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Generic Cartan Invariants for Chevalley Groups

LEONARD CHASTKOFSKY*

Department of Mathematics, University of Georgia, Athens, Georgia 30602

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INTRODUCTION

The most successful approach to determining the characteristic p Cartan invariants of a finite Chevalley group defined in the same characteristic has been to consider the corresponding algebraic group G. This method was used by Humphreys [2, p. 57] to compute the generic p-Cartan invariants of $SL_3(p)$. A module for the algebraic group with restricted highest weight is constructed, which when restricted to the finite group is generically a projective indecomposable module. The G-composition factors are determined and then restricted to the finite group to get the Cartan invariants.

In this paper we describe a variation on this approach. The modules for G that we consider do not have restricted highest weight, but on the contrary are "generic" in the sense that their highest weights are far away from the walls of the fundamental region. We define "generic Cartan invariants" which can be computed fairly easily if the generic decomposition of Weyl modules into irreducible is known. (This decomposition is presently known for small rank groups from Jantzen's work, e.g., [4], but Lusztig [5] has a conjecture giving it for all groups.) We then show precisely how these invariants correspond to the generic Cartan invariants of the finite groups. This gives an alternative and usually easier way of computing these invariants. In rank 2 they can be computed graphically and we give examples.

1. PRELIMINARIES

Let G be a semi-simple simply connected algebraic group defined over K, the algebraic closure of a finite field of order p. Let σ be an endomorphism of G such that the set of fixed points, G_σ, is finite. Let X be the weight lattice of G, X^+ the set of dominant weights. σ acts also on X and we shall

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write this action functionally, while the induced action on the group ring $\mathbb{Z}[X]$ will be written exponentially. Let δ be half the sum of the positive roots and X_σ the subset of X^+ consisting of those weights whose co-ordinates in terms of the fundamental dominant weights are all less than those of $\sigma\delta$. For example, if σ is the nth power of the Frobenius endomorphism composed with an automorphism of the Dynkin diagram of G, then X_σ is just the set of weights whose co-ordinates are between 0 and $p^n - 1$.

For λ in X, let λ_0 be the element of X_σ such that $\lambda \equiv \lambda_0 \pmod{\sigma X}$ and let λ_1 be defined by $\sigma\lambda_1 = \lambda - \lambda_0$.

Let W be the Weyl group of G. There is a dot action of W on X given by $w \cdot \lambda = w(\lambda + \delta) - \delta$ for $w \in W$, $\lambda \in X$. As in [1] we also have a star action given by $w * \lambda = w \cdot (\lambda_0 - \sigma\delta) + \sigma(\lambda_1 + \delta)$. Geometrically, the dot action is the action of W with origin shifted to $-\delta$, while for the star action it is shifted to $\sigma(\lambda_1 + \delta) - \delta$. If w_1 is also in W then $w_1 \cdot (w \cdot \lambda) = w_1 w \cdot \lambda$ and $w_1 \cdot (w * \lambda) = w_1 w * (\lambda_0 + \sigma w_1 \cdot \lambda_1)$.

Let $\chi(\lambda)$ and $\chi_p(\lambda)$ be the formal characters of the Weyl module and irreducible KG module of highest weight λ, respectively. Let Ψ_λ be the character constructed in [1]. It has the property that on restriction to G_σ it is projective character which contains the projective cover of $\chi_p(\lambda)$ exactly once and generically, if λ is far enough away from the walls of an alcove, is equal to this projective cover.

For λ in X^+ we can express Ψ_λ as a linear combination with integer coefficients of the $\chi(\lambda_1)^\sigma \chi_p(\lambda_0)$, where $\lambda_1 \in X^+$, $\lambda_0 \in X_\sigma$. It is these coefficients, which we can think of as "Cartan invariants," which form the main object of study of this paper. We will show that for λ deep enough inside X^+ these Cartan invariants take on generic values which correspond to the generic Cartan invariants of G_σ.

2. Decomposition Formulas

In [1] formulas showing how the $\chi(\mu)$ decompose into $\chi(\lambda_1)^\sigma \chi_p(\lambda_0)$ as λ varies over X^+ were given. These are reformulations of formulas originally given by Jantzen [4]. For λ far enough inside X^+ the decomposition pattern becomes uniform. Our objective here is to modify these formulas so that the pattern is uniform for all λ by allowing λ and μ to range over all of X.

