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# Varieties of nilpotent elements for simple Lie algebras I: Good primes

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**Abstract**

Let $G$ be a simple algebraic group over $k = \mathbb{C}$, or $\overline{\mathbb{F}}_p$ where $p$ is good. Set $\mathfrak{g} = \operatorname{Lie} G$. Given $r \in \mathbb{N}$ and a faithful (restricted) representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, one can define a variety of nilpotent elements $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \{x \in \mathfrak{g}\colon\ \rho(x)^r = 0\}$. In this paper we determine this variety when $\rho$ is an irreducible representation of minimal dimension or the adjoint representation.


## 1. Introduction

*1.1.* Let $G$ be a simple algebraic group scheme defined over $\mathbb{F}_p$ and $\mathfrak{g} = \operatorname{Lie} G$ with $p$th power map $[p]$. Early work in the 1980s, by Jantzen [8], Friedlander and Parshall [4,5], and subsequent results by Suslin, Friedlander and Bendel [17,18] can be used to show that the spectrum of the cohomology ring for the first Frobenius kernel $\mathrm{H}^2(G_1, k)$ can

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be identified with the restricted nullcone $\mathcal{N}_1(\mathfrak{g}) := \{x \in \mathfrak{g}\colon\ x^{[p]} = 0\}$. In concrete terms, this algebraic variety can be identified with a set of matrices where each matrix in the set when multiplied by itself $p$ times is zero. Even though the algebraic variety $\mathcal{N}_1(\mathfrak{g})$ plays a crucial role in the cohomology theory for restricted Lie algebras, only recently has a concrete description of this variety been given via orbit closures. Carlson, Lin, Nakano and Parshall [1] have computed the restricted nullcone in the case when the underlying field has good characteristic. The description of the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$ is given as the closure of a certain Richardson orbit. In particular, the restricted nullcone is an irreducible variety. The underlying methods employed in [1] heavily relied on a conjecture of Jantzen on the support variety of Weyl modules [10] which was verified by Nakano, Parshall and Vella [15].

Now assume that $G$ is a simple algebraic group over an arbitrary algebraically closed field $k$. Let $\mathcal{N}(\mathfrak{g})$ be the set of nilpotent elements in $\mathfrak{g}$. The variety $\mathcal{N}(\mathfrak{g})$ is often referred to as the (ordinary) nullcone. The computation of the restricted nullcone motivates us to formulate a more general problem. Let $\Psi : G \to \mathrm{GL}(V)$ be a faithful representation of $G$. The derivative of $\Psi$, $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, is a faithful (restricted) representation of $\mathfrak{g}$. Define

$$\mathcal{N}_{r,\rho}(\mathfrak{g}) = \left\{x \in \mathfrak{g}\colon\ \rho(x)^r = 0\right\}.$$

Then $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is an algebraic variety contained in $\mathcal{N}(\mathfrak{g})$ which is invariant under the action of $G$. Since there are only finitely many $G$-orbits on $\mathcal{N}(\mathfrak{g})$, it is reasonable to try to describe $\mathcal{N}_{r,\rho}(\mathfrak{g})$ as the union of closures of $G$-orbits. We note that if the characteristic of the field is $p > 0$, then $\rho$ is a restricted representation (i.e., $\rho(x^{[p]}) = \rho(x)^p$ for all $x \in \mathfrak{g}$). Consequently,

$$\mathcal{N}_{p,\rho}(\mathfrak{g}) = \mathcal{N}_1(\mathfrak{g}).$$

The general varieties $\mathcal{N}_{r,\rho}(\mathfrak{g})$ might be relevant in the future investigation of the cohomology for the quantum groups at roots of unity and for affine Lie algebras.

Our method in approaching this problem involves extending ideas used in the work of Nakano and Tanisaki [16] on realizing orbit closures as support varieties. In that paper the authors show how to compute orbit closures in $\mathfrak{gl}(V)$ after intersecting with $\mathfrak{g}$. For fields of characteristic zero, one can then use the Jacobson–Morozov theorem, weighted Dynkin diagrams, as well as the knowledge about the characters of irreducible representations to calculate these varieties. In characteristic $p > 0$ one has to be more careful, and use tables given in [11,12] as a replacement for the Jacobson–Morozov theorem.

The main results in this paper are the computation of $\mathcal{N}_{r,\rho}(\mathfrak{g})$ when $\rho$ is an irreducible representation of minimal dimension or the adjoint representation of $\mathfrak{g}$ when the characteristic of the field is either zero or a good prime. When $\mathfrak{g}$ is a classical Lie algebra (i.e., the root system is of type $A_l$, $B_l$, $C_l$ or $D_l$), the minimal dimension representation is the standard matrix representation of the Lie algebra on column vectors. For these algebras the results on the minimal representation and the adjoint representation are presented in Section 3. Our results for the exceptional Lie algebras are given in Section 4. As a corollary of our work, when the characteristic of the field is $p$, one can set $r = p$ to recover all the calculations of the restricted nullcone in [1]. It should be noted that the computations in this paper do not make use of the verification of the Jantzen conjecture.

## 2. Intersections

### 2.1. Notation

Let $G$ be a simple algebraic group defined over an algebraically closed field $k$. Let $T$ be a maximal torus of $G$. The root system $\Phi$ with respect to the pair $(G, T)$ is identified with a subset of the set of weights $X(T)$. Let $\Phi^+$ be a set of positive roots with the negative roots being denoted by $\Phi^-$. The set of simple roots determined by $\Phi^+$ is $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. Our convention is to use the ordering of simple roots given in [9] following Bourbaki.

*Throughout this paper we will always assume that $p$ is a good prime for $\Phi$ or zero.* A list of good primes is provided below.

- $\Phi$ of type $A_l$, all primes;
- $\Phi$ of type $B_l$, $C_l$, $D_l$, $p \geqslant 3$;
- $\Phi$ of type $E_6$, $E_7$, $F_4$, $G_2$, $p \geqslant 5$;
- $\Phi$ of type $E_8$, $p \geqslant 7$.

Let $W$ be the Weyl group, $h$ be the Coxeter number for $\Phi$, and $\{\omega_i\colon 1 \leqslant i \leqslant \ell\}$ be the fundamental dominant weights. If $J \subset \Delta$ then let $P_J = L_J \ltimes U_J$ be the parabolic subgroup determined by $J$ with Levi factor $L_J$. Set $\mathfrak{u}_J = \operatorname{Lie} U_J$.

Let $\mathcal{N}(\mathfrak{g})$ be the variety of nilpotent elements of $\mathfrak{g}$ which is often called the nullcone. The group $G$ acts on $\mathcal{N}(\mathfrak{g})$ via conjugation and $\mathcal{N}(\mathfrak{g})$ has finitely many $G$-orbits [13]. Furthermore, the nullcone is an irreducible variety of dimension equal to $|\Phi|$.

When $k$ is an algebraically closed field and the characteristic is a good prime the classification and structures of these orbits coincide with the orbit theory for complex simple Lie algebras (see [2,3,7]). If $J \subseteq \Delta$ then $G \cdot \mathfrak{u}_J$ is a closed, irreducible subvariety of $\mathcal{N}(\mathfrak{g})$ and there exists a unique dense open $G$-orbit in $G \cdot \mathfrak{u}_J$. Orbits which arise in this way are the Richardson orbits in $\mathfrak{g}$.

*2.2.* Let $G$ be a simple algebraic group over an algebraically closed field $k$. Set $\mathfrak{g} = \operatorname{Lie} G$. Let $\Psi : G \hookrightarrow \mathrm{GL}(V)$ be a faithful finite-dimensional representation of $G$. The derivative $\rho = d\Psi : \mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$ is also a faithful finite-dimensional representation of $\mathfrak{g}$, with $N = \dim V$. The goal of this section is to describe the intersection of a nilpotent orbit of $\mathfrak{gl}(V)$ (or its closure) with $\mathfrak{g}$.

