

Decomposition Numbers of $SL_3(p^n)$ and $SU_3(p^n)$

Leonard Chastkofsky

Department of Mathematics, University of Georgia, Athens, Georgia 30602

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INTRODUCTION

In Decomposition numbers of Chevalley groups, *J. Algebra* **240** (2001), 589–607, we gave a technique for determining certain characteristic p decomposition numbers of a Chevalley group $G(p^n)$. We showed that in certain cases, the multiplicities with which an irreducible character occurs in a projective indecomposable character $\Phi_{\lambda, n}$ can be determined by computing the trace of the product of certain matrices. In this paper we apply the results to the groups $SL_3(p^n)$ and $SU_3(p^n)$ to get explicit formulas for the multiplicities of decomposition numbers. We will show that for λ a weight deep enough in the interior of a p -restricted alcove, the number of times a given integer k occurs as a decomposition number in the column corresponding to $\Phi_{\lambda, n}$ has a generating function which is a combination of a rational function and a Dirichlet series. This result gives us an alternative approach to determining the Cartan invariants.

1. SUMMARY OF PREVIOUS RESULTS AND STATEMENT OF MAIN THEOREM

We review the results obtained in [1], as applied to $SL_3(p^n)$ and $SU_3(p^n)$.

Let Γ_n be either the group $SL_3(p^n)$ or $SU_3(p^n)$. Let X be the weight lattice for Γ_n . It is isomorphic to $\mathbb{Z}^+ \times \mathbb{Z}^+$. Let X_{p^n} be the subset of p^n -restricted weights, consisting of those elements of X whose coordinates in terms of the fundamental dominant weights lie between 0 and $p^n - 1$. The projective indecomposable characters are indexed by weights λ in X_{p^n} . Let $\Phi_{\lambda, n}$ be the projective indecomposable character indexed by λ .

Let W be the Weyl group. Let $W_{a, p}$ be the affine Weyl group, that is, the semidirect product of W with translations by elements of p times the



root lattice. Let $\tilde{W}_{a,p}$ be the extended affine Weyl group, that is, the semidirect product of W with translations by elements of p times the weight lattice. There are two $W_{a,p}$ -alcoves in X_p . The top alcove consists of (r, s) in X_p with $r + s \geq p - 1$, while the bottom alcove consists of those with $r + s < p - 1$. The results on decomposition numbers of $\Phi_{\lambda,n}$ in [1] assume that λ is deep enough in a $\tilde{W}_{a,p}$ alcove. In this paper we shall always assume that λ satisfies this condition.

Let \mathcal{S}_3 be the following set of 12 matrices:

$$\mathcal{S}_3 = \left\{ \begin{aligned} &\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \\ &\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ &\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{aligned} \right\}.$$

From [1], we have the following:

THEOREM 1.1. *There is a coding of Deligne–Lusztig characters of Γ_n by $(X_p \times W)^n$. Denote the character coded by $(\alpha_1, \dots, \alpha_n)$, $\alpha_i \in X_p \times W$ by $\chi(\alpha_1, \dots, \alpha_n)$. For each i with $1 \leq i \leq n$ there is a subset \mathcal{A}_i of $X_p \times W$ such that the set $\{\chi(\alpha_1, \dots, \alpha_n) \mid \alpha_i \in \mathcal{A}_i\}$ gives a full set of irreducible characters, without repetition, which appear in $\Phi_{\lambda,n}$. For each i with $1 \leq i \leq n$, there is a map \mathcal{M}_i from $X_p \times W$ to \mathcal{S}_3 such that the multiplicity of the irreducible character $\chi(\alpha_1, \dots, \alpha_n)$ in $\Phi_{\lambda,n}$ is equal to $\text{Tr}(\prod_{i=1}^n \mathcal{M}_i(\alpha_i))$. Moreover,*

- (1) *If $2 \leq i \leq n$ and*

$$M_i \in \mathcal{S}_3, \quad M_i \neq \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

then there is exactly one $\alpha_i \in \mathcal{A}_i$ with $\mathcal{M}_i(\alpha_i) = M_i$.

- (2) *If $2 \leq i \leq n$ and*

$$M_i \in \mathcal{S}_3, \quad M_i = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

then there are exactly two $\alpha_i \in \mathcal{A}_i$ with $\mathcal{M}_i(\alpha_i) = M_i$.

(3) If λ is in the top alcove of X_p , the only matrix which can occur as $\mathcal{M}_1(\alpha_1)$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

which occurs for exactly six such α_1 .

(4) If λ is in the bottom alcove of X_p , the only matrices which can occur as $\mathcal{M}_1(\alpha_1)$ are

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

each of which occur for exactly six such α_1 .

Remark. In the case of $SU_3(p^n)$, there is actually another “twisting” matrix,

$$M_\pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

which occurs as a factor in the product. However, as the argument in [1, end of 5.2] shows, adding this matrix as a factor merely permutes the set of matrices occurring for $SL_3(p^n)$. Thus the theorem is also true for $SU_3(p^n)$, as stated.

For a sequence $(M_1, \dots, M_n) \in (\mathcal{S}_3)^n$ define $\beta(M_1, \dots, M_n)$ to be the number of M_i , $2 \leq i \leq n$, which are equal to

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

COROLLARY 1.2. (1) Suppose that λ is in the top alcove of X_p . The number of times which the integer k occurs as a decomposition number in $\Phi_{\lambda, n}$ is equal to 6 times $\sum 2^{\beta(M_1, \dots, M_n)}$, where the sum is over sequences $(M_1, \dots, M_n) \in (\mathcal{S}_3)^n$ with

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and with $\text{Tr}(\prod M_i) = k$.

(2) Suppose that λ is in the bottom alcove of X_p . The number of times which the integer k occurs as a decomposition number in $\Phi_{\lambda, n}$ is equal to 6 times $\sum 2^{\beta(M_1, \dots, M_n)}$, where the sum is over sequences $(M_1, \dots, M_n) \in (\mathcal{S}_3)^n$

with

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

or

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and with $Tr(\prod M_i) = k$.

Define $\tilde{d}_n^1(k)$ to be the number of times that k occurs as a decomposition number in $\Phi_{\lambda, n}$, when λ is in the top alcove, and $\tilde{d}_n^2(k)$ to be the number of times that k occurs as a decomposition number in $\Phi_{\lambda, n}$, when λ is in the bottom alcove. For $i = 1, 2$, define

$$\tilde{g}_i(x, s) = \sum_{k, n} \tilde{d}_n^i(k) x^n k^{-s}.$$

If a and b are positive integers with $a \geq b$, let $l(a, b)$ be the number of steps needed to find the greatest common divisor of a and b by the subtractive algorithm. For $k \geq 1$ define a polynomial $f_k(x)$ as

$$f_k(x) = \sum_{\substack{1 \leq a \leq k \\ (k, a) = 1}} x^{l(k, a)}.$$

Define $\xi(x, s)$ to be

$$\sum_{k \geq 2} f_k(x) k^{-s}.$$

These functions will be discussed in more detail in Section 3.