Let us start by recalling some formulas from [1]. (These were stated there for the case when σ is the Frobenius map, but they generalize easily to the general case.) For $\lambda, \mu \in X$ define

$$h_{\lambda,\mu} = \sum \operatorname{sgn} w \tag{1}$$

where the sum is over $w \in W$ such that $w \cdot \mu = y * \lambda$ for some $y \in W$. Then if μ is in X^+

$$\chi(\mu) = \sum b_{\mu,\lambda} \chi(\lambda_1)^\sigma \chi_p(\lambda_0) \tag{2}$$

where

$$b_{\mu,\lambda} = \sum \operatorname{sgn} w_1 m_{\lambda_0,\gamma}(\theta)\, h_{\gamma+\sigma\xi,\mu} \tag{3}$$

where $\lambda \in X^+$ the sum is over $w_1 \in W$, $\gamma \in X_\sigma$, $\xi \in X^+$, $\theta \in X$ such that $\theta + w_1 \cdot \xi = \lambda_1$, and where $m_{\lambda_0,\gamma}(\theta)$ is the multiplicity of θ in the $\lambda_0 - \gamma$ entry of a certain matrix, indexed by X_σ and with entries in $\mathbb{Z}[X]^W$, which can be computed once we know the decomposition of $\chi(\mu)$ for μ in X_σ, or any translate of it. We also have for λ in X^+

$$\Psi_\lambda = \sum_{\mu \in X^+} b_{\mu,\lambda} \chi(\mu). \tag{4}$$

Now if μ is an element of X not necessarily in X^+, $\chi(\mu)$ can still be defined by Weyl's character formula. Let us call an element μ in X dot-regular, if there is an element w in W, necessarily unique, such that $w \cdot \mu$ belongs to X^+. Then if μ is dot-regular with $w \cdot \mu \in X^+$ then $\chi(\mu) = \operatorname{sgn} w\ \chi(w \cdot \mu)$, while if μ is not dot-regular then $\chi(\mu) = 0$. Similarly if $\lambda \in X$ and λ_1 is dot-regular with $w \cdot \lambda_1 \in X^+$, define $\Psi_\lambda = \operatorname{sgn} w\ \Psi_{\sigma w \cdot \lambda_1 + \lambda_0}$, while if λ_1 is not dot-regular define Ψ_λ to be 0.

LEMMA 1. *With the definitions of* $\chi(\mu)$ *and* Ψ_λ *as above, formulas* (2) *and* (4) *hold for all* λ, μ *in* X.

Proof. From the definition of $h_{\lambda,\mu}$ it follows easily that $h_{\lambda_1, w \cdot \mu} = \operatorname{sgn} w\ h_{\lambda,\mu}$, while $h_{\sigma w \cdot \lambda_1 + \lambda_0, \mu} = \operatorname{sgn} w\ h_{\lambda,\mu}$. The lemma now follows immediately from the definitions. ▌

We now modify these definitions slightly. For λ and μ in X define

$$\eta_{\lambda,\mu} = \begin{cases} 1 & \text{if } \mu = w * \lambda \text{ some } w \in W \\ 0 & \text{otherwise.} \end{cases}$$

Define $\beta_{\mu,\lambda} = \sum m_{\lambda_0,\gamma}(\theta)\, \eta_{\gamma+\sigma\zeta,\mu}$, where the sum is over $\theta, \zeta \in X$, $\gamma \in X_\rho$ such that $\theta + \zeta = \lambda_1$.

LEMMA 2. *For* $\zeta \in X$, $\eta_{\lambda+\sigma\zeta,\mu+\sigma\zeta} = \eta_{\lambda,\mu}$ *and* $\beta_{\mu+\sigma\zeta,\lambda+\sigma\zeta} = \beta_{\mu\lambda}$.

Proof. The first formula follows from $w * (\lambda + \sigma\zeta) = w * \lambda + \sigma\zeta$, and the second then follows from the definition of $\beta_{\mu,\lambda}$. ▌

We now use these definitions to modify the decomposition formulas (2) and (4).

Theorem 1. (A) *Let* $\lambda \in X$. *Then* $\Psi_\lambda = \sum_{\mu \in X} \beta_{\mu,\lambda}\chi(\mu)$.

(B) *Let* $\mu \in X$. *Then* $\chi(\mu) = \sum_{\lambda \in X} \beta_{\mu,\lambda}\chi(\lambda_1)^\sigma \chi_p(\lambda_0)$.