Since there are finitely many $G$ orbits on $\mathcal{N}(\mathfrak{g})$, we can let $S := \{z_1, z_2, \dots, z_t\}$ be a set of orbit representatives of $G$ on $\mathcal{N}(\mathfrak{g})$. There are two partial orderings related to orbit closures that we need to introduce. First, if $x, y \in S$ then $x \preccurlyeq y$ if and only if $\overline{G \cdot x} \subseteq \overline{G \cdot y}$. The other ordering of relevance is the dominance ordering on partitions of $N$ denoted by $\trianglelefteq$. If $\lambda$ is a partition of $N$ then let $\mathcal{O}_\lambda$ be the orbit in $\mathcal{N}(\mathfrak{gl}(V))$ having Jordan blocks of size $\lambda$ as an orbit representative. We have $\overline{\mathcal{O}}_\mu \subseteq \overline{\mathcal{O}}_\lambda$ if and only if $\mu \trianglelefteq \lambda$. For $z \in S$, let $\epsilon(z)$ be the partition of $N$ corresponding to the orbit containing $\rho(z) \in \mathcal{N}(\mathfrak{gl}(V))$.

For $r \in \mathbb{N}$, set $\mathcal{N}_r(\mathfrak{gl}(V)) = \{x \in \mathfrak{gl}(V)\colon x^r = 0\}$. We note that under the identification of $\mathfrak{g}$ with $\rho(\mathfrak{g})$, we have

$$\mathcal{N}_{r,\rho}(\mathfrak{g}) \cong \mathfrak{g} \cap \mathcal{N}_r\big(\mathfrak{gl}(V)\big). \tag{2.2.1}$$

Now suppose that $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ factors as $\rho = \phi \circ \iota$, where $\phi : \mathfrak{gl}_m(k) \hookrightarrow \mathfrak{gl}(V)$ is a faithful representation for $\mathfrak{gl}_m(k)$ and $\iota : \mathfrak{g} \hookrightarrow \mathfrak{gl}_m(k)$ is a Lie algebra monomorphism. Note that this condition means that the action of $\mathfrak{g}$ on $V$ can be extended to an action of $\mathfrak{gl}_m(k)$ on $V$. Then it follows that under the identification of $\mathfrak{g}$ in $\mathfrak{gl}_m(k)$ under $\iota$ one has

$$\mathcal{N}_{r,\rho}(\mathfrak{g}) \cong \mathfrak{g} \cap \mathcal{N}_{r,\phi}\big(\mathfrak{gl}_m(k)\big). \tag{2.2.2}$$

This formula will be useful when we study these nilpotent varieties for the adjoint representations for classical Lie algebras.

*2.3.* We first show what happens when we intersect orbits in $\mathcal{N}(\mathfrak{gl}(V))$ with $\mathfrak{g}$.

**Proposition.** *Let $z \in S$ and $\mu$ be a partition of $N$ where $N = \dim V$.*

(a) *If $\mu \notin \{\epsilon(z_1), \epsilon(z_2), \dots, \epsilon(z_t)\}$ then $\mathcal{O}_\mu \cap \mathfrak{g} = \emptyset$.*
(b) *$\mathcal{O}_{\epsilon(z)} \cap \mathfrak{g} = \bigcup_{j \in I} G \cdot z_j$ where $I = \{j\colon \epsilon(z_j) = \epsilon(z)\}$.*

**Proof.** (a) Let $y \in \mathcal{O}_\mu \cap \mathfrak{g}$. Then $y$ is conjugate to $x_\mu$ under GL(V). Moreover, $y \in \mathcal{N}(\mathfrak{g})$ so $y$ is conjugate under $G$ to some $z_j$ where $j = 1, 2, \dots, t$. Hence, $z_j$ is conjugate to $x_\mu$ under GL(V), and $x_\mu = x_{\epsilon(z_j)}$. It follows that $\mu = \epsilon(z_j)$ for some $j$, which contradicts our hypothesis. Consequently, $\mathcal{O}_\mu \cap \mathfrak{g} = \emptyset$.

(b) The set $\mathcal{O}_{\epsilon(z)} \cap \mathfrak{g}$ is $G$-stable, so

$$\mathcal{O}_{\epsilon(z)} \cap \mathfrak{g} = \bigcup_{i \in I} G \cdot z_i$$

for some $I \subseteq S$. But from part (a) we have seen that if $y \in \mathcal{O}_{\epsilon(z)} \cap \mathfrak{g}$ then $\epsilon(z) = \epsilon(z_j)$ for some $j$. So this condition completely characterizes $I$. □

*2.4.* The problem of intersecting closures of orbits in $\mathcal{N}(\mathfrak{gl}(V))$ with $\mathfrak{g}$ is more delicate. We need to take into account the dominance ordering and the ordering on orbits in $\mathcal{N}(\mathfrak{g})$. For this purpose, we introduce two indexing sets. First, let $\epsilon(S) = \{\epsilon(z_j)\colon\ j = 1, 2, \dots, t\}$. For $\mu \vdash N$ set

$$\Omega(\mu) = \big\{\gamma \vdash N\colon\ \gamma \text{ is maximal with respect to } \trianglelefteq \text{ such that } \gamma \in \epsilon(S) \text{ and } \gamma \trianglelefteq \mu\big\}.$$

Furthermore, for $\gamma \vdash N$ let

$$\Pi(\gamma) = \big\{j\colon\ z_j \text{ is maximal with respect to } \preccurlyeq \text{ such that } z_j \in S \text{ and } \epsilon(z_j) \trianglelefteq \gamma\big\}.$$

**Proposition.** *Let $\mu \vdash N$ where $N = \dim V$ and $\mathcal{O}_\mu$ be the corresponding orbit in $\mathfrak{gl}(V)$. Then*

$$\overline{\mathcal{O}}_\mu \cap \mathfrak{g} = \bigcup_{\gamma \in \Omega(\mu)} \bigcup_{j \in \Pi(\gamma)} \overline{G \cdot z_j}.$$

**Proof.** First observe that

$$\overline{\mathcal{O}}_\mu \cap \mathfrak{g} = \left( \bigcup_{\sigma \trianglelefteq \mu} \mathcal{O}_\sigma \right) \cap \mathfrak{g} = \bigcup_{\sigma \trianglelefteq \mu} (\mathcal{O}_\sigma \cap \mathfrak{g}) = \bigcup_{\sigma \trianglelefteq \mu:\ \sigma \in \epsilon(S)} (\mathcal{O}_\sigma \cap \mathfrak{g}).$$

The last equality follows by Proposition 2.3(a). Now

$$\bigcup_{\sigma \trianglelefteq \mu:\ \sigma \in \epsilon(S)} (\mathcal{O}_\sigma \cap \mathfrak{g}) = \left( \bigcup_{\sigma \trianglelefteq \mu:\ \sigma \in \epsilon(S)} \mathcal{O}_\sigma \right) \cap \mathfrak{g} \subseteq \left( \bigcup_{\gamma \in \Omega(\mu)} \overline{\mathcal{O}}_\gamma \right) \cap \mathfrak{g} = \bigcup_{\gamma \in \Omega(\mu)} (\overline{\mathcal{O}}_\gamma \cap \mathfrak{g}).$$

On the other hand,

$$\bigcup_{\gamma \in \Omega(\mu)} \overline{\mathcal{O}}_\gamma \cap \mathfrak{g} = \left( \bigcup_{\gamma \in \Omega(\mu)} \overline{\mathcal{O}}_\gamma \right) \cap \mathfrak{g} \subseteq \overline{\mathcal{O}}_\mu \cap \mathfrak{g}.$$

This shows that $\overline{\mathcal{O}}_\mu \cap \mathfrak{g} = \bigcup_{\gamma \in \Omega(\mu)} \overline{\mathcal{O}}_\gamma \cap \mathfrak{g}$.