MAIN THEOREM. For $i = 1, 2$,

$$\tilde{g}_i(x, s) = g_i\left(\frac{2x}{1 - 2x}, s\right),$$

where

$$g_1(x, s) = \frac{(1 - x)^2(1 - 2x)}{(1 + x)(1 - 4x + x^2 - (x + x^2)\xi(x, s))} - \frac{1 - x}{1 + x}$$

while

$$g_2(x, s) = \frac{(1 - x)^2}{1 - 4x + x^2 - (x + x^2)\xi(x, s)} - 1.$$

2. REDUCTION OF RANK

We have seen that the problem of determining how many times an integer k occurs as a decomposition number in $\Phi_{\lambda, n}$ is equivalent to determining how many times it occurs as the trace of a product of certain 3×3 matrices. We will now show that the problem can be reduced further to determining how many times it occurs as the trace of a product of certain 2×2 matrices, belonging to a smaller set.

Let $\mathcal{S}_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$.

Let $\tilde{\mathcal{S}}_2 = \mathcal{S}_2 \cup \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We partition \mathcal{S}_3 into 4 subsets $\mathcal{S}_\emptyset, \mathcal{S}_{\sigma_1}, \mathcal{S}_{\sigma_2}$, and $\mathcal{S}_{\sigma_1, \sigma_2}$, as

$$\mathcal{S}_\emptyset = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}$$

$$\mathcal{S}_{\sigma_1} = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

$$\mathcal{S}_{\sigma_2} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$\mathcal{S}_{\sigma_1, \sigma_2} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

We consider the matrices in \mathcal{S}_3 as being indexed by the elements 1, σ_1 , σ_2 of the Weyl group. Note that in the columns corresponding to σ_1 and σ_2 , there is at most one non-zero entry. A matrix is placed in \mathcal{S}_{σ_1} if there is a 1 in the σ_1 column, but not in the σ_2 column. A matrix is placed in \mathcal{S}_{σ_2} if there is a 1 in the σ_2 column, but not in the σ_1 column. It is placed in $\mathcal{S}_{\sigma_1, \sigma_2}$ if there is a 1 in both columns, and in \mathcal{S}_\emptyset if there is a 1 in neither.

We will now define a map $\theta: \mathcal{S}_3 \rightarrow \mathcal{S}_2$. Consider $(M_1, \dots, M_n) \in \mathcal{S}_3^n$. Assign to this element the element $\theta(M_1, \dots, M_n) = (M'_1, \dots, M'_n) \in \mathcal{S}_2^n$ as follows.

We consider the matrices in \mathcal{S}_2 as being indexed by 1 and the set $\{\sigma_1, \sigma_2\}$.

If there is a 1 (resp. 0) in the 1-1 entry of M_i , there is a 1 (resp. 0) in the 1-1 entry of M'_i .

If there are ones (resp. zeros) in *both* the σ_1 -1 and σ_2 -1 entries of M_i , there is a 1 (resp. 0) in the $\{\sigma_1, \sigma_2\}$ -1 entry of M'_i .

If there is a 1 in *either* the 1- σ_1 or the 1- σ_2 entry of M_i , there is a 1 in the 1- $\{\sigma_1, \sigma_2\}$ entry of M'_i . Otherwise there is a 0 there.

If there are ones in the $\sigma_1\text{-}\sigma_1$ and $\sigma_2\text{-}\sigma_2$ entries of M_i , there is a 1 in the $\{\sigma_1, \sigma_2\}\text{-}\{\sigma_1, \sigma_2\}$ entry of M'_i .

If there are zeros in all the $\sigma_k\text{-}\sigma_l$ entries of M_i , $k, l = 1, 2$, there is a 0 in the $\{\sigma_1, \sigma_2\}\text{-}\{\sigma_1, \sigma_2\}$ entry of M'_i .

This assigns M'_i to 6 of the matrices in \mathcal{S}_3 . Thus we have:

$$\text{If } M_i = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ then } M'_i = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

$$\text{If } M_i = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \text{ then } M'_i = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

$$\text{If } M_i = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ then } M'_i = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$\text{If } M_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ then } M'_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\text{If } M_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ then } M'_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The assignment for each of the remaining 6 matrices depends on its position relative to the other M_j . We consider the M_i as being arranged on a circle, with nodes numbered 1 to n .

Suppose that there is a 1 (resp. 0) in the $\sigma_1\text{-}1$ entry and a 0 (resp. 1) in the $\sigma_2\text{-}1$ entry of M_i . Consider the first M_j preceding M_i on the circle which is not in $\mathcal{S}_{\sigma_1, \sigma_2}$; if M_j is in \mathcal{S}_{σ_1} , then there is a 1 (resp. 0) in the $\{\sigma_1, \sigma_2\}\text{-}1$ entry of M'_i ; if it is in \mathcal{S}_{σ_2} , then there is a 0 (resp. 1) in that entry of M'_i ; if it is in \mathcal{S}_{\emptyset} , then we assign the $\{\sigma_1, \sigma_2\}\text{-}1$ entry of M'_i arbitrarily; but we do so in such a way that in half the cases the assignments are equal to 1, while in half they are equal to 0. (This is not really essential, because the trace of the product in this case will be 0, but we do this for aesthetic reasons.) Note that there always is an M_j preceding M_i on the circle which is not in $\mathcal{S}_{\sigma_1, \sigma_2}$, even though we may need to go all the way around the circle to reach M_i itself.

Suppose that there is a 1 in the $\sigma_k\text{-}\sigma_l$ position, with $k = 1, 2$ and $k \neq l$. There is at most one such pair (k, l) for these 6 matrices. If the first M_j preceding M_i which is not in $\mathcal{S}_{\sigma_1, \sigma_2}$ is in \mathcal{S}_{σ_k} , then there is a 1 in the $\{\sigma_1, \sigma_2\}\text{-}\{\sigma_1, \sigma_2\}$ entry of M'_i ; if it is in \mathcal{S}_{σ_l} , then there is a 0 in that entry; if it is in \mathcal{S}_{\emptyset} , then we assign the $\{\sigma_1, \sigma_2\}\text{-}\{\sigma_1, \sigma_2\}$ entry arbitrarily in such a way that half the assignments are 1, and half are 0.

To summarize, we have the following assignments, where for each choice of assignment, half the assignments are to the first matrix, and half to the second:

$$\text{If } M_i = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \text{ then } M'_i = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

$$\text{If } M_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ then } M'_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\text{If } M_i = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ then } M'_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We will show that this assignment preserves traces of products. To show this we will use graphs associated with the sets \mathcal{S}_2 and \mathcal{S}_3 .

Suppose that \mathcal{S} is a set of 0-1 matrices each of which is indexed by a set V . We associate with \mathcal{S} a colored directed graph $\mathcal{G}(\mathcal{S})$ as follows. We assume that there is a color $\mathcal{E}(M)$ for each $M \in \mathcal{S}$. The vertices of $\mathcal{G}(\mathcal{S})$ are the elements of V . There is an edge of color $\mathcal{E}(M)$ from v to w if and only if the $v - w$ entry of $M \in \mathcal{S}$ is 1.

It is easy to see that $\text{Tr}(\prod M_i)$, $M_i \in \mathcal{S}$, is equal to the number of closed paths in $\mathcal{G}(\mathcal{S})$ with successive edges of colors $\mathcal{E}(M_1)\mathcal{E}(M_2)\cdots\mathcal{E}(M_n)$.