Proof of A. By (4) and definitions (1) and (3) we have

$$\Psi_\lambda = \sum \operatorname{sgn} w_1 m_{\lambda_0,\gamma}(\theta)\, h_{\gamma+\sigma\zeta,\mu}\chi(\mu) \tag{5}$$

where the sum is over $w_1 \in W_1$, $\gamma \in X_\sigma$, $\theta \in X$, ζ, $\mu \in X^+$ such that $w_1 \cdot \zeta + \theta = \lambda_1$.

Now there is a one-to-one correspondence between 4-tuples (w_1, w_2, ζ, μ), where w_1 and w_2 are in W and ζ and μ are in X^+ , and pairs (ζ', μ'), where ζ' and μ' are dot-regular elements of X, such that

$$\zeta' = w_1 \cdot \zeta \qquad \text{and} \qquad \mu' = w_1 w_2 \cdot \mu. \tag{6}$$

Moreover in this notation, $w_2 \cdot \mu = w * (\gamma + \sigma\zeta)$ is equivalent to $\mu' = w_1 w_2 \cdot \mu = w_1 w * (\gamma + \sigma w_1 \cdot \zeta) = w_1 * (\gamma + \sigma\zeta')$. Thus (5) is equal to $\sum \operatorname{sgn} w_2 \operatorname{sgn} w_1\, m_{\lambda_0,\gamma}(\theta)\, \eta_{\gamma+\sigma\zeta',\mu'}\chi(\mu)$, where the sum is over γ in X_σ, θ, ζ' and μ' in X such that $\theta + \zeta' = \lambda_1$ and where $\mu \in X^+$ and w_1 and w_2 are defined by (6). Since $\chi(\mu) = \operatorname{sgn} w_1 w_2 \chi(\mu')$, this sum is then equal to the expression for Ψ_λ stated in the theorem.

Proof of B. We have by (2), (1) and (3),

$$\chi(\mu) = \sum \operatorname{sgn} w_1 m_{\lambda_0,\gamma}(\theta)\, h_{\gamma+\sigma\zeta,\mu}\chi(\lambda_1)^\sigma \chi_p(\lambda_0) \tag{7}$$

where the sum is over w_1 in W, γ and λ_0 in X_σ, ζ and λ_1 in X^+, and θ in X such that $\theta + w_1 \cdot \zeta = \lambda_1$.

Now there is a one-to-one correspondence between 5-tuples $(w_1, w_2, \zeta, \lambda_1, \theta)$ and triples $(\zeta', \lambda_1', \theta')$, where $w_1, w_2 \in W$, ζ and λ_1 are in X^+, θ, $\theta' \in X$, and ζ' and λ' are dot-regular elements of X such that

$$\lambda_1 = w_1 w_2 \cdot \lambda_1', \qquad \zeta = w_2 \cdot \zeta', \qquad \text{and} \qquad \theta = w_1 w_2 \theta'. \tag{8}$$

Under this correspondence the condition $w_2 \cdot \mu = w * (\gamma + \sigma\zeta)$ is equivalent to $\mu = w_2^{-1} w * (\gamma + \sigma w_2^{-1} \cdot \zeta) = w_2^{-1} w * (\gamma + \sigma\zeta')$. Moreover, with the same notation, $\theta + w_1 \cdot \zeta = \lambda$, is equivalent to $\theta' + \zeta' = \lambda_1'$. Since $m_{\lambda_0,\gamma}(\theta) = m_{\lambda_0,\gamma}(\theta')$, the sum in (7) is equal to

$$\sum \operatorname{sgn} w_2 \operatorname{sgn} w_1\, m_{\lambda_0,\gamma}(\theta')\, \eta_{\gamma+\sigma\zeta',\mu}\chi(\lambda_1)^\sigma \chi_p(\lambda_0),$$

where the sum is over λ_0 and γ in X_σ, θ', ζ', λ_1' in X and dot-regular such that $\theta' + \zeta' = \lambda_1'$ and where w_1, w_2 in W and λ_1 in X^+ are defined by (8). Since $\chi(\lambda_1) = \operatorname{sgn} w_1 w_2 \chi(\lambda_1')$ we see that this sum is equal to $\sum_{\lambda \in X} \beta_{\mu,\lambda}\chi(\lambda_1)^\sigma \chi_p(\lambda_0)$ as required. ∎

3. Cartan Invariants

For λ and λ' belonging to X define

$$C_{\lambda,\lambda'} = \sum_{\mu \in X} \beta_{\mu,\lambda} \beta_{\mu,\lambda'}.$$

It is then immediate that $C_{\lambda,\lambda'} = C_{\lambda',\lambda}$ and

$$\Psi_\lambda = \sum_{\lambda' \in X} C_{\lambda,\lambda'} \chi(\lambda'_1)^\sigma \chi_p(\lambda'_0).$$

If λ is far enough inside X^+ then $C_{\lambda,\lambda'}$ is non-zero only for λ' also in X^+. Moreover by Lemma 2, $C_{\lambda+\sigma\zeta,\lambda'+\sigma\zeta} = C_{\lambda,\lambda'}$. Thus we do in fact get generic values.