Next we need to analyze $\overline{\mathcal{O}}_\gamma \cap \mathfrak{g}$ where $\gamma \in \epsilon(S)$. From Proposition 2.3, we have

$$\overline{\mathcal{O}}_\gamma \cap \mathfrak{g} = \left( \bigcup_{\sigma \trianglelefteq \gamma} \mathcal{O}_\sigma \right) \cap \mathfrak{g} = \bigcup_{\sigma \trianglelefteq \gamma} (\mathcal{O}_\sigma \cap \mathfrak{g}) = \bigcup_{\sigma \in \mathcal{X}} \left[ \bigcup_{i \in \mathcal{Y}(\sigma)} G \cdot z_i \right]$$

where $\mathcal{X} = \{\sigma \in \epsilon(S):\ \sigma \trianglelefteq \gamma\}$ and $\mathcal{Y}(\sigma) = \{i:\ \epsilon(z_i) = \sigma\}$. Since $\overline{\mathcal{O}}_\gamma \cap \mathfrak{g}$ is closed, we can express the last union as

$$\bigcup_{\sigma \in \mathcal{X}} \left[ \bigcup_{i \in \mathcal{Y}(\sigma)} G \cdot z_i \right] = \bigcup_{j \in \Pi(\gamma)} \overline{G \cdot z_j}. \qquad \square$$

## 3. Classical groups

### 3.1. Partitions

Let $G$ be a classical group with underlying irreducible root system of type $A_l$, $B_l$, $C_l$, or $D_l$. Set $N = l + 1$ (respectively $2l + 1$, $2l$, $2l$) when $\Phi$ is of type $A_l$ (respectively $B_l$, $C_l$, $D_l$). For classical groups, the nilpotent orbits are parameterized by certain partitions of $N$. If $X = A$ (respectively $B$, $C$, $D$), let $\mathcal{P}_X(N)$ be the set of partitions of $N$ parameterizing the set of nilpotent orbits for $A_l$ (respectively $B_l$, $C_l$, $D_l$). A precise description of $\mathcal{P}_X(N)$ [3, Theorems 5.1.2–5.1.4] is given as follows.

- $\mathcal{P}_A(N)$: all partitions of $N$.
- $\mathcal{P}_B(N)$: partitions of $N$ such that even parts occur with even multiplicity.
- $\mathcal{P}_C(N)$: partitions of $N$ such that odd parts occur with even multiplicity.
- $\mathcal{P}_D(N)$: partitions of $N$ such that even parts occur with even multiplicity.

A *very even partition* in $\mathcal{P}_D(N)$ is a partition of $N$ with only even parts. For $\lambda \in \mathcal{P}_X(N)$, let $\mathcal{O}_\lambda$ be the corresponding nilpotent orbit with the following additional proviso in the case of type $D$. When $\Phi$ is of type $D$, for very even partitions $\lambda$, there are two orbits corresponding to $\lambda$. The two orbits will be denoted by $\mathcal{O}_\lambda^I$ and $\mathcal{O}_\lambda^{II}$. Finally, for a given $X = B, C$, or $D$ if $\lambda \in \mathcal{P}_A(N)$, let $\lambda_X \in \mathcal{P}_X(N)$ denote the $X$-collapse of $\lambda$. The partition given by the $X$-collapse $\lambda_X$ is the unique maximal partition in $\mathcal{P}_X(N)$ dominated by $\lambda$ [3, Lemma 6.3.3].

### 3.2. Minimal representation

For classical simple algebraic groups, one has a realization of these groups via matrices which preserve certain symmetric and skew symmetric forms. One also has a standard representation for these groups given by the action of the group on column vectors. This irreducible representation is the minimal dimensional nontrivial representation for $G$. The standard representation induces an embedding $\rho : \mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, where $\dim V := N = l + 1$ (respectively $N = 2l + 1$, $N = 2l$, $N = 2l$) for type $A_l$ (respectively $B_l$, $C_l$, $D_l$). For $0 \leqslant r \leqslant N$, express $N = dr + s$ where $0 \leqslant s \leqslant r - 1$, and set $\lambda_{N,r} = (r^d, s)$, and for $r > N$ set $\lambda_{N,r} = (N)$.

The following result shows that when $\rho$ is the standard representation, the closed subvariety $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is an irreducible variety in types $A_l$, $B_l$, or $C_l$. An explicit description of this variety is also provided below.

**Theorem.** *Let $G$ be a simple algebraic group of classical type and $\rho$ be the minimal representation.*

(a) *If $\Phi$ is of type $A$ then $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_{\lambda_{N,r}}$.*
(b) *If $\Phi$ is of type $X$ where $X$ is $B$ or $C$ then $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_{(\lambda_{N,r})_X}$.*
(c) *If $\Phi$ is of type $D$ then*
   (i) *$\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_{(\lambda_{N,r})_X}$ when $(\lambda_{N,r})_X$ is not very even*;
   (ii) *$\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}^I_{(\lambda_{N,r})_X} \cup \overline{\mathcal{O}}^{II}_{(\lambda_{N,r})_X}$ when $(\lambda_{N,r})_X$ is very even.*

**Proof.** Set $\lambda := \lambda_{N,r} = (r^d, s)$ to be the partition with $d$ parts of size $r$ and one part of size $s$. Then $\lambda$ is maximal in the dominance ordering of partitions of $N$ which have parts of size at most $r$. First assume that $\Phi$ is type $A$. The variety $\mathcal{N}_{r,\rho}(\mathfrak{g})$ in this situation consists of matrices with Jordan block sizes at most $r$. Therefore, $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_\lambda$.

Now assume that $G$ has type $B$, $C$, or $D$. For the sake of notation (in this proof), for $\lambda \in \mathcal{P}_A(N)$, let $\mathcal{O}_\lambda^{\mathfrak{gl}(V)}$ be the corresponding nilpotent orbit in $\mathfrak{gl}(V)$. Since the $X$-collapse $\lambda_X$ is the unique maximal partition in $\mathcal{P}_X(N)$ dominated by $\lambda$, we have by Proposition 2.4,

$$\overline{\mathcal{O}}_\lambda^{\mathfrak{gl}(V)} \cap \mathfrak{g} = \bigcup_{j \in \Pi(\lambda_X)} \overline{G \cdot z_j}. \tag{3.2.1}$$

Since $\mathcal{N}_{r,\rho}(\mathfrak{g}) \cong \mathfrak{g} \cap \mathcal{N}_r(\mathfrak{gl}(V)))$, one can use the computation in type $A$ along with the formula above to deduce that $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_{\lambda_X}$ as long as $\lambda_X$ is not very even in type $D$. In the case that $\lambda_X$ is very even in type $D$, we can conclude using the same reasoning that $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}^I_{\lambda_X} \cup \overline{\mathcal{O}}^{II}_{\lambda_X}$. $\square$

### *3.3. Adjoint representation, characteristic zero*

In this subsection we determine $\mathcal{N}_{r,\rho}(\mathfrak{g})$ when the characteristic of $k$ is zero and $\rho$ is the adjoint representation, using a tensor-product realization of the adjoint representation.

**Theorem.** *Let $\mathfrak{g}$ be a classical Lie algebra of type $X = A, B, C$, or $D$ over an algebraically closed field $k$ of characteristic* 0. *Let $\rho$ be the adjoint representation of $\mathfrak{g}$. Let $N$ be the dimension of the standard representation of $\mathfrak{g}$. Given a positive integer $r$, set $a = \lfloor (r+1)/2 \rfloor$, and $\lambda = \lambda_{N,a}$.*

(a) *If $X = A$ or $C$, then $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_{\lambda_X}$.*
(b) *If $X = B$ or $D$ and $r \geqslant 2N-1$, then $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_{\lambda_X}$.*
(c) *If $X = B$ or $D$ and $r$ is odd with $r \leqslant 2N-3$, write $N-(a+1) = d'(a-1)+s'$ where $0 \leqslant s' < a-1$, and let $\lambda' = (a+1, (a-1)^{d'}, s')$. Then $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_{\lambda_X} \cup \overline{\mathcal{O}}_{\lambda'_X}$.*
(d) *If $X = B$ or $D$ and $r$ is even with $r \leqslant 2N-2$, write $N-(a+1) = d'a+s'$ where $0 \leqslant s' < a$, and let $\lambda'' = (a+1, a^{d'}, s')$. Then $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_{\lambda''_X}$.*

*In our notation, if $\mu$ is very even* (*in type $D$*) *then $\overline{\mathcal{O}}_\mu = \overline{\mathcal{O}}^I_\mu \cup \overline{\mathcal{O}}^{II}_\mu$.*

**Proof.** We first do the computation for $\mathfrak{gl}(V)$, after extending the adjoint representation from $\mathfrak{g}$ to $\mathfrak{gl}(V)$. We then apply (2.2.2) to the embedding $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, along with the collapse argument in (3.2.1), to obtain the theorem for general classical $\mathfrak{g}$.