PROPOSITION 2.1. *Let $(M_1, \dots, M_n) \in \mathcal{S}_3^n$ and let $\theta(M_1, \dots, M_n) = (M'_1, \dots, M'_n) \in \mathcal{S}_2^n$ be defined as above. Assume that not all M_i are in $\mathcal{S}_{\sigma_1, \sigma_2}$. Then $\text{Tr}(\prod M_i) = \text{Tr}(\prod M'_i)$.*

Proof. We will show that the number of closed paths in $\mathcal{G}(\mathcal{S}_3)$ with successive edges of colors $\mathcal{E}(M_1), \mathcal{E}(M_2), \dots, \mathcal{E}(M_n)$ is equal to the number of closed paths in $\mathcal{G}(\mathcal{S}_2)$ with successive edges of colors $\mathcal{E}(M'_1), \mathcal{E}(M'_2), \dots, \mathcal{E}(M'_n)$. Let E_1, E_2, \dots, E_n be edges in $\mathcal{G}(\mathcal{S}_3)$, E_i of color $\mathcal{E}(M_i)$, which form a closed path in $\mathcal{G}(\mathcal{S}_3)$. We will associate to each E_i an edge E'_i of color $\mathcal{E}(M'_i)$ in $\mathcal{G}(\mathcal{S}_2)$ such that E'_1, E'_2, \dots, E'_n form a closed path in $\mathcal{G}(\mathcal{S}_2)$.

If $E_i: 1 \rightarrow 1$ is an edge of color $\mathcal{E}(M_i)$, then there is an edge $E'_i: 1 \rightarrow 1$ of color $\mathcal{E}(M'_i)$.

There is at most one edge $E_i: 1 \rightarrow \sigma_1$ or $E_i: 1 \rightarrow \sigma_2$ of color $\mathcal{E}(M_i)$. If there is such an edge, then there is an edge $E'_i: 1 \rightarrow \{\sigma_1, \sigma_2\}$ of color $\mathcal{E}(M'_i)$.

Suppose that there is an edge $E_i: \sigma_1 \rightarrow 1$ (resp. $E_i: \sigma_2 \rightarrow 1$) of color $\mathcal{E}(M_i)$. If such an edge is part of a closed path in $\mathcal{G}(\mathcal{S}_3)$, then the first M_j preceding M_i on the circle which is not in $\mathcal{S}_{\sigma_1, \sigma_2}$ is in \mathcal{S}_{σ_1} (resp. \mathcal{S}_{σ_2}). Thus there is at most one such edge, and if there is one, there is also an edge $E'_i: \{\sigma_1, \sigma_2\} \rightarrow 1$ of color $\mathcal{E}(M'_i)$.

There is an edge $E_i: \sigma_1 \rightarrow \sigma_1$ of color $\mathcal{E}(M_i)$ if and only if there is an edge $E_i: \sigma_2 \rightarrow \sigma_2$ of color $\mathcal{E}(M_i)$. Suppose there are such edges and suppose that the first M_j preceding M_i on the circle which is not in $\mathcal{S}_{\sigma_1, \sigma_2}$ is in \mathcal{S}_{σ_k} . Then the only one of the 2 edges which can be part of a closed path is the $E_i: \sigma_k \rightarrow \sigma_k$. There is a corresponding edge $E'_i: \{\sigma_1, \sigma_2\} \rightarrow \{\sigma_1, \sigma_2\}$ of color $\mathcal{E}(M'_i)$.

There is at most one edge $E_i: \sigma_k \rightarrow \sigma_l$ with $k = 1, 2$ and $k \neq l$ of color $\mathcal{E}(M_i)$. If there is such an edge, then the first M_j preceding M_i on the circle which is not in $\mathcal{S}_{\sigma_1, \sigma_2}$ is in \mathcal{S}_{σ_k} . There is then also an edge $E'_i: \{\sigma_1, \sigma_2\} \rightarrow \{\sigma_1, \sigma_2\}$ of color $\mathcal{E}(M'_i)$.

Conversely, it is clear that every edge E' in $\mathcal{E}(\mathcal{S}_2)$ which is part of a closed path arises in exactly one way from an edge E_i in the way just described. This proves the proposition.

COROLLARY 2.2. (1) *Suppose that λ is in the top alcove of X_p . The number of times which the integer k occurs as a decomposition number in $\Phi_{\lambda, n}$ is equal to 3×2^n times the number of sequences $(M'_1, \dots, M'_n) \in (\mathcal{S}_2)^n$ with $M'_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and with $Tr(\prod M'_i) = k$.*

(2) *Suppose that λ is in the bottom alcove of X_p . The number of times which the integer k occurs as a decomposition number in $\Phi_{\lambda, n}$ is equal to 3×2^n times the number of sequences $(M'_1, \dots, M'_n) \in (\mathcal{S}_2)^n$ with $M'_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$ or $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and with $Tr(\prod M'_i) = k$.*

Proof. (1) By 1.2, the number of times which the integer k occurs as a decomposition number in $\Phi_{\lambda, n}$ is equal to 6 times $\sum 2^{\beta(M_1, \dots, M_n)}$, where the sum is over sequences $(M_1, \dots, M_n) \in (\mathcal{S}_3)^n$ with

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and with $Tr(\prod M_i) = k$. Let $\beta(M'_1, \dots, M'_n)$ be the number of $M'_i, 2 \leq i \leq n$, which are equal to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Note that $\beta(M_1, \dots, M_n) = \beta\theta(M_1, \dots, M_n)$. For each sequence $(M'_1, \dots, M'_n) \in (\mathcal{S}_2)^n$ with $M'_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, there are $2^{n-1-\beta(M'_1, \dots, M'_n)}$ sequences $(M_1, \dots, M_n) \in (\mathcal{S}_3)^n$ with

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and $\theta(M_1, \dots, M_n) = (M'_1, \dots, M'_n)$. Thus the number of times which k occurs is equal to the sum over $(M'_1, \dots, M'_n) \in (\mathcal{S}_2)^n$ with $M'_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ of $6 \times 2^{n-1-\beta(M'_1, \dots, M'_n)} \times 2^{\beta(M'_1, \dots, M'_n)} = 3 \times 2^n$, as claimed.

(2) This is proved similarly.

This gives us the following result:

PROPOSITION 2.3. *The largest decomposition number occurring in $\Phi_{\lambda, n}$ is F_{n+1} , where F_n is the n th Fibonacci number ($F_1 = F_2 = 1$), and this occurs as a decomposition number 3×2^n times.*

Proof. The largest trace which can occur as $\text{Tr}(\prod M_i)$ with $(M_1, \dots, M_n) \in (\mathcal{S}_2)^n$ and $M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is when $M_i = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ for $2 \leq i \leq n$. The product is equal to $\begin{pmatrix} F_n & F_{n-1} \\ F_n & F_{n-1} \end{pmatrix}$, whose trace is equal to F_{n+1} .

3. SUBTRACTIVE LENGTH POLYNOMIALS AND DIRICHLET SERIES

The problem of determining the decomposition numbers occurring in $\Phi_{\lambda, n}$ has been reduced to determining the integers which occur as traces of a product of n matrices for \mathcal{S}_2 . We will obtain recursions for the number of times that an integer k occurs as a product of n such matrices. The solution to these recursions involves certain polynomials which we now define.

If a and b are positive integers with $a \geq b$, let $l(a, b)$ be the number of steps needed to find the greatest common divisor of a and b by the subtractive algorithm. Recall that the subtractive algorithm proceeds from the observation that the g.c.d. of a and b is the same as the g.c.d. of b and $a - b$. Thus $l(a, b)$ can be defined inductively as

$$(3.1) \quad l(a, b) = \begin{cases} 0, & \text{if } a = b \\ l(a - b, b) + 1, & \text{if } a - b \geq b \\ l(b, a - b) + 1, & \text{if } a - b < b. \end{cases}$$

The subtractive length in this definition is 1 less than in the more common one, where the algorithm proceeds one more step until the pair $(1, 0)$ is reached. It is also 1 less than the sum of the partial quotients in the continued fraction representation of a/b .