The result which allows us to relate these "Cartan invariants" to those of G_σ is the following symmetry property:

Lemma 3. *Let $y \in W$, λ, λ', $\lambda'' \in X$, with $\lambda''_1 = y(\lambda'_1 - \lambda_1) + \lambda_1$ and $\lambda'_0 = \lambda''_0$. Then $C_{\lambda,\lambda'} = C_{\lambda,\lambda''}$.*

Proof. We have by definition

$$C_{\lambda,\lambda'} = \sum m_{\lambda_0\gamma}(\theta)\, m_{\lambda'_0,\gamma'}(\theta')\, \eta_{\gamma+\sigma\zeta,\mu} \eta_{\gamma'+\sigma\zeta',\mu'}$$

the sum over $\gamma, \gamma' \in X_\sigma$, θ, θ', ζ, $\zeta' \in X$ such that $\zeta + \theta = \lambda_1$, $\zeta' + \theta' = \lambda'_1$. Now $\eta_{\gamma+\sigma\zeta,\mu} \eta_{\gamma'+\sigma\zeta',\xi'}$ is equal to the number of μ such that

$$\mu = w * (\gamma + \sigma(\lambda_1 - \theta)) = w' * (\gamma' + \sigma(\lambda'_1 - \theta')), \qquad \text{some } w, w' \in W. \tag{9}$$

An easy computation shows $w * (\gamma + \sigma(\lambda_1 - \theta)) = w' * (\gamma' + \sigma(\lambda'_1 - \theta'))$ if and only if

$$yw * (\gamma + \sigma(\lambda_1 - y\theta)) = yw' * (\gamma' + \sigma(\lambda''_1 - y\theta')) \tag{10}$$

and the number of μ satisfying (9) is the same as the number satisfying (10). Since $m_{\lambda_0,\gamma}(\theta) = m_{\lambda_0,\gamma'}(y\theta)$ and $m_{\lambda'_0,\gamma'}(\theta') = m_{\lambda_0,\gamma'_1}(y\theta')$ we see that $C_{\lambda,\lambda'} = \sum m_{\lambda_0,\gamma}(\theta)\, m_{\lambda'_0,\gamma'}(\theta'')\, \eta_{\gamma+\sigma\zeta,\mu} \eta_{\gamma'+\sigma\zeta'',\mu}$, where the sum is over $\gamma, \gamma' \in X_\sigma$, θ, θ'', ζ, $\zeta'' \in X$ such that $\theta + \zeta = \lambda_1$ and $\theta'' + \zeta'' = \lambda''_1$. This expression is then equal to $C_{\lambda,\lambda''}$ as required. ▮

Taking $\lambda_1 = 0$ in the lemma we have the following:

Corollary. *If $\lambda \in X_\sigma$ then $C_{\lambda,\lambda'} = C_{\lambda, w\lambda'_1 + \lambda'_0}$ for all $w \in W$.* ▮

For $\lambda \in X$ let $s(\lambda)$ be the element of $\mathbb{Z}[X]^W$ obtained by summing the exponentials of all W-conjugates of λ. We have the following well-known formula of Brauer:

For

$$\lambda, \mu \in X, \qquad s(\lambda)\,\chi(\mu) = \sum \chi(\mu + \lambda') \tag{11}$$

the sum being over all λ' in the W-orbit of λ. As a corollary:

LEMMA 4. *For* $\lambda \in X$, $s(\lambda) = \sum \chi(\lambda')$, *the sum being over all* λ' *in the* W*-orbit of* λ.