Assume first that $\mathfrak{g} = \mathfrak{gl}(V)$ with $\dim V = N$. It is well known that the adjoint representation of $\mathfrak{g}$ is naturally isomorphic to $V \otimes V^*$ where $V^* = \mathrm{Hom}_k(V, k)$ is the contragredient (dual) module.

Let $x \in \mathfrak{g}$ be a nilpotent matrix, corresponding to the partition $\mu = (\mu_1, \ldots, \mu_t) \vdash N$. Given $v \in V$, $w \in V^*$ we have

$$x^r \cdot (v \otimes w) = \sum_{k=0}^{r} \binom{r}{k} x^k v \otimes x^{r-k} w. \tag{3.3.1}$$

Then $x \in \mathcal{N}_{r,\rho}(\mathfrak{g})$ if and only if $x^r(v \otimes w) = 0$ for all $v \in V, w \in V^*$.

Choose a Jordan basis for $V$ with respect to $x$, and index the basis vectors corresponding to the largest $(\mu_1 \times \mu_1)$ block as $v_0, v_1 = xv_0, \ldots, v_{\mu_1-1} = x^{\mu_1-1}v_0$ where $x^{\mu_1}v_0 = 0$. The largest Jordan block of $x$ acting on $V^*$ has Jordan basis $w_i = (-1)^i v^*_{\mu_1-1-i}$, $0 \leqslant i \leqslant \mu_1 - 1$. Clearly (3.3.1) is 0 for all $v, w$ if and only if it is 0 for $v = v_0$, $w = w_0$. When $v = v_0$, $w = w_0$, the terms in the sum in (3.3.1) are linearly independent (or 0); thus every term must be 0. Thus for each $k$ $(0 \leqslant k \leqslant r)$, either $x^k v_0 = 0$ or $x^{r-k}w_0 = 0$; equivalently, $\max(k, r-k) \geqslant \mu_1$. The key term is $k = \mu_1 - 1$, in the sense that if the condition is satisfied for this $k$, then it is satisfied for all larger and smaller $k$. So the condition is reduced to simply $r - (\mu_1 - 1) \geqslant \mu_1$; i.e., $\mu_1 \leqslant (r+1)/2$. There is evidently a unique maximal (with respect to $\trianglelefteq$) partition $\mu$ of $N$ satisfying this condition. Set $a = \min(\lfloor (r+1)/2 \rfloor, N)$ and $\mu = \lambda_{N,a}$. Then $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_\mu$. Note that the answer is the same for $\mathfrak{g} = \mathfrak{sl}(V)$ because, as $\mathfrak{g}$-module, $V \otimes V^* \cong k \oplus \mathfrak{g}$.

In types $B$, $C$, and $D$, the natural representation $V$ of $\mathfrak{g}$ is self-dual (because $\mathfrak{g}$ preserves a nondegenerate bilinear form on $V$), so the adjoint representation $\rho$ occurs as a summand of $V \otimes V$. Assume $\mathfrak{g}$ is of type $C$ (respectively $B$ or $D$). Using the fact that $\mathfrak{g}$ preserves a skew-symmetric (respectively symmetric) form, and following through the identifications of $\mathfrak{gl}(V)$ with $V \otimes V^*$ and of $V^*$ with $V$, one finds that $\mathfrak{g}$ corresponds to $S^2(V)$, the degree 2 symmetric algebra on $V$ (respectively $\Lambda^2(V)$, the degree 2 exterior algebra on $V$). Thus $\rho$ can be realized as the standard action of $\mathfrak{g}$ on $S^2(V)$ (respectively $\Lambda^2(V)$). Note that this action lifts to $\mathfrak{gl}(V)$. Thus it suffices to determine the order $r$ nilpotent elements for the action $\rho$ of $\mathfrak{gl}(V)$ on $S^2(V)$ (respectively $\Lambda^2(V)$), and then intersect with $\mathfrak{g}$.

Consider first the obvious representation $\phi$ of $\mathfrak{gl}(V)$ on $S^2(V)$. Let $x \in \mathfrak{gl}(V)$ be nilpotent, corresponding to a partition $\mu \vdash N$. The action of $x^r$ on $S^2(V)$ is again given by (3.3.1), where now $v, w \in V$. Let $v_0, \ldots, v_{\mu_1 - 1}$ again be the portion of a Jordan basis for $V$ corresponding to the largest block of $x$. Then $x \in \mathcal{N}_{r,\phi}(\mathfrak{gl}(V))$ if and only if (3.3.1) is 0 when $v = w = v_0$. In this case, (3.3.1) becomes

$$x^r \cdot (v_0 \otimes v_0) = \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} 2\binom{r}{k} x^k v_0 \otimes x^{r-k} v_0 + \delta_{r,\text{even}} \binom{r}{r/2} x^{r/2} v_0 \otimes x^{r/2} v_0, \tag{3.3.2}$$

where $\delta_{r,\text{even}}$ is 1 if $r$ is even and 0 otherwise. The terms written are all linearly independent (if they are nonzero), so $x \in \mathcal{N}_{r,\phi}(\mathfrak{gl}(V))$ if and only if $\mu_1 \leqslant (r+1)/2$. This condition characterizes a unique maximal partition $\mu = \lambda_{N,a}$ as before. This completes the proof in type $C$.

Assume now that $\phi$ is the representation of $\mathfrak{gl}(V)$ on $\Lambda^2(V)$. Let $x = x_\mu \in \mathfrak{gl}(V)$ be a nilpotent element corresponding to a partition $\mu \vdash N$, and let $v_0, \ldots, v_{\mu_1 - 1}$ as above be the portion of a Jordan basis for $V$ corresponding to the $\mu_1 \times \mu_1$ block of $x$. Also let $w_0, \ldots, w_{\mu_2 - 1}$ be the portion of the Jordan basis for $V$ corresponding to the $\mu_2 \times \mu_2$ block of $x$. Then $x \in \mathcal{N}_{r,\phi}(\mathfrak{gl}(V))$ if and only if

$$0 = x^r \cdot (v_0 \wedge w_0) = \sum_{k=0}^{r} \binom{r}{k} x^k v_0 \wedge x^{r-k} w_0 \tag{3.3.3}$$

and

$$\begin{aligned} 0 = x^r \cdot (v_0 \wedge v_1) &= \sum_{k=0}^{r} \binom{r}{k} x^k v_0 \wedge x^{r+1-k} v_0 \\ &= \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \left[ \binom{r}{k} - \binom{r}{r+1-k} \right] x^k v_0 \wedge x^{r+1-k} v_0. \end{aligned} \tag{3.3.4}$$

The terms in (3.3.3) are linearly independent or 0, so either $k \geqslant \mu_1$ or $r - k \geqslant \mu_2$, which reduces to $r - (\mu_1 - 1) \geqslant \mu_2$; i.e.,

$$\mu_1 + \mu_2 \leqslant r + 1. \tag{3.3.5}$$

The terms in (3.3.4) are linearly independent or 0, and the difference of binomial coefficients is never 0, so $\mu_1 \leqslant r + 1 - \lfloor r/2 \rfloor$; i.e.,

$$\mu_1 \leqslant \lfloor (r+3)/2 \rfloor. \tag{3.3.6}$$

When $r$ is odd, there may be two maximal partitions $\mu$ of $N$ which satisfy (3.3.5) and (3.3.6). The first is $\lambda_{N,a}$ where $a = (r+1)/2$ as above. The second, $\mu'$ (which only occurs when $r \leqslant 2N - 3$), has $\mu'_1 = (r+3)/2 = a + 1$ . Write $N - (a+1) = d'(a-1) + s'$ with $0 \leqslant s' < a - 1$. Then $\mu' = (a+1, (a-1)^{d'}, s') = \lambda'$ (as in (c) of the statement of the theorem).