Note that we have not defined $l(a, b)$ for $a < b$. We leave $l(a, b)$ undefined in this case. Thus, for example, when we have a summation wherein $l(a, b)$ makes an appearance, it is understood that the summation is over (a, b) with $a \geq b$.

LEMMA 3.2. *If a and b are positive integers with $a > b$ then $l(a, b) = l(a, a - b)$.*

Proof. By (3.1), $l(a, a - b)$ is equal to either $1 + l(a - b, b)$ or $1 + l(b, a - b)$, which is the same as $l(a, b)$.

For $k \geq 1$ define a polynomial $f_k(x)$ as

$$f_k(x) = \sum_{\substack{1 \leq a \leq k \\ (k, a) = 1}} x^{l(k, a)}.$$

Here we are writing (k, a) for the g.c.d. of k and a .

EXAMPLES. We give the values of $f_k(x)$ for $1 \leq k \leq 8$,

$$\begin{aligned} f_1(x) &= 1 \\ f_2(x) &= x \\ f_3(x) &= 2x^2 \\ f_4(x) &= 2x^3 \\ f_5(x) &= 2x^3 + 2x^4 \\ f_6(x) &= 2x^5 \\ f_7(x) &= 4x^4 + 2x^6 \\ f_8(x) &= 2x^4 + 2x^7. \end{aligned}$$

We shall need the following later:

LEMMA 3.3. *For any d dividing k we have*

$$f_{k/d}(x) = \sum_{\substack{d \leq a \leq k \\ (k, a) = d}} x^{l(k, a)}.$$

Proof. By definition,

$$f_{k/d}(x) = \sum_{\substack{1 \leq a' \leq k/d \\ (k/d, a') = 1}} x^{l(k/d, a')}.$$

But $(k, a) = d$ if and only if $(k/d, a/d) = 1$, and $l(k, a) = l(k/d, a/d)$. We thus get the formula stated.

LEMMA 3.4. *For any d dividing k , $d \neq k/2, k$, we have*

$$f_{k/d}(x) = 2x \sum_{\substack{a+b=k \\ (a, b)=d}} x^{l(a, b)}.$$

Proof. If $(k, a) = d$, then $(k, k - a) = d$ and $l(k, a) = l(k, k - a)$. Since d is not equal to either $k/2$ or k , $k - a$ is not equal to either 0 or a . Thus

$$\begin{aligned} f_{k/d}(x) &= \sum_{\substack{d \leq a \leq k \\ (k, a) = d}} x^{l(k, a)} \\ &= 2 \sum_{\substack{k/2 < a < k \\ (k, a) = d}} x^{l(k, a)} \\ &= 2 \sum_{\substack{k/2 < a < k \\ (k, a) = d}} x^{l(a, k-a)+1} \\ &= 2x \sum_{\substack{a+b=k \\ (a, b)=d}} x^{l(a, b)}. \end{aligned}$$

Define $\xi(x, s)$ to be

$$\sum f_k(x) k^{-s},$$

the sum being over all $k \geq 2$.

PROPOSITION 3.5. *We have*

- (1) $\xi(x, 0) = 1/(1 - 2x)$,
- (2) $\xi(x, -1) = 2x/(1 - 3x)$,
- (3) $\xi(x, -2) = 2x(2 - x)/(1 - 5x + 2x^2)$.

Proof. The number of coprime pairs (k, a) with $k > a$ and $l(k, a) = n$ is 2^{n-1} (see [3, Corollary to Theorem 1]), from which (1) follows. The sum of (k, a) over k, a with $k > a$ and $l(k, a) = n$ is $(2 \times 3^{n-1}, 2 \times 3^{n-2})$ (see [3, Theorem 2]), from which (2) follows.

Let $\xi(x, -2) = \sum c_n x^n$. We claim that for $n \geq 1$, $c_{n+2} = 5c_{n+1} - 2c_n$. To see this, note that $c_n = \sum k^2$, the sum being over all pairs of positive integers (k, a) with k and a coprime, $k > a$, and $l(k, a) = n$. If (k, a) is such a pair, so is $(k, k - a)$. These 2 pairs give rise to 4 pairs, $(k + a, a)$, $(k + a, k)$, $(2k - a, k - a)$, $(2k - a, k)$, each of length $n + 1$, and 8 pairs, $(k + 2a, a)$, $(k + 2a, k + a)$, $(2k + a, k)$, $(2k + a, k + a)$, $(3k - 2a, k - a)$, $(3k - 2a, 2k - a)$, $(3k - a, k)$, $(3k - a, 2k - a)$, each of length $n + 2$. Thus $c_{n+1} = \sum (k + a)^2 + (2k - a)^2 = \sum 5k^2 - 2ka + 2a^2$, the sum over (k, a) with the same conditions as above; while $c_{n+2} = \sum (k + 2a)^2 + (2k - a)^2 + (3k - a)^2 = \sum 23k^2 - 10ka + 10a^2$, with the sum over (k, a) also with the same conditions. Thus $c_{n+2} = 5c_{n+1} - 2c_n$ as claimed.

This shows that the coefficient of x^n in $(1 - 5x + 2x^2)\xi(x, -2)$ is 0 for $n \geq 3$. Calculation then shows that $(1 - 5x + 2x^2)\xi(x, -2) = 4x - 2x^2$, as claimed.

4. RECURSIONS

Throughout this section a, b, c, d are assumed to be non-negative integers. If n is an integer define $d_n \binom{a \ b}{c \ d}$ to be the number of sequences M_1, M_2, \dots, M_n with $M_i \in \mathcal{S}_2$ and $\prod M_i = \binom{a \ b}{c \ d}$. If $n \leq 0$ then this is defined to be 0. Define $\mathcal{P}_{n-1}(\mathcal{S}_2)$ to be the set of 2×2 matrices which can be written as a product $\prod_{i=1}^n M_i$ with $M_i \in \mathcal{S}_2$.

We record the effect of multiplying a matrix by an element of \mathcal{S}_2 :

LEMMA 4.1.

$$(1) \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

$$(1') \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$$

$$(2) \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$$

$$(2') \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix}$$

$$(3) \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$$

$$(3') \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a + b & 0 \\ c + d & 0 \end{pmatrix}$$

$$(4) \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ a + c & b + d \end{pmatrix}$$

$$(4') \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a + b & b \\ c + d & d \end{pmatrix}$$

$$(5) \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + c & b + d \\ a & b \end{pmatrix}$$

$$(5') \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a + b & a \\ c + d & c \end{pmatrix}.$$

LEMMA 4.2. *Suppose that $d_n\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-zero. Then $a \geq b$ and $c \geq d$.*

Proof. This follows by induction using 4.1(1)–(5).

LEMMA 4.3. *Assume that $n \geq 1$. Suppose that $d_n\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ is non-zero with $a \neq 0$ and $d \neq 0$. Then $a = d = 1$. We have*

$$d_n\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{cases} 1, & \text{if } c = n \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This is certainly true for $n = 1$. Assume that $n \geq 2$. Write $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} = M_1 \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, with $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathcal{P}_{n-1}(\mathcal{S}_2)$, and $M_1 \in \mathcal{S}_2$. By 4.1, we must have $M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $a' = a$, $b' = 0$, and $d' = d$. By induction, $a' = d' = 1$, $c' = n - 1$, and $d_{n-1}\begin{pmatrix} 1 & 0 \\ c' & 1 \end{pmatrix} = 1$. Thus $c = c' + 1 = n$ and $d_n\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = 1$.