Proof. Take $\mu = 0$ in (11). ▌

We now state our main result, which shows that the generic Cartan invariants for G_σ are the same as those for G. For $Y \in \mathbb{Z}[X]^W$ let Y_σ be the restriction of Y to G_σ. For λ, ν in X_σ define $(C_{\lambda,\nu})_\sigma$ by

$$(\Psi_\lambda)_\sigma = \sum_{\nu \in X_\sigma} (C_{\lambda,\nu})_\sigma\,(\chi_p(\nu))_\sigma.$$

THEOREM 2. *Suppose* λ *is in* X_σ, *and that for all* ν *for which* $C_{\lambda,\nu}$ *is non-zero,* $\nu_0 + w\nu_1$ *is in the same alcove as* ν_0 *for every* w *in* W. *Then* $(\Psi_\lambda)_\sigma = \sum_{\nu \in X^+} \sum_{w \in W} C_{\lambda,\nu}\chi_p(\nu_0 + w\nu_1)_\sigma$.

Proof. By Theorem 1 and the definition of $C_{\lambda,\nu}$ we have

$$\Psi_\lambda = \sum_{\nu \in X} C_{\lambda,\nu}\chi(\nu_1)^\sigma\,\chi_p(\nu_0). \tag{12}$$

By the corollary to Lemma 3, (12) can be written as

$$\sum_{\nu \in X^+} \sum_{\nu_1' \sim \nu_1} C_{\lambda,\nu}\chi(\nu_1')^\sigma\,(\chi_p(\nu_0)$$

where the inner sum is over all ν_1' in X which are W-conjugate to ν_1. Restricting to G_σ gives

$$(\Psi_\lambda)_\sigma = \sum_{\nu \in X^+} \sum_{\nu_1' \sim \nu_1} C_{\lambda,\nu}\chi(\nu_1')_\sigma\,\chi_p(\nu_0)_\sigma$$

which by Lemma 4 is equal to

$$\sum_{\nu \in X^+} C_{\lambda,\nu} s(\nu_1)_\sigma\,\chi_p(\nu_0)_\sigma.$$

If $\nu_0 + w\nu_1$ belongs to the same alcove as ν_0 for every w in W, then by (11) and Jantzen's translation principle [3 Theorem 1], $s(\nu_1)\,\chi_p(\nu_0)$ is equal to $\sum_{\nu_1'\sim\nu_1}\chi_p(\nu_0+\nu_1')$. This gives

$$(\Psi_\lambda)_\sigma = \sum_{\nu\in X^+}\sum_{\nu_1'\sim\nu_1} C_{\lambda,\nu}\chi_p(\nu_0+\nu_1')_\sigma$$

proving the theorem. ▌

The coefficients in Theorem 2 do not necessarily give the G_σ Cartan invariants even when $(\Psi_\lambda)_\sigma$ is indecomposable. The problem is that $\nu_0+w\nu_1$ may equal $\nu_0'+w'\nu_1'$ even when ν is not equal to ν'. However, if this does not happen then the theorem says that the $C_{\lambda,\nu}$ are indeed the G_σ Cartan invariants. We state this explicitly.

COROLLARY. *Assume the hypotheses of Theorem* 2 *and let* $\nu\in X^+$. *Suppose that* $(\Psi_\lambda)_\sigma$ *is indecomposable and that whenever* $C_{\lambda,\nu'}$ *is non-zero and* $\nu_0+w\nu_1=\nu_0'+w\nu_1'$ *for some* w *in* W, *then* $\nu=\nu'$. *Then* $(C_{\lambda,\nu_0+\nu_1})_\sigma=C_{\lambda,\nu}$. ▌

4. EXAMPLES

We will give the values of all the generic Cartan invariants for G in rank 2 when σ is the Frobenius endomorphism. We assume that λ is in the interior of an alcove and that it is deep inside X^+, deep enough so that $C_{\lambda,\nu}$ is non-zero only for ν in X^+. The values of $C_{\lambda,\nu}$ can be computed using the definition and the values of $\beta_{\mu,\lambda}$ which are known from Jantzen's work [4]. Lusztig's diagrams in [5] give a visual presentation of the patterns of the $\beta_{\mu,\lambda}$. These can be used to compute the $C_{\lambda,\nu}$ by examining the alcoves for which the patterns for λ and ν, respectively, coincide.

The resulting pattern for the non-zero $C_{\lambda,\nu}$ in case A_2 is shown in Figs. 1 and 2. The alcove with the circled number in it is the one to which λ belongs. A number in an alcove indicates that if ν is the weight in the interior of that alcove which is conjugate under the affine Weyl group to λ then the value of $C_{\lambda,\nu}$ is equal to that number. The values of all other $C_{\lambda,\nu}$ are zero.