Assume $r$ is even. We will show there is a unique maximal partition $\mu$ satisfying (3.3.5) and (3.3.6). When $(r+2)/2 > N$ we are forced to take $\mu_1 \leqslant N \leqslant r/2 = a$, and then $\mu = \lambda_{N,a}$. Assuming $(r+2)/2 \leqslant N$, we may take $\mu_1 = (r+2)/2 = a + 1$. Write $N - (a+1) = d'a + s'$ with $0 \leqslant s' < a$. Then $\mu = (a+1, a^{d'}, s') = \lambda''$ (as in (d) of the statement of the theorem). (Note that $\lambda'' \trianglerighteq \lambda_{N,a}$. For instance, when $r$ is small enough, $\lambda'' = ((r+2)/2, (r/2)^{d'}, s') \trianglerighteq \lambda_{N,a} = ((r/2)^{d'+1}, s'+1)$. The situation for larger $r$ is simpler.) This completes the verification of the theorem in types $B$ and $D$. □

We remark that there are instances in the previous theorem where $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is reducible and has three irreducible components. Consider the case $D_4$ where $N = 8$ and $r = 3$. Then $\lambda = (2^4)$ and $\lambda' = (3, 1^5)$. Since $\lambda$ is very even and $\lambda' \in \mathcal{P}_D(8)$ we have $\mathcal{N}_{3,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}^I_{(2^4)} \cup \overline{\mathcal{O}}^{II}_{(2^4)} \cup \overline{\mathcal{O}}_{(3,1^5)}$.

### *3.4. Adjoint representation, prime characteristic*

We now use the tensor-product techniques of the previous subsection to determine $\mathcal{N}_{r,\rho}(\mathfrak{g})$ when the characteristic of $k$ is a prime $p$ and $\rho$ is the adjoint representation. The issue that we need to deal with is that certain binomial coefficients are now zero in $k$. First, we need a lemma describing binomial coefficients upon reduction modulo $p$.

**Lemma.** *Let $p$ be prime, $m \in \mathbb{N}$, $n \in \mathbb{Z}_+$, and assume $p^m \leqslant r < p^{m+1}$.*

(a) $p \mid \binom{p^m}{k}$ *for* $0 < k < p^m$.
(b) $p \nmid \binom{r}{k}$ *for* $2^n p^m \leqslant r \leqslant 2^{n+1} p^m$, $k = 2^n p^m$.
(c) $p \nmid \left[\binom{r}{k} - \binom{r}{k-1}\right]$ *for* $2^n p^m \leqslant r \leqslant 2^{n+1} p^m - 2$, $k = 2^n p^m$.
(d) $p \nmid \left[\binom{r}{k} - \binom{r}{k-1}\right]$ *for* $r = 2^{n+1} p^m - 1$ *or* $2^{n+1} p^m$ *when* $k = 2^n p^m + 1$ *and* $p \neq 2$.

**Proof.** The general fact we need is the following: if $r = ap^m + b$ with $0 \leqslant b < p^m$, and $k = cp^m + d$ with $0 \leqslant d < p^m$ then $\binom{r}{k}$ is congruent to $\binom{a}{c}\binom{b}{d}$ modulo $p$. This can be seen by looking at the identity $(x+1)^r = (x^{p^m}+1)^a(x+1)^b$ over $\mathbb{F}_p$ and comparing coefficients of $x^k = (x^{p^m})^c x^d$.

(a) We have $a = 1$, $b = 0$, $c = 0$ and $d = k$, so $p$ divides $\binom{b}{d}$. (b) This is a special case of the more general statement that if $k = cp^m \leqslant r < p^{m+1}$ then $p$ does not divide $\binom{r}{k}$. One can use the fact that $c \leqslant a < p$ and $d = 0$. (c) To compute $\binom{r}{k-1}$, note that $d = p^m - 1$ and $b < p^m - 1$ so $\binom{r}{k-1}$ is zero modulo $p$. Use (b) to see that $\binom{r}{k}$ is nonzero modulo $p$. For part (d), there are two cases. If $r = 2^{n+1}p^m - 1$ then $\binom{r}{k}$ is congruent to $-\binom{2^{n+1}-1}{2^n}$ and $\binom{r}{k-1}$ is congruent to $\binom{2^{n+1}-1}{2^n}$. Since $r < p^{m+1}$ it follows that $2^{n+1} - 1 < p$ and these binomial coefficients are nonzero modulo $p$. Their difference is twice the first one, which is still nonzero modulo $p$. For the second case when $r = 2^{n+1}p^m$, $\binom{r}{k}$ is zero modulo $p$ and $\binom{r}{k-1}$ is congruent to $\binom{2^{n+1}}{2^n}$ modulo $p$. $\square$

The theorem below gives a concrete description of $\mathcal{N}_{r,\rho}(\mathfrak{g})$ for a classical Lie algebra over an algebraically closed field of characteristic $p > 0$ where $\rho$ is the adjoint representation.

**Theorem.** *Let $\mathfrak{g}$ be a classical Lie algebra of type $X = A, B, C$, or $D$ over an algebraically closed field $k$ of characteristic $p > 0$, where $p$ is a good prime for $\mathfrak{g}$. Let $\rho$ be the adjoint representation of $\mathfrak{g}$ and* $\mathrm{std} : \mathfrak{g} \hookrightarrow \mathfrak{gl}_N(k)$ *the standard representation.*

(a) *When $1 \leqslant r < p$, $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is the union of the closures of the orbits given by the same partitions as in characteristic zero.*
(b) *When $p^m \leqslant r < p^{m+1}$, $m > 0$,*

$$\mathcal{N}_{r,\rho}(\mathfrak{g}) = \mathcal{N}_{p^m,\rho}(\mathfrak{g}) = \mathcal{N}_{p^m,\mathrm{std}}(\mathfrak{g}) = \overline{\mathcal{O}}_{(\lambda_{N,p^m})_X}. \tag{3.4.1}$$

**Proof.** Let $V$ be a vector space over $k$ of dimension $N$. In view of Section 2, it suffices to consider the representation of $\mathfrak{gl}(V)$ on $V \otimes V^*$ for $X = A$, on $S^2(V)$ for $X = C$, and on $\Lambda^2(V)$ for $X = B, D$. (In all but the first case, we are assuming $p > 2$, so the identifications of the adjoint representation in the previous subsection remain valid in characteristic $p$.) Let $x \in \mathfrak{gl}(V)$ be nilpotent, associated to the partition $\mu \vdash N$.

Let $\phi$ be the representation of $\mathfrak{gl}(V)$ on $V \otimes V^*$. Then $\phi(x^r)$ is given by (3.3.1). As in characteristic zero, choose a Jordan basis for $V$ with respect to $x$, labelled as before. Then $\phi(x^r) = 0$ if and only if every term in $x^r \cdot (v_0 \otimes w_0)$ is zero. In characteristic $p$, this may happen because $\binom{r}{k}$ may be 0 in $k$. However, when $1 \leqslant r < p$ this does not occur, and the analysis is identical to that in characteristic zero, with the answer given by the same partition(s) $\mu$.

Suppose that $r = p^m$, $m > 0$. All the binomial coefficients $\binom{p^m}{k} = 0$ for $0 < k < p^m$, so

$$x^{p^m} \cdot (v_0 \otimes w_0) = x^{p^m} v_0 \otimes w_0 + v_0 \otimes x^{p^m} w_0,$$

which is zero if and only if $p^m \geqslant \mu_1$. Thus $\mu = \lambda_{N,p^m}$. This verifies the last two equalities in (3.4.1) (when $X = A$).