LEMMA 4.4. $d_n\begin{pmatrix} a & b \\ c & d \end{pmatrix} = d_n\begin{pmatrix} c & d \\ a & b \end{pmatrix}$ for all n, a, b, c, d .

Proof. This follows from the fact that the set \mathcal{S}_2 is left unchanged when the rows of all its matrices are interchanged.

Let D be the operator acting on a sequence $\{a_n : n \in \mathbb{Z}\}$ indexed by \mathbb{Z} defined by $Da_n := a_{n-1}$. Denote the identity operator by I .

LEMMA 4.5. *For all n ,*

$$(1) \text{ If } a > c \text{ then } d_n\begin{pmatrix} a & b \\ c & d \end{pmatrix} = Dd_n\begin{pmatrix} a & c & b & d \\ c & b & a & d \end{pmatrix}.$$

$$(2) \text{ If } c > a \text{ then } d_n\begin{pmatrix} a & b \\ c & d \end{pmatrix} = Dd_n\begin{pmatrix} a & a & b & b \\ c & c & a & d - b \end{pmatrix}.$$

Proof. Write $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = M_1 \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, with $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathcal{P}_{n-1}(\mathcal{S}_2)$, and $M_1 \in \mathcal{S}_2$. In case (1), by 4.1, we must have $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $a' = c$, $b' = d$, $c' = a - c$, and $d' = b - d$. In case (2), by 4.1, we must have $M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $a' = a$, $b' = b$, $c' = c - a$, and $d' = d - b$. In either case the value of d_n is as stated.

LEMMA 4.6. *Suppose that $a > 1$. Then*

$$d_n\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = D\left(2d_n\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \sum_{c \neq 0} d_n\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}\right).$$

Proof. Write $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = M_1 \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, with $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathcal{P}_{n-1}(\mathcal{S}_2)$, and $M_1 \in \mathcal{S}_2$. By 4.1, we must have either (1) $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $a' = a$, $b' = 0$, or (2) $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $a' = b' = d' = 0$, and $c' = a$. In case (1), we must also have

$d' = 0$ by 4.3. Thus $d_n \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = d_{n-1} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \sum_c d_{n-1} \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = 2d_{n-1} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \sum_{c \neq 0} d_{n-1} \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$.

For $n \in \mathbb{Z}$, define δ_n as

$$\delta_n = \begin{cases} 1, & \text{if } n \geq 1 \\ 0, & \text{if } n \leq 0. \end{cases}$$

LEMMA 4.7. For all $n \in \mathbb{Z}$,

$$d_n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = D \left(2d_n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{c \neq 0} d_n \begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix} \right) + \delta_n.$$

Proof. The proof is the same as for 4.5, except that in case (1) there is also the possibility that $a' = d' = 1$, $b' = 0$, and $c' = n - 1$. This gives the extra $+\delta_n$ for $n \geq 2$. The formula is easily checked directly for $n = 1$.

LEMMA 4.8. Suppose that $a > 1$. Then for all $n \in \mathbb{Z}$,

$$d_n \begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix} = D \left(3d_n \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \sum_{c \neq 0} d_n \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \right).$$

Proof. Write $\begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix} = M_1 \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, with $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathcal{P}_{n-1}(\mathcal{S}_2)$, and $M_1 \in \mathcal{S}_2$. By 4.1, we must have either (1) $M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $a' = a$, $b' = 0$, or (2) $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and $b' = c' = d' = 0$ and $a' = a$. In case (1), we must also have $d' = 0$ by 4.3. Thus $d_n \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = 2d_{n-1} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \sum_c d_{n-1} \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = 3d_{n-1} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \sum_{c \neq 0} d_{n-1} \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$.

LEMMA 4.9. For all $n \in \mathbb{Z}$,

$$d_n \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = D \left(3d_n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{c \neq 0} d_n \begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix} \right) + \delta_n.$$

Proof. Repeat the proof of 4.7, using 4.8 instead of 4.5.

LEMMA 4.10. For all $n, a \in \mathbb{Z}$,

$$d_n \begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix} = (I + D) d_n \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. By 4.6–4.9,

$$d_n \begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix} - d_n \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = d_{n-1} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

LEMMA 4.11. Suppose that $a > c > 0$ and that the g.c.d. of a and c is d . For all $n \in \mathbb{Z}$,

$$d_n \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = D^{l(a,c)} d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix}.$$

Proof. By 4.4 and 4.5,

$$d_n \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = \begin{cases} Dd_n \begin{pmatrix} a & 0 \\ a-c & 0 \end{pmatrix}, & \text{if } a > a-c \\ Dd_n \begin{pmatrix} a-c & 0 \\ a & 0 \end{pmatrix}, & \text{if } a < a-c. \end{cases}$$

The lemma follows using the definition of $l(a, c)$ in 3.1.

LEMMA 4.12. *Suppose that $a > 0$ and $d \mid a$. For all $n \in \mathbb{Z}$,*

$$\sum_{\substack{d \leq c \leq a \\ (a, c) = d}} d_n \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = f_{a/d}(D) d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix}.$$

Proof. By 4.11,

$$\begin{aligned} \sum_{\substack{d \leq c \leq a \\ (a, c) = d}} d_n \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} &= \sum_{\substack{d \leq c \leq a \\ (a, c) = d}} D^{l(a, c)} d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix} \\ &= f_{a/d}(D) d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix} \end{aligned}$$

by 3.3.

LEMMA 4.13. *Suppose that $a > 0$. For all $n \in \mathbb{Z}$,*

$$\sum_{1 \leq c \leq a} d_n \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = \sum_{d \mid a} f_{a/d}(D) d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix}.$$

Proof. By 4.11,

$$\begin{aligned} \sum_{1 \leq c \leq a} d_n \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} &= \sum_{d \mid a} \sum_{d \leq c \leq a} d_n \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \\ &= \sum_{d \mid a} f_{a/d}(D) d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix} \end{aligned}$$

by 4.12.

LEMMA 4.14. *Suppose that $a > 0$. For all $n, k \in \mathbb{Z}$ with $k \geq 0$,*

$$\sum_{ka+1 \leq c \leq (k+1)a} d_n \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = D^k \sum_{1 \leq c \leq a} d_n \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}.$$

Proof.

$$\begin{aligned} \sum_{ka+1 \leq c \leq (k+1)a} d_n \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} &= D \sum_{ka+1 \leq c \leq (k+1)a} d_n \begin{pmatrix} a & 0 \\ c-a & 0 \end{pmatrix} \\ &= D \sum_{(k-1)a+1 \leq c \leq ka} d_n \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}. \end{aligned}$$

The result follows by induction.

LEMMA 4.15. Assume that $a > 0$. For all $n \in \mathbb{Z}$,

$$(I - D) \sum_{c \neq 0} d_n \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = \sum_{d|a} f_{a/d}(D) d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix}.$$

Proof.

$$\begin{aligned} \sum_{c \neq 0} d_n \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} &= \sum_{k=0}^{\infty} \sum_{ka+1 \leq c \leq (k+1)a} d_n \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \\ &= \sum_{k=0}^{\infty} D^k \sum_{1 \leq c \leq a} d_n \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \\ &= \sum_{k=0}^{\infty} D^k \sum_{d|a} f_{a/d}(D) d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix}. \end{aligned}$$

The result follows upon multiplying both sides by $I - D$.