For cases B_2 and G_2 we present the generic values of $C_{\lambda,\nu}$ by means of tables. The alcoves in the box which contains weights whose co-ordinates lie between 0 and p-1 are numbered as in Fig. 3. The pattern depends on which type of alcove λ_0 lies in. For each type we present a matrix giving the non-zero $C_{\lambda,\nu}$, where the columns represent the type of alcove to which ν_0 belongs and the rows the weight in X^+ to which $\nu_1-\lambda_1$ is conjugate to under W. (By Lemma 3 the values are the same for fixed ν_0 and λ_0 for all such conjugates of $\nu_1-\lambda_1$.) In both cases we take α_1 to be the short root and α_2 the long one.

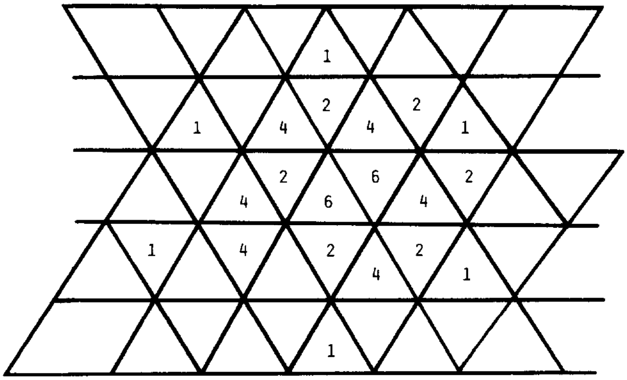

FIG. 1. Generic Cartan invariants for G of type A_2 when $\langle \lambda_0+\delta, \alpha_0^\vee \rangle > p$.

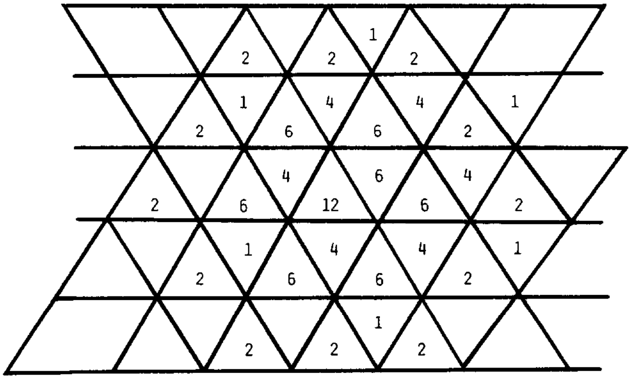

FIG. 2. Generic Cartan invariants for G of type A_2 when $\langle \lambda_0+\delta, \alpha_0^\vee \rangle > p$.

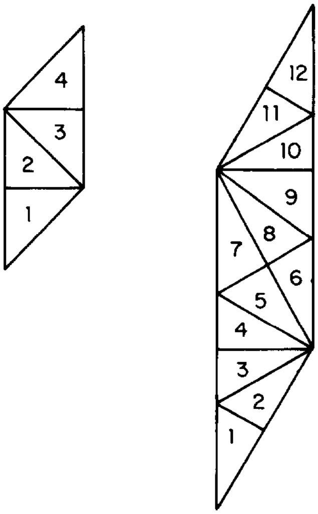

FIG. 3. Labelling of alcoves for types B_2 and G_2.

B_2, λ_0 *in alcove of type* 4

	1	2	3	4
(0, 0)	8	8	8	8
(1, 0)	6	6	4	2
(0, 1)	6	4	2	
(1, 1)	1			

B_2, λ_0 *in alcove of type* 3

	1	2	3	4
(0, 0)	16	16	16	8
(1, 0)	12	10	8	4
(0, 1)	8	6	4	2
(1, 1)	2	1		
(2, 0)	4	2		

B_2, λ_0 *in alcove of type* 2

	1	2	3	4
(0, 0)	24	24	16	8
(1, 0)	16	14	10	6
(0, 1)	10	8	6	4
(1, 1)	4	2	1	
(2, 0)	6	4	2	
(0, 2)	2			

B_2, λ_0 *in alcove of type* 1

	1	2	3	4
(0, 0)	32	24	16	8
(1, 0)	18	16	12	6
(0, 1)	12	10	8	6
(1, 1)	6	4	2	1
(2, 0)	8	6	4	
(0, 2)	4	2		
(3, 0)	2			