Assume that $p^m \leqslant r < p^{m+1}$ and choose $n \geqslant 0$ such that $2^n p^m \leqslant r \leqslant 2^{n+1} p^m$. We will show that $\mathcal{N}_{r,\phi}(\mathfrak{gl}(V)) = \mathcal{N}_{2^n p^m,\phi}(\mathfrak{gl}(V))$. By induction on $r$, this will verify the first equality in (3.4.1), and thus prove the theorem (in type $A$). Since $\supset$ is obvious, it remains to check $\subset$. Let $x \in \mathcal{N}_{r,\phi}(\mathfrak{gl}(V))$ and consider the term in $x^r \cdot (v_0 \otimes w_0)$ with $k = 2^n p^m$ (recall (3.3.1)):

$$\binom{r}{2^n p^m} x^{2^n p^m} v_0 \otimes x^{r-2^n p^m} w_0 = 0.$$

By the lemma, the binomial coefficient is not zero. Since $r - 2^n p^m \leqslant 2^{n+1} p^m - 2^n p^m = 2^n p^m$, we must have $2^n p^m \geqslant \mu_1$. This proves $x \in \mathcal{N}_{2^n p^m,\rho}(\mathfrak{g})$, and hence the inclusion $\mathcal{N}_{r,\phi}(\mathfrak{gl}(V)) \subset \mathcal{N}_{2^n p^m,\phi}(\mathfrak{gl}(V))$.

Consider $X = C$ and let $\phi$ be the representation of $\mathfrak{gl}(V)$ on $S^2(V)$, with $p > 2$, and with notation as in characteristic zero. Then $\phi(x^r) = 0$ if and only if every term of (3.3.2) is zero. When $r < p$, the conditions are the same as in characteristic zero (because none of the binomial coefficients are zero). When $r = p^m$, (3.3.2) reduces to

$$x^{p^m} \cdot (v_0 \otimes v_0) = 2v_0 \otimes x^{p^m} v_0,$$

which is zero if and only if $p^m \geqslant \mu_1$. Thus $\mu = \lambda_{N,p^m}$, and this verifies the last two equalities in (3.4.1) for $X = C$. A similar argument to the one given in the previous paragraph shows that $\mathcal{N}_{r,\phi}(\mathfrak{gl}(V)) = \mathcal{N}_{2^n p^m,\phi}(\mathfrak{gl}(V))$ when $2^n p^m \leqslant r \leqslant \min(2^{n+1} p^m, p^{m+1} - 1)$. One needs to look at the term $k = r - 2^n p^m$ in (3.3.2).

Finally, let $X = B$ or $D$ and let $\phi$ be the representation of $\mathfrak{gl}(V)$ on $\Lambda^2(V)$, with $p > 2$. Then $\phi(x^r) = 0$ if and only if (3.3.3) and (3.3.4) hold. When $r < p$, the resulting conditions on $\mu$ are identical to the characteristic zero results, which proves (a).

When $r = p^m$ we get

$$x^{p^m} v_0 \wedge w_0 + v_0 \wedge x^{p^m} w_0 = 0 = x^{p^m} v_0 \wedge x v_0 + v_0 \wedge x^{p^m+1} v_0,$$

which is true if and only if $p^m \geqslant \mu_1$. Thus $\mu = \lambda_{N,p^m}$, and this verifies the last two equalities in (3.4.1) for $X = B$ or $D$.

Assume $2^n p^m \leqslant r \leqslant \min(2^{n+1} p^m, p^{m+1} - 1)$. We wish to show that $\mathcal{N}_{r,\phi}(\mathfrak{gl}(V)) = \mathcal{N}_{2^n p^m,\phi}(\mathfrak{gl}(V))$. Since we need to focus on the higher of the powers of $x$ in (3.3.4) (that being the "factor" which must be 0), let us rewrite (3.3.4) as

$$x^r \cdot (v_0 \wedge v_1) = \sum_{k=\lceil \frac{r+2}{2} \rceil}^{r+1} \left[ \binom{r}{k} - \binom{r}{k-1} \right] x^k v_0 \wedge x^{r+1-k} v_0. \tag{3.4.2}$$

When $r < 2^{n+1} p^m - 1$, the term $k = 2^n p^m$ occurs in the sum, and its coefficient is nonzero by the lemma, so we deduce that $2^n p^m \geqslant \mu_1$, as desired. When $r = 2^{n+1} p^m - 1$ or $2^{n+1} p^m$, the term $k = 2^n p^m + 1$ occurs, and its coefficient is nonzero, so we deduce that $\mathcal{N}_{r,\phi}(\mathfrak{gl}(V)) \subset \mathcal{N}_{2^n p^m+1,\phi}(\mathfrak{gl}(V))$. But we have already shown (inductively on $r$) that the latter set equals $\mathcal{N}_{2^n p^m,\phi}(\mathfrak{gl}(V))$, so we are done. □

### 3.5. Adjoint representation, via roots and Jacobson–Morozov

In this subsection we present an alternate method to compute $\mathcal{N}_{r,\rho}(\mathfrak{g})$ when the characteristic of $k$ is 0, $\mathfrak{g}$ is classical, and $\rho$ is the adjoint representation, using roots and the Jacobson–Morozov theorem.

Embed $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, with $\dim V = N$ via the standard representation. Let $x \in \mathfrak{gl}(V)$ be nilpotent. As usual, $x$ corresponds to a partition $\mu = (\mu_1, \ldots, \mu_t) \vdash N$. Using the Jacobson–Morozov theorem, $x$ belongs to a standard $\mathfrak{sl}_2$-triple whose semisimple element $h_\mu \in \mathfrak{sl}(V)$ is a diagonal matrix, with diagonal blocks $D(\mu_1), \ldots, D(\mu_t)$, where $D(\mu_i) = \operatorname{diag}(\mu_i - 1, \mu_i - 3, \ldots, -\mu_i + 3, -\mu_i + 1)$ [3]. In fact, the semisimple element of the $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ corresponding to $x$ has a $W$-conjugate $\tilde{h}_\mu$ in the dominant chamber, with diagonal entries a permutation of those of $h_\mu$, in the following form:

$$\tilde{h}_\mu = \begin{cases} \operatorname{diag}(a_1, \ldots, a_{l+1}), & a_1 \geqslant \cdots \geqslant a_{l+1} \geqslant 0 \text{ in type } A_l, \\ \operatorname{diag}(0, a_1, \ldots, a_l, -a_1, \ldots, -a_l), & a_1 \geqslant \cdots \geqslant a_l \geqslant 0 \text{ in type } B_l, \\ \operatorname{diag}(a_1, \ldots, a_l, -a_1, \ldots, -a_l), & a_1 \geqslant \cdots \geqslant a_l \geqslant 0 \text{ in type } C_l, \\ \operatorname{diag}(a_1, \ldots, a_l, -a_1, \ldots, -a_l), & a_1 \geqslant \cdots \geqslant a_{l-1} \geqslant |a_l| \text{ in type } D_l. \end{cases}$$

The weighted Dynkin diagram associated to $x$ is obtained by applying the simple roots to $\tilde{h}_\mu$ [3], and thus the $\mathfrak{sl}_2$-weights of the adjoint representation of $\mathfrak{g}$ are obtained by applying all the roots to $\tilde{h}_\mu$ (together with $l$ copies of the weight 0 coming from $\mathfrak{h}$). When these weights are organized into irreducible representations of $\mathfrak{sl}_2$, the dimensions of these irreducible $\mathfrak{sl}_2$-modules form the partition $\lambda(x) = (\lambda_1, \ldots, \lambda_n) \vdash \dim \mathfrak{g}$ corresponding to the nilpotent element $\rho(x)$ in the adjoint representation. In particular, $\lambda_1 = 1 + \alpha_0(\tilde{h}_\mu)$, where $\alpha_0$ is the highest root of $\mathfrak{g}$. Write roots in terms of the standard $\varepsilon_i$ basis of $\mathfrak{h}^*$.