PROPOSITION 4.16. Assume that $a > 0$. For all $n \in \mathbb{Z}$,

$$(I - 4D + D^2) d_n \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = (D + D^2) \sum_{\substack{d|a \\ d \neq a}} f_{a/d}(D) d_n \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. By 4.6, 4.10, and 4.15,

$$\begin{aligned} (I - D) d_n \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} &= 2D(I - D) d_n \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + D \sum_{d|a} f_{a/d}(D) d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix} \\ &= 2D(I - D) d_n \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + (D + D^2) d_n \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad + D \sum_{\substack{d|a \\ d \neq a}} f_{a/d}(D) d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix} \\ &= (3D - D^2) d_n \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + D \sum_{\substack{d|a \\ d \neq a}} f_{a/d}(D) d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix}. \end{aligned}$$

The result now follows from 4.10.

PROPOSITION 4.17. Assume that $a > 0$. For all $n \in \mathbb{Z}$,

$$(I - 4D + D^2) d_n \begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix} = (D + D^2) \sum_{\substack{d|a \\ d \neq a}} f_{a/d}(D) d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix}.$$

Proof. This follows from 4.10 and 4.16.

PROPOSITION 4.18. For all $n \in \mathbb{Z}$,

$$(I - 4D + D^2)d_n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{cases} 0, & \text{if } n \neq 1 \\ 1, & \text{if } n = 1. \end{cases}$$

Proof. A computation similar to 4.16 gives

$$(I - 4D + D^2)d_n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (I - D)\delta_n$$

from which the result follows.

PROPOSITION 4.19. For all $n \in \mathbb{Z}$,

$$(I - 4D + D^2)d_n \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{cases} 0, & \text{if } n \neq 1, 2 \\ 1, & \text{if } n = 1, 2. \end{cases}$$

Proof. This time we have

$$(I - 4D + D^2)d_n \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = (I - D^2)\delta_n.$$

LEMMA 4.20. Assume that $b > 0$. For all $n \in \mathbb{Z}$,

$$d_n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} Dd_n \begin{pmatrix} a - b & b \\ c - d & d \end{pmatrix}, & \text{if } a - b > b \\ Dd_n \begin{pmatrix} b & a - b \\ d & c - d \end{pmatrix}, & \text{if } a - b < b \\ 2Dd_n \begin{pmatrix} b & d \\ b & d \end{pmatrix}, & \text{if } a - b = b \text{ and } c - d = d. \end{cases}$$

Proof. Write $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \cdot M_n$, with $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathcal{P}_{n-1}(\mathcal{S}_2)$, and $M_n \in \mathcal{S}_2$. By 4.1, we must have either (1) $M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $a' = a - b$, $b' = b$, $c' = c - d$, and $d' = d$, or (2) $M_n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $a' = b$, $b' = a - b$, $c' = d$, and $d' = c - d$. Thus

$$d_n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = Dd_n \begin{pmatrix} a - b & b \\ c - d & d \end{pmatrix} + Dd_n \begin{pmatrix} b & a - b \\ d & c - d \end{pmatrix}.$$

If $a - b > b$, the second term on the right hand side is 0. If $a - b < b$, the first term on the right hand side is 0. If $a - b = b$ and $c - d = d$, the two terms are equal. This proves the lemma.

LEMMA 4.21. Assume that $a > b > 0$ and that the g.c.d. of a and b is d . For all $n \in \mathbb{Z}$,

$$d_n \begin{pmatrix} a & b \\ a & b \end{pmatrix} = 2D^{l(a,b)+1} d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix}$$

and

$$d_n \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = 2D^{l(a,b)+1} d_n \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. This follows from 4.20 and the definition of length in 3.1.

LEMMA 4.22. For all $n \in \mathbb{Z}$,

$$\sum_{a+b=k} d_n \begin{pmatrix} a & b \\ a & b \end{pmatrix} = \sum_{d|k} f_{k/d}(D) d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix},$$

and

$$\sum_{a+b=k} d_n \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \sum_{d|k} f_{k/d}(D) d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix}.$$

Proof.

$$\begin{aligned} \sum_{a+b=k} d_n \begin{pmatrix} a & b \\ a & b \end{pmatrix} &= \sum_{d|k} \sum_{\substack{a+b=k \\ (a,b)=d}} d_n \begin{pmatrix} a & b \\ a & b \end{pmatrix} \\ &= \sum_{\substack{d|k \\ d \neq k/2, k}} \sum_{\substack{a+b=k \\ (a,b)=d}} 2D^{l(a,b)+1} d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix} \\ &\quad + D d_n \begin{pmatrix} k/2 & 0 \\ k/2 & 0 \end{pmatrix} + d_n \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix} \\ &= \sum_{\substack{d|k \\ d \neq k/2, k}} \sum_{\substack{a+b=k \\ (a,b)=d}} f_{k/d}(D) d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix} \\ &\quad + f_2(D) d_n \begin{pmatrix} k/2 & 0 \\ k/2 & 0 \end{pmatrix} + d_n \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix} \quad (\text{by 3.4}) \\ &= \sum_{\substack{d|k \\ d \neq k}} f_{k/d}(D) d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix} + d_n \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix} \\ &= \sum_{d|k} f_{k/d}(D) d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix}. \end{aligned}$$

The proof of the second statement is similar.

LEMMA 4.23. For all $n \in \mathbb{Z}$,

$$\sum_b d_n \begin{pmatrix} k & b \\ 0 & 0 \end{pmatrix} = (I + D) d_n \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} + 2D \sum_{\substack{d|k \\ d \neq 0}} f_{k/d}(D) d_n \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. As in the proof of 4.13, using 4.21 we get

$$\sum_{b \neq 0, k} d_n \begin{pmatrix} k & b \\ 0 & 0 \end{pmatrix} = 2D \sum_{\substack{d|k \\ d \neq 0}} f_{k/d}(D) d_n \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}.$$

Since

$$d_n \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} + d_n \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix} = (I + D) d_n \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix},$$

the result is proved.

LEMMA 4.24. For all $n \in \mathbb{Z}$,

$$\begin{aligned} (I - D) \sum_c d_n \begin{pmatrix} 0 & 0 \\ c & k \end{pmatrix} &= (D + D^2) d_n \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} \\ &+ 2D^2 \sum_{\substack{d|k \\ d \neq 0}} f_{k/d}(D) d_n \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Proof. As in the proof of 4.15, using 4.21 we get

$$\begin{aligned} \sum_{c \neq 0 \pmod{k}} d_n \begin{pmatrix} 0 & 0 \\ c & k \end{pmatrix} &= \sum_{c \neq 0 \pmod{k}} d_n \begin{pmatrix} c & k \\ 0 & 0 \end{pmatrix} \\ &= \sum_{k+1 \leq c < k} \sum_{r=0}^{\infty} D^r d_n \begin{pmatrix} c & k \\ 0 & 0 \end{pmatrix} \\ &= \sum_{1 \leq c < k} \sum_{r=0}^{\infty} D^{r+1} d_n \begin{pmatrix} c & k \\ 0 & 0 \end{pmatrix} \\ &= \sum_{r=0}^{\infty} 2D^{r+2} \sum_{\substack{d|k \\ d \neq 0}} f_{k/d}(D) d_n \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{c \equiv 0 \pmod{k}} d_n \begin{pmatrix} 0 & 0 \\ c & k \end{pmatrix} &= \sum_{c \equiv 0 \pmod{k}} d_n \begin{pmatrix} c & k \\ 0 & 0 \end{pmatrix} \\ &= \left(D + \sum_{r=2}^{\infty} 2D^r \right) d_n \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} \\ &= \left(\sum_{r=1}^{\infty} D^r + \sum_{r=2}^{\infty} D^r \right) d_n \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

The result follows upon multiplying both sides by $I - D$.

