \begin{pmatrix} 10 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 6 & 2 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 6 & 2 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 6 & 1 & 1 & 2 & 0 & 0 & 2 & 0 & 0 \\ 8 & 1 & 1 & 1 & 2 & 0 & 1 & 0 & 2 \\ 6 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & 0 \\ 6 & 1 & 1 & 2 & 0 & 0 & 2 & 0 & 0 \\ 6 & 1 & 1 & 1 & 0 & 0 & 1 & 2 & 0 \\ 8 & 1 & 1 & 2 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

Call this matrix $M^{(2)}$.

First consider the case when π is the identity. By 5.1, $c(\Phi_{\nu,n}, \Phi_{\nu,n}) = \mathrm{Tr}(M^{(2)}_0 (M^{(2)})^{n-1})$. A computation using Mathematica shows that this trace is equal to $(2/11)[(33 + 5\sqrt{33})\alpha^{n-1} + (33 - 5\sqrt{33})\beta^{n-1}]$ where $\alpha = 9 + \sqrt{33}$ and $\beta = 9 - \sqrt{33}$ (see the Appendix). An elementary computation gives the generating function stated in the theorem.

Now consider the case of $SU_3(p^n)$. Then

$$c(\Phi_{\nu,n}, \Phi_{\nu,n}) = Tr\Big(M^{(2)}_0 (M^{(2)})^{n-2} M^{(2)}_\pi M^{(2)}\Big),$$

where $M^{(2)}_\pi$ denotes $M_\pi \otimes M_\pi$. Let S be the set of 12 matrices given above as occurring as $M_{a_k}(\mu, z)$, and let S_0 be the set of four matrices occurring as $M_{a_0}(\mu, z)$. First note that multiplication of the elements of S

on the left by M_π gives the same set of matrices as multiplying S on the right by M_π. Next note that multiplying on the left by M_π permutes S_0. Thus we get $c(\Phi_{\nu,n}, \Phi_{\nu,n}) = \mathrm{Tr}(M_0^{(2)}(M^{(2)})^{n-2}M_\pi^{(2)}M^{(2)}) = \mathrm{Tr}(M_0^{(2)}(M^{(2)})^{n-1}M_\pi^{(2)}) = \mathrm{Tr}(M_\pi^{(2)}M_0^{(2)}(M^{(2)})^{n-1}) = \mathrm{Tr}(M_0^{(2)}(M^{(2)})^{n-1})$. Thus the Cartan invariant is the same as for $SL_3(p^n)$.

THEOREM 5.3. *Suppose that ν is in the top $W_{a,p}$-alcove of $SL_3(p)$ and sufficiently deep in the interior of a $\tilde{W}_{a,p}$-alcove. The Cartan invariant $c(\Phi_{\nu,n}, \Phi_{\nu,n})$ of $SL_3(p^n)$ or $SU_3(p^n)$ is equal to the coefficient of x^n in the series expansion of* $6x(1-2x)/(48x^2-18x+1)$.

Proof. The matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

occurs 6 times as $M_{a_0}(\mu, z)$. Thus $\sum_{(\mu,z)\in\mathscr{S}_0} M_{a_0}(\mu,z) \otimes M_{a_0}(\mu,z)$ is the 9×9 matrix with entries of 6 in the first column, and all other entries 0. A similar Mathematica computation as in 5.2 gives the trace equal to $\frac{1}{11}[(33+7\sqrt{33})\alpha^{n-1} + (33-7\sqrt{33})\beta^{n-1}]$. This gives the result.

6. CONCLUSION

It is possible to get much more information about the decomposition numbers using the description we have given. In particular, we can give a formula for the number of times a given integer k occurs as a decomposition number for a projective indecomposable Φ. We will give the details elsewhere.

APPENDIX

We give here the Mathematica calculation for $\mathrm{Tr}(M_0^{(2)}(M^{(2)})^{n-1})$ cited above. It is equal to $\sum_{j=1}^{9} b_j \lambda_i^n (v_i w_i^T)_{i,j}$ where $(b_1, \dots, b_9)^T$ is the first column of $M_0^{(2)}$; v_i is the ith eigenvector of $M^{(2)}$, written as a column vector; λ_i is the ith eigenvalue of $M^{(2)}$; and $[w_1, \dots, w_9]$ is the inverse of the matrix $[v, \dots, v_9]$.

```
r1 := {10,1,1,1,1,0,1,0,1}
r2 := {6,2,2,1,0,0,1,0,0}

r3 := {6,2,2,1,0,0,1,0,0}
r4 := {6,1,1,2,0,0,2,0,0}
```

```
r5 := {8, 1, 1, 1, 2, 0, 1, 0, 2}
r6 := {6, 1, 1, 1, 0, 2, 1, 0, 0}
r7 := r3
r8 := {6, 1, 1, 1, 0, 0, 1, 2, 0}
r9 := {8, 1, 1, 1, 2, 0, 1, 0, 2}
m := {r1, r2, r3, r4, r5, r6, r7, r8, r9}
c := Eigenvectors[m]
d := Inverse[Transpose[c]]
b := {12, 6, 6, 6, 12, 6, 6, 6, 12}
Simplify[Sum[b[[j]] * (((Eigenvalues[m])[[i]])^n)
    ((Transpose[{c[[i]]}].{d[[i]]})[[1, j]]), {i, 1, 9}, {j, 1, 9}]]
```

$$6(9-\sqrt{33})^n - 10\sqrt{\frac{3}{11}}(9-\sqrt{33})^n + 6(9+\sqrt{33})^n + 10\sqrt{\frac{3}{11}}(9+\sqrt{33})^n$$

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