In type $A_l$, $\alpha_0 = \varepsilon_1 - \varepsilon_{l+1}$, so $\alpha_0(\tilde{h}_\mu) = (\mu_1 - 1) - (-\mu_1 + 1) = 2\mu_1 - 2$ and $\lambda_1 = 2\mu_1 - 1$. Now $\rho(x)^r = 0 \Leftrightarrow \lambda_1 \leqslant r \Leftrightarrow \mu_1 \leqslant (r+1)/2$. This is the same condition as the one obtained in Section 3.3, so we get the same maximal partition $\lambda_{N,a}$ where $a = \lfloor (r+1)/2 \rfloor$.

Next let us consider type $C_l$. Then $\alpha_0 = 2\varepsilon_1$, so $\alpha_0(\tilde{h}_\mu) = 2(\mu_1 - 1)$ and again $\lambda_1 = 2\mu_1 - 1 \leqslant r$. As before, there is a unique maximal partition $\mu = \lambda_{N,a}$ satisfying this fcondition. Thus $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_{\mu_C}$, where $\mu_C$ is the $C$-collapse of $\mu$ as in Section 3.1.

Assume now that $\mathfrak{g}$ is of type $B_l$ or $D_l$. The highest root is $\alpha_0 = \varepsilon_1 + \varepsilon_2$. We have

$$\lambda_1 = 1 + \alpha_0\big(\tilde{h}_\mu\big) = \begin{cases} 2\mu_1 - 1, & \text{if } \mu_2 = \mu_1, \\ 2\mu_1 - 2, & \text{if } \mu_2 = \mu_1 - 1, \\ 2\mu_1 - 3, & \text{if } \mu_2 \leqslant \mu_1 - 2; \end{cases} \tag{3.5.1}$$

in the first and second cases, $\varepsilon_2(\tilde{h}_\mu) = \mu_2 - 1$, whereas in the third case, $\varepsilon_2(\tilde{h}_\mu) = \mu_1 - 3$.

We now apply the computation given in (3.5.1) in conjunction with the condition that $\lambda_1 \leqslant r$. When $r$ is odd, there may be two possible maximal partitions $\mu$ of $N$ which satisfy this condition (arising from the first and third cases of (3.5.1)). The first is $\lambda_{N,a}$. The second (which only occurs when $r \leqslant 2N - 3$), is the $\lambda'$ defined in Theorem 3.3(c).

When $r$ is even, we get a unique maximal partition $\mu$ satisfying (3.5.1), arising from the second case. When $r \geqslant 2N$, $\mu = \lambda_{N,a}$; otherwise, $\mu = \lambda''$ as defined in Theorem 3.3(d).

## 4. Exceptional groups

In this section we will provide tables listing the orbit closures which describe $\mathcal{N}_{r,\rho}(\mathfrak{g})$ for exceptional groups in good characteristics. The first column consists of a list of orbits. The remaining columns indicate the characteristic of a field under consideration. The column labelled "other" corresponds to those fields which have characteristic 0 or a good prime which is not listed separately. The cell values provide the $r$s for which $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is the orbit closure of the orbit(s) listed in that row. For example: for type $E_6$ with $\rho$ a minimal representation,

$$\mathcal{N}_{8,\rho}(E_6) = \overline{\mathcal{O}\big(D_5(a_1)\big)} \quad \text{for } p \neq 7 \quad \text{and}$$
$$\mathcal{N}_{8,\rho}(E_6) = \overline{\mathcal{O}\big(E_6(a_3)\big)} \quad \text{for } p = 7.$$

Here $\mathcal{O}(X)$ denotes the orbit corresponding to the Bala–Carter label $X$. In the tables, we have also indicated by $(*)$ the non-Richardson orbits which arise. The even orbits can be read off from the table in [2] and the odd Richardson orbits are listed in [6].

In order to obtain these results for exceptional groups we use the tables given in [11] on unipotent elements along with work of [14] to make the transition to nilpotent elements. Recall that for a faithful representation $\rho : \mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, we have $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \mathfrak{g} \cap \mathcal{N}_r(\mathfrak{gl}(V))$. For $\rho$ a minimal dimensional representation or the adjoint representation, given a nilpotent element $x$ in $\mathfrak{g}$, one can read off the corresponding partition $\lambda(x)$ of $N := \dim V$ in [11]. We can then invoke Proposition 2.4 by comparing the dominance ordering with the ordering on orbit closures given by the Hasse diagrams (see [2, pp. 439–445]) for exceptional groups to express $\mathcal{N}_{r,\rho}(\mathfrak{g})$ as a union of orbit closures.

### *4.1. Minimal representation*

We first consider the minimal dimensional irreducible representations for the exceptional groups. For $E_6$ there are two irreducible representations of minimal dimension. They are dual to one another, but since $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \mathcal{N}_{r,\rho^*}(\mathfrak{g})$, where $\rho^*$ is the dual representation of $\rho$, it suffices to consider only one of them. For the minimal dimensional representations $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is irreducible except in four cases involving only $E_6$ and $E_7$:

$$\text{Type } E_6: \quad \mathcal{N}_{7,\rho}(\mathfrak{g}) = \overline{\mathcal{O}(A_4 + A_1)} \cup \overline{\mathcal{O}(D_4)} \quad \text{for } p \neq 5, 7.$$
$$\text{Type } E_7: \quad \mathcal{N}_{7,\rho}(\mathfrak{g}) = \overline{\mathcal{O}(A_4 + A_2)} \cup \overline{\mathcal{O}(D_4)} \quad \text{for } p \neq 7,$$
$$\mathcal{N}_{11,\rho}(\mathfrak{g}) = \overline{\mathcal{O}(D_5 + A_1)} \cup \overline{\mathcal{O}(A_6)} \quad \text{for } p \neq 11,$$
$$\mathcal{N}_{17,\rho}(\mathfrak{g}) = \overline{\mathcal{O}(E_6)} \cup \overline{\mathcal{O}(E_7(a_3))} \quad \text{for } p \neq 17, 13.$$

The tables below describe $\mathcal{N}_{r,\rho}(\mathfrak{g})$ when $\rho$ is a minimal dimensional representation and $\mathfrak{g}$ is an exceptional Lie algebra. The conventions are explained at the beginning of Section 4.

Type $E_6$

| Orbits | other | $p=13$ | $p=11$ | $p=7$ | $p=5$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| $A_1$ (∗) | 2 | 2 | 2 | 2 | 2 |
| $A_2$ | 3 | 3 | 3 | 3 | 3 |
| $A_2+2A_1$ | 4 | 4 | 4 | 4 | 4 |
| $D_4(a_1)$ | 5–6 | 5–6 | 5–6 | 5–6 | |
| $A_4+A_1$ | | | | | 5–7 |
| $A_4+A_1,\ D_4$ | 7 | 7 | 7 | | |
| $D_5(a_1)$ | 8 | 8 | 8 | | 8 |
| $E_6(a_3)$ | 9–10 | 9–10 | 9–10 | 7–10 | 9–10 |
| $D_5$ | 11–12 | 11–12 | | 11–12 | 11–12 |
| $E_6(a_1)$ | 13–16 | | 11–16 | 13–16 | 13–16 |
| $E_6$ | 17+ | 13+ | 17+ | 17+ | 17+ |