5. MORE RECURSIONS

If n and k are integers and $M \in \mathcal{S}_2$, define $d_n(M, k)$ to be the number of sequences M_1, M_2, \dots, M_n with $M_i \in \mathcal{S}_2$, $M_1 = M$, and $\text{Tr}(\prod_{i=1}^n M_i) = k$.

LEMMA 5.1. *For all $n, k \in \mathbb{Z}$,*

(1) *If $A = d_n \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ then*

$$d_n(A, k) = \sum_{a+b=k} d_n \begin{pmatrix} a & b \\ a & b \end{pmatrix} - 2D \sum_{a+b=k} d_n \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$

(2) *If $A = d_n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then*

$$d_n(A, k) = (I - D) \sum_b d_n \begin{pmatrix} k & b \\ 0 & 0 \end{pmatrix}.$$

(3) *If $A = d_n \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ then*

$$d_n(A, k) = (I - D) \sum_c d_n \begin{pmatrix} 0 & 0 \\ c & k \end{pmatrix}.$$

Proof. (1) A matrix equal to $\prod_{i=1}^n M_i$ with $M_i \in \mathcal{S}_2$, trace equal to k , and $M_1 = A$ is of the form $\begin{pmatrix} a & b \\ a & b \end{pmatrix}$ with $a + b = k$. On the other hand, if $\begin{pmatrix} a & b \\ a & b \end{pmatrix} = \prod_{i=1}^n M_i$ with $M_i \in \mathcal{S}_2$ then either (1) $M_1 = A$ or (2) $M_1 =$

$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\prod_{i=2}^n M_i = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. Thus

$$\sum_{a+b=k} d_n \begin{pmatrix} a & b \\ a & b \end{pmatrix} = d_n(A, k) + 2D \sum_{a+b=k} d_n \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$

This proves (1).

(2) A matrix equal to $\prod_{i=1}^n M_i$ with $M_i \in \mathcal{S}_2$, trace equal to k , and $M_1 = A$ is of the form $\begin{pmatrix} k & b \\ 0 & 0 \end{pmatrix}$. On the other hand, if $\begin{pmatrix} k & b \\ 0 & 0 \end{pmatrix} = \prod_{i=1}^n M_i$ with $M_i \in \mathcal{S}_2$ then either (1) $M_1 = A$ or (2) $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\prod_{i=2}^n M_i = \begin{pmatrix} 0 & 0 \\ k & b \end{pmatrix}$. Thus

$$\begin{aligned} \sum_b d_n \begin{pmatrix} k & b \\ 0 & 0 \end{pmatrix} &= d_n(A, k) + D \sum_b d_n \begin{pmatrix} 0 & 0 \\ k & b \end{pmatrix} \\ &= d_n(A, k) + D \sum_b d_n \begin{pmatrix} k & b \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

This proves (2).

(3) A matrix equal to $\prod_{i=1}^n M_i$ with $M_i \in \mathcal{S}_2$, trace equal to k , and $M_1 = A$ is of the form $\begin{pmatrix} 0 & 0 \\ c & k \end{pmatrix}$. On the other hand, if $\begin{pmatrix} 0 & 0 \\ c & k \end{pmatrix} = \prod_{i=1}^n M_i$ with $M_i \in \mathcal{S}_2$ then either (1) $M_1 = A$ or (2) $M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\prod_{i=2}^n M_i = \begin{pmatrix} 0 & 0 \\ c & k \end{pmatrix}$. Thus

$$\sum_c d_n \begin{pmatrix} 0 & 0 \\ c & k \end{pmatrix} = d_n(A, k) + D \sum_c d_n \begin{pmatrix} 0 & 0 \\ c & k \end{pmatrix}.$$

This proves (3).

PROPOSITION 5.2. For all $n, k \in \mathbb{Z}$,

(1) If $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ then

$$d_n(A, k) = (I - D) \sum_{d|k} f_{k/d}(D) d_n \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}.$$

(2) If $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then

$$d_n(A, k) = (I - D^2) d_n \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} + 2D(I - D) \sum_{\substack{d|k \\ d \neq k}} f_{k/d}(D) d_n \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}.$$

(3) If $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ then

$$d_n(A, k) = D(I + D) d_n \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} + 2D^2 \sum_{\substack{d|k \\ d \neq k}} f_{k/d}(D) d_n \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. (1) By part (1) of 5.1 and 4.22

$$\begin{aligned} d_n(A, k) &= \sum_{d|k} f_{k/d}(D) d_n \begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix} - 2D \sum_{d|k} f_{k/d}(D) d_n \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} \\ &= (I - D) \sum_{d|k} f_{k/d}(D) d_n \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

(2) This follows from 5.1 part (2) and 4.23.

(3) This follows from 5.1 part (3) and 4.24.

For $n, k \in \mathbb{Z}$ define

$$d_n^1(k) = d_n \left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, k \right).$$

and

$$d_n^2(k) = d_n \left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, k \right) + d_n \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, k \right) + d_n \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, k \right).$$

PROPOSITION 5.3. For all $n, k \in \mathbb{Z}$ with $k \geq 2$,

$$D(I + D)d_n^1(k) = (I - D)^2(I - 2D)d_n \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$Dd_n^2(k) = (I - D)^2 d_n \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}.$$

On the other hand,

$$d_n^1(1) = (I - D)d_n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$d_n^2(1) = 2d_n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. The values for $k = 1$ follow from 5.2.

Suppose that $k \geq 2$. By 5.2,

$$d_n^2(k) = 2d_n \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} + (I + D) \sum_{\substack{d|k \\ d \neq k}} f_{k/d}(D) d_n \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, using 4.16, we get

$$\begin{aligned} Dd_n^2(k) &= 2Dd_n\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} + (D + D^2) \sum_{\substack{d|k \\ d \neq k}} f_{k/d}(D) d_n\begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} \\ &= 2Dd_n\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} + (I - 4D + D^2) d_n\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} \\ &= (I - D)^2 d_n\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

The formula for $d_n^1(k)$ is proved similarly.

6. GENERATING FUNCTIONS

We will now give generating functions for the multiplicities of decomposition numbers.

PROPOSITION 6.1.

$$\sum_{k,n} d_n\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} x^n k^{-s} = \frac{x}{(1 - 4x + x^2 - (x + x^2)\xi(x, s))}.$$

Proof. We have

$$\begin{aligned} (1 - 4x + x^2) \sum_n d_n\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^n &= (I - 4D + D^2) \sum_n d_n\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^n \\ &= x \end{aligned}$$

by 4.18. On the other hand,

$$\begin{aligned} (1 - 4x + x^2) \sum_{\substack{n,k \\ k \geq 2}} d_n\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} x^n k^{-s} \\ &= (I - 4D + D^2) \sum_{\substack{n,k \\ k \geq 2}} d_n\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} x^n k^{-s} \\ &= (D + D^2) \sum_n \sum_{k \geq 2} \sum_{\substack{d|k \\ d \neq k}} f_{k/d}(D) d_n\begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} x^n k^{-s} \end{aligned}$$

by 4.17. We also have

$$\begin{aligned} & (x + x^2) \xi(x, s) \sum_{k,n} d_n \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} x^n k^{-s} \\ &= (x + x^2) \sum_{k \geq 2} f_k(x) k^{-s} \sum_{k,n} d_n \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} x^n k^{-s} \\ &= (D + D^2) \sum_n \sum_{k \geq 2} \sum_{\substack{d|k \\ d \neq k}} f_{k/d}(D) d_n \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} x^n k^{-s}. \end{aligned}$$

Thus

$$(1 - 4x + x^2 - (x + x^2) \xi(x, s)) \sum_{k,n} d_n \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} x^n k^{-s} = x,$$

which gives us the result.