G_2, λ_0 *in alcove of type* 12

	1	2	3	4	5	6	7	8	9	10	11	12
(0, 0)	48	36	36	36	24	12	12	12	12	12	12	12
(1, 0)	28	16	20	24	16	8	10	8	6	4	2	
(0, 1)	14	10	8	6	4	2						
(2, 0)	6											
(1, 1)	1											

G_2, λ_0 *in alcove of type* 11

	1	2	3	4	5	6	7	8	9	10	11	12
(0, 0)	96	60	72	48	24	24	24	24	24	24	24	12
(1, 0)	56	32	38	44	32	16	16	14	12	8	4	2
(0, 1)	28	14	12	12	8	4	4	2				
(2, 0)	12	6	6	4								
(1, 1)	2	1										

G_2, λ_0 *in alcove of type* 10

	1	2	3	4	5	6	7	8	9	10	11	12
(0, 0)	132	84	96	108	72	36	36	36	36	36	24	12
(1, 0)	84	48	56	64	44	24	22	20	16	12	8	4
(0, 1)	36	18	18	18	12	6	6	4	2			
(2, 0)	24	12	10	8	4	0	2					
(1, 1)	4	2	1									

G_2, λ_0 *in alcove of type* 9

	1	2	3	4	5	6	7	8	9	10	11	12
(0, 0)	156	96	120	132	96	48	48	48	48	36	24	12
(1, 0)	112	64	74	80	56	28	28	24	20	16	12	6
(0, 1)	46	24	24	26	16	8	8	6	4	2		
(2, 0)	32	16	14	14	8	4	4	2				
(1, 1)	8	4	2	1								

G_2, λ_0 *in alcove of type* 8

	1	2	3	4	5	6	7	8	9	10	11	12
(0, 0)	180	108	132	156	108	60	60	60	48	36	24	12
(1, 0)	132	80	88	96	64	32	32	28	24	20	14	8
(0, 1)	60	30	32	34	22	10	12	8	6	4	2	
(2, 0)	40	20	20	20	14	8	6	4	2			
(1, 1)	12	6	4	2	1							
(3, 0)	4											

G_2, λ_0 *in alcove of type* 7

	1	2	3	4	5	6	7	8	9	10	11	12
(0, 0)	204	120	144	168	120	60	72	60	48	36	24	12
(1, 0)	148	92	102	108	72	36	36	32	28	22	16	10
(0, 1)	74	36	40	44	28	14	16	12	8	6	4	
(2, 0)	52	28	26	28	20	10	8	6	4	2		
(1, 1)	16	8	6	4	2	1						
(3, 0)	6											
(0, 2)	2											

G_2, λ_0 *in alcove of type* 6

	1	2	3	4	5	6	7	8	9	10	11	12
(0, 0)	204	120	144	168	120	72	60	60	48	36	24	12
(1, 0)	152	88	98	108	72	36	36	32	28	24	16	8
(0, 1)	74	40	42	44	28	12	14	10	8	6	4	2
(2, 0)	48	24	26	28	20	12	10	8	4			
(1, 1)	16	8	6	4	2	1						
(3, 0)	8	4	2									

G_2, λ_0 *in alcove of type* 5

	1	2	3	4	5	6	7	8	9	10	11	12
(0, 0)	408	240	288	324	240	120	120	108	96	72	48	24
(1, 0)	296	170	196	210	144	72	72	64	56	44	32	16
(0, 1)	148	80	84	88	56	28	28	22	16	12	8	4
(2, 0)	104	54	52	58	40	20	20	14	8	4		
(1, 1)	32	16	12	10	4	2	2	1				
(3, 0)	12	6	4	2								
(0, 2)	4	2										

G_2, λ_0 *in alcove of type* 4

	1	2	3	4	5	6	7	8	9	10	11	12
(0, 0)	588	360	420	480	324	168	168	156	132	108	72	36
(1, 0)	430	252	282	312	210	108	108	96	80	64	44	24
(0, 1)	222	120	126	132	88	44	44	34	26	18	12	6
(2, 0)	158	84	86	88	58	28	28	20	14	8	4	
(1, 1)	52	24	20	16	10	4	4	2	1			
(3, 0)	22	8	6	4	2							
(0, 2)	6	4	2									