For $i = 1, 2$, define $g_i(x, s) = \sum_{k,n} d_n^i(k) x^n k^{-s}$.

PROPOSITION 6.2.

$$g_1(x, s) = \frac{(1 - x)^2(1 - 2x)}{(1 + x)(1 - 4x + x^2 - (x + x^2) \xi(x, s))} - \frac{1 - x}{1 + x}$$

while

$$g_2(x, s) = \frac{(1 - x)^2}{1 - 4x + x^2 - (x + x^2) \xi(x, s)} - 1.$$

Proof. We have

$$\begin{aligned} g_1(x, s) &= \sum_n d_n^1(1) x^n + \sum_n \sum_{k \geq 2} d_n^1(k) x^n k^{-s} \\ &= (1 - x) \sum_n d_n^1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^n \\ &\quad + \frac{(1 - x)^2(1 - 2x)}{x(1 + x)} \sum_n \sum_{k \geq 2} d_n^1 \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} x^n k^{-s} \\ &= \frac{x(1 - x)}{1 - 4x + x^2} + \frac{(1 - x)^2(1 - 2x)}{x(1 + x)} \sum_n \sum_{k \geq 1} d_n^1 \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} x^n k^{-s} \end{aligned}$$

$$\begin{aligned}
& - \frac{(1-x)^2(1-2x)}{x(1+x)} \sum_n d_n^1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^n \\
& = \frac{x(1-x)}{1-4x+x^2} + \frac{(1-x)^2(1-2x)}{(1+x)(1-4x+x^2 - (x+x^2)\xi(x,s))} \\
& \quad - \frac{(1-x)^2(1-2x)}{(1+x)(1-4x+x^2)}
\end{aligned}$$

which simplifies to the formula stated. Also,

$$\begin{aligned}
g_2(x,s) & = \sum_n d_n^2(1)x^n + \sum_n \sum_{k \geq 2} d_n^2(k)x^n k^{-s} \\
& = 2 \sum_n d_n^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^n + \frac{(1-x)^2}{x} \sum_n \sum_{k \geq 2} d_n^2 \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} x^n k^{-s} \\
& = \frac{2x}{1-4x+x^2} + \frac{1-x^2}{x} \sum_n \sum_{k \geq 1} d_n^2 \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} x^n k^{-s} \\
& \quad - \frac{(1-x)^2}{x} \sum_n d_n^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^n \\
& = \frac{2x}{1-4x+x^2} + \frac{(1-x)^2}{1-4x+x^2 - (x+x^2)\xi(x,s)} \\
& \quad - \frac{(1-x)^2}{1-4x+x^2}
\end{aligned}$$

which again simplifies to the formula stated.

If n and k are integers, define $\bar{d}_n^1(k)$ to be the number of sequences M_1, M_2, \dots, M_n with $M_i \in \bar{\mathcal{F}}_2$, $M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, and $\text{Tr}(\prod_{i=1}^n M_i) = k$. Define $\bar{d}_n^2(k)$ to be the number of sequences M_1, M_2, \dots, M_n with $M_i \in \bar{\mathcal{F}}_2$, $M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, or $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\text{Tr}(\prod_{i=1}^n M_i) = k$.

For $i = 1, 2$, define $\bar{g}_i(x, s) = \sum_{k,n} \bar{d}_n^i(k) x^n k^{-s}$.

PROPOSITION 6.3. For $i = 1, 2$, $\bar{g}_i(x, s) = g_i(x/(1-x), s)$.

Proof. There are $\binom{n-1}{j}$ choices of where to put exactly j identity matrices among M_1, M_2, \dots, M_n (M_1 is not the identity matrix). Thus

$$\bar{d}_n^i(k) = \sum_{j=0}^{n-1} \binom{n-1}{j} d_{n-j}^i(k).$$

We therefore have

$$\begin{aligned}
 \bar{g}_i(x, s) &= \sum_{k, n} \bar{d}_n^i(k) x^n k^{-s} \\
 &= \sum_{k, n} \sum_{j=0}^{n-1} \binom{n-1}{j} d_{n-j}^i(k) x^n k^{-s} \\
 &= \sum_{k, n} \sum_{j=0}^{\infty} \binom{n-1+j}{j} d_n^i(k) x^{n+j} k^{-s} \\
 &= \sum_{k, n} d_n^i(k) \left(\frac{x}{1-x} \right)^n k^{-s},
 \end{aligned}$$

where we are using the identity

$$\left(\frac{1}{1-x} \right)^n = \sum_{j=0}^{\infty} \binom{n-1+j}{j} x^j.$$

This proves the proposition.

We can now prove our main theorem, stated in Section 1.

Proof of Main Theorem. By 2.2, $\tilde{d}_n^i(k) = 3 \times 2^n \bar{d}_n^i(k)$. Thus $\tilde{g}_i(x, s) = 3\bar{g}_i(2x, s)$. The theorem thus follows from 6.2 and 6.3.

7. CARTAN INVARIANTS

Our results allow us to compute certain Cartan invariants.

If Φ_1 and Φ_2 are projective indecomposable characters, let $c(\Phi_1, \Phi_2)$ be the Cartan invariant corresponding to Φ_1 and Φ_2 , i.e., the number of times the irreducible character corresponding to Φ_1 occurs in Φ_2 . Compare the following with Theorems 5.2 and 5.3 in [1]. Compare also with the results in [2, 4].

THEOREM 7.1. (1) *Suppose that λ is in the top $W_{a,p}$ -alcove of $SL_3(p)$ and sufficiently deep in the interior of a $\tilde{W}_{a,p}$ -alcove. The Cartan invariant $c(\Phi_{\lambda, n}, \Phi_{\lambda, n})$ of $SL_3(p^n)$ or $SU_3(p^n)$ is equal to the coefficient of x^n in the series expansion of $6x(1 - 2x)/(48x^2 - 18x + 1)$.*

(2) *Suppose λ is in the bottom alcove of $SL_3(p)$ and sufficiently deep in the interior of a $\tilde{W}_{a,p}$ -alcove. Then the Cartan invariant $c(\Phi_{\lambda, n}, \Phi_{\lambda, n})$ of $SL_3(p^n)$ or $SU_3(p^n)$ is equal to the coefficient of x^n in the series expansion of $12x(1 - 4x)/(48x^2 - 18x + 1)$.*

Proof. We have $c(\Phi_{\lambda, n}, \Phi_{\lambda, n}) = \sum_{k, n} \tilde{d}_n^i(k) x^n k^2 = \tilde{g}_i(x, -2) = 3g_i(2x/(1-2x), -2)$, where $i = 1$ if λ is in the top alcove, and $i = 2$ if λ is in the bottom alcove. Using 6.2 and 3.5, part (3), we get the formula stated.

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