G_2, λ_0 *in alcove of type* 3

	1	2	3	4	5	6	7	8	9	10	11	12
(0, 0)	540	336	396	420	288	144	144	132	120	96	72	36
(1, 0)	392	224	256	282	196	98	102	88	74	56	38	20
(0, 1)	212	118	120	126	84	40	42	32	24	18	12	8
(2, 0)	148	80	84	86	52	26	26	20	14	10	6	
(1, 1)	52	24	20	20	12	6	6	4	2	1		
(3, 0)	24	12	8	6	4	2						
(0, 2)	8	6	4	2								
(2, 1)	4											

G_2, λ_0 *in alcove of type* 2

	1	2	3	4	5	6	7	8	9	10	11	12
(0, 0)	480	312	336	360	240	120	120	108	96	84	60	36
(1, 0)	342	196	224	252	170	88	92	80	64	48	36	16
(0, 1)	198	116	118	120	80	40	36	30	24	18	14	10
(2, 0)	144	76	80	84	54	24	28	20	16	12	6	
(1, 1)	54	24	24	24	16	8	8	6	4	2	1	
(3, 0)	30	16	12	8	6	4						
(0, 2)	12	8	6	4	2							
(2, 1)	6											
(4, 0)	2											

G_2, λ_0 *in alcove of type* 1

	1	2	3	4	5	6	7	8	9	10	11	12
(0, 0)	828	480	540	588	408	204	204	180	156	132	96	48
(1, 0)	600	342	392	430	296	152	148	132	112	84	56	28
(0, 1)	360	198	212	222	148	74	74	60	46	36	28	14
(2, 0)	264	144	148	158	104	48	52	40	32	24	12	6
(1, 1)	104	54	52	52	32	16	16	12	8	4	2	1
(3, 0)	60	30	24	22	12	8	6	4				
(0, 2)	24	12	8	6	4	2						
(2, 1)	12	6	4	2								
(4, 0)	4	2										

The corresponding generic Cartan invariants for G_σ can be read off of these diagrams and tables by means of Theorem 2. For example, in the case of $SL_3(p)$ we get the following values for $(C_{\lambda,\nu})_\sigma$, assuming the hypotheses of the theorem are satisfied.

Let the coordinates of λ in terms of the fundamental dominant weights be (r, s). If $r+s<p-2$, we get

ν	$(C_{\lambda,\nu})_\sigma$
(r, s)	12
$(p-s-2, p-r-2)$	6
$(p-r-s-3, r)+w(0, 1)$	6
$(s, p-r-s-3)+w(1, 0)$	6
$(p-r-2, r+s+1)+w(0, 1)$	4
$(r+s+1, p-s-2)+w(1, 0)$	4
$(s, p-r-s-3)+w(0, 2)$	2
$(p-r-s-3, r)+w(2, 0)$	2
$(r, s)+w(1, 1)$	2
$(p-s-2, p-r-2)+w(1, 1)$	1

Here w ranges over all elements of the Weyl group. On the other hand, if $r+s>p-2$, we obtain:

ν	$(C_{\lambda,\nu})_\sigma$
(r, s)	6
$(p-s-2, p-r-2)$	6
$(r+s+1-p, p-s-2)+w(0, 1)$	4
$(p-r-2, r+s+1-p)+w(1, 0)$	4
$(s, 2p-r-s-3)+w(0, 1)$	2
$(2p-r-s-3, r)+w(1, 0)$	2
$(p-s-2, p-r-2)+w(1, 1)$	1

If G_σ is $SU_3(p)$ then the above tables for $(C_{\lambda,\nu})_\sigma$ are still valid, except that $(1, 0)$ and $(0, 1)$ are interchanged, as are $(2, 0)$ and $(0, 2)$.

REFERENCES

1. L. CHASTKOFSKY, Variations on Hulsurkar's matrix with applications to representations of algebraic Chevalley groups, *J. Algebra* **82** (1983), 255–274.
2. J. E. HUMPHREYS, "Ordinary and Modular Representations of Chevalley Groups," Lecture Notes in Mathematics, Vol. 528, Springer-Verlag, Berlin, 1976.
3. J. C. JANTZEN, Zun Charakterformel gewisser Darstellungen halbeinfacher Gruppen und ihrer Lie-Algebren, *Math. Z.* **140** (1974), 127–149.
4. J. C. JANTZEN, Uber das Dekompositionsverhalten gewisser modularer Darstellingen halbeinfacher Gruppen und ihrer Lie-Algebren, *J. Algebra* **49** (1977) 441–469.
5. G. LUSZTIG, Hecke algebras and Jantzen's generic decomposition patterns, *Adv. in Math.* **37** (1980), 121–164.