Type $E_7$

| Orbits | other | $p=23$ | $p=19$ | $p=17$ | $p=13$ | $p=11$ | $p=7$ | $p=5$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $A_1$ (∗) | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $A_2$ | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| $A_2+3A_1$ | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| $D_4(a_1)$ | 5 | 5 | 5 | 5 | 5 | 5 | 5 | |
| $A_3+A_2+A_1$ | 6 | 6 | 6 | 6 | 6 | 6 | 6 | |
| $A_4+A_2$ | | | | | | | | 5–6 |
| $A_4+A_2,\ D_4$ | 7 | 7 | 7 | 7 | 7 | 7 | | 7 |
| $D_5(a_1)+A_1$ | 8 | 8 | 8 | 8 | 8 | 8 | | 8 |
| $E_6(a_3)$ | 9 | 9 | 9 | 9 | 9 | 9 | | 9 |
| $E_7(a_5)$ | 10 | 10 | 10 | 10 | 10 | 10 | | 10 |
| $A_6$ | | | | | | | 7–10 | |
| $D_5+A_1,\ A_6$ | 11 | 11 | 11 | 11 | 11 | | 11 | 11 |
| $E_7(a_4)$ | 12 | 12 | 12 | 12 | 12 | | 12 | 12 |
| $E_6(a_1)$ | 13–15 | 13–15 | 13–15 | 13–15 | | | 13–15 | 13–15 |
| $E_7(a_3)$ | 16 | 16 | 16 | 16 | | 11–16 | 16 | 16 |
| $E_6,\ E_7(a_3)$ | 17 | 17 | 17 | | | 17 | 17 | 17 |
| $E_7(a_2)$ | 18–21 | 18–21 | 18 | | 13–21 | 18–21 | 18–21 | 18–21 |
| $E_7(a_1)$ | 22–27 | 22 | | 17–27 | 22–27 | 22–27 | 22–27 | 22 |
| $E_7$ | 28+ | 23+ | 19+ | 28+ | 28+ | 28+ | 28+ | 23+ |

Type $F_4$

| Orbits | other | $p=13$ | $p=7$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| $A_1$ (∗) | 2 | 2 | 2 |
| $A_2$ | 3 | 3 | 3 |
| $A_2+\widetilde{A}_1$ (∗) | 4 | 4 | 4 |
| $F_4(a_3)$ | 5–6 | 5–6 | 5–6 |
| $B_3$ | 7–8 | 7–8 | |
| $F_4(a_2)$ | 9–10 | 9–10 | 7–10 |
| $F_4(a_1)$ | 11–16 | 11–12 | 11–16 |
| $F_4$ | 17+ | 13+ | 17+ |

Type $G_2$

| Orbits | other |
|---|---|
| 1 | 1 |
| $A_1$ (∗) | 2 |
| $G_2(a_1)$ | 3–6 |
| $G_2$ | 7+ |

Type $E_8$

| Orbits | other | $p=53$ | $p=47$ | $p=43$ | $p=41$ | $p=37$ | $p=31$ | $p=29$ | $p=23$ | $p=19$ | $p=17$ | $p=13$ | $p=11$ | $p=7$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 |
| $2A_1$ $(*)$ | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| $4A_1$ $(*)$ | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| $2A_2$ | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| $2A_2+2A_1$ $(*)$ | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| $D_4(a_1)+A_2$ | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | |
| $2A_3$ $(*)$ | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | |
| $A_4+A_2+A_1$ | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | |
| $A_4+A_3$ $(*)$ | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | |
| $E_8(a_7)$ | 11–12 | 11–12 | 11–12 | 11–12 | 11–12 | 11–12 | 11–12 | 11–12 | 11–12 | 11–12 | 11–12 | 11–12 | | |
| $A_6+A_1$ | 13–14 | 13–14 | 13–14 | 13–14 | 13–14 | 13–14 | 13–14 | 13–14 | 13–14 | 13–14 | 13–14 | | | 7–14 |
| $D_7(a_2)$ | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | | | |
| $A_7$ $(*)$ | 16 | 16 | 16 | 16 | 16 | 16 | 16 | 16 | 16 | 16 | 16 | | | |
| $E_8(b_6)$ | 17–18 | 17–18 | 17–18 | 17–18 | 17–18 | 17–18 | 17–18 | 17–18 | 17–18 | 17–18 | | | | 15–18 |
| $E_8(a_6)$ | 19–22 | 19–22 | 19–22 | 19–22 | 19–22 | 19–22 | 19–22 | 19–22 | 19–22 | | | | 11–22 | 19–21 |
| $D_7$ $(*)$ | | | | | | | | | | | | | | 22 |
| $E_8(a_5)$ | 23–26 | 23–26 | 23–26 | 23–26 | 23–26 | 23–26 | 23–26 | 23–26 | | | | 13–26 | | 23–26 |
| $E_8(b_4)$ | 27–28 | 27–28 | 27–28 | 27–28 | 27–28 | 27–28 | 27–28 | 27–28 | | | | | 23–28 | 27–28 |
| $E_8(a_4)$ | 29–34 | 29–34 | 29–34 | 29–34 | 29–34 | 29–34 | 29–30 | | | | 17–34 | 27–34 | 29–34 | 29–34 |
| $E_8(a_3)$ | 35–38 | 35–38 | 35–38 | 35–38 | 35–38 | 35–36 | | | | 19–38 | | 35–38 | 35–38 | 35–38 |
| $E_8(a_2)$ | 39–46 | 39–46 | 39–46 | 39–42 | 39–40 | | | | 23–46 | | 35–46 | 39–46 | 39–46 | 39–46 |
| $E_8(a_1)$ | 47–58 | 47–53 | | | | | | 29–58 | 47–58 | 39–58 | 47–58 | 47–58 | 47–58 | 47–48 |
| $E_8$ | 59+ | 53+ | 47+ | 43+ | 41+ | 37+ | 31+ | 59+ | 59+ | 59+ | 59+ | 59+ | 59+ | 49+ |

### *4.2. Adjoint representation*

We first remark that the adjoint representation for $E_8$ coincides with the minimal dimensional (nontrivial) irreducible representation. When $\rho$ is the adjoint representation, $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is irreducible for exceptional Lie algebras in all but 2 cases. Again these cases occur in $E_6$ and $E_7$.

$$\text{Type } E_6\text{:} \quad \mathcal{N}_{5,\rho}(\mathfrak{g}) = \overline{\mathcal{O}(2A_2)} \cup \overline{\mathcal{O}(A_2 + 2A_1)} \quad \text{for } p \neq 5.$$

$$\text{Type } E_7\text{:} \quad \mathcal{N}_{5,\rho}(\mathfrak{g}) = \overline{\mathcal{O}(A_2 + 3A_1)} \cup \overline{\mathcal{O}(2A_2)} \quad \text{for } p \neq 5.$$

The tables below describe $\mathcal{N}_{r,\rho}(\mathfrak{g})$ for $\mathfrak{g}$ an exceptional Lie algebra and $\rho$ the adjoint representation. For $E_8$, the reader is referred back to the previous section for the table.

Type $E_6$

| Orbits | other | $p=19$ | $p=17$ | $p=13$ | $p=11$ | $p=7$ | $p=5$ |
|---|---|---|---|---|---|---|---|
| 1 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 |
| $2A_1$ | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| $3A_1$ $(*)$ | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| $2A_2,\ A_2+2A_1$ | 5 | 5 | 5 | 5 | 5 | 5 | |
| $2A_2+A_1$ $(*)$ | 6 | 6 | 6 | 6 | 6 | 6 | |
| $D_4(a_1)$ | 7–8 | 7–8 | 7–8 | 7–8 | 7–8 | | |
| $A_4+A_1$ | 9–10 | 9–10 | 9–10 | 9–10 | 9–10 | | 5–10 |
| $E_6(a_3)$ | 11–14 | 11–14 | 11–14 | 11–12 | | 7–14 | 11–14 |
| $D_5$ | 15–16 | 15–16 | 15–16 | | | | 15–16 |
| $E_6(a_1)$ | 17–22 | 17–18 | | | 11–22 | 15–22 | 17–22 |
| $E_6$ | 23+ | 19+ | 17+ | 13+ | 23+ | 23+ | 23+ |

Type $E_7$

| Orbits | other | $p=23$ | $p=19$ | $p=17$ | $p=13$ | $p=11$ | $p=7$ | $p=5$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 |
| $(3A_1)''$ | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| $4A_1$ $(*)$ | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| $A_2+3A_1,\ 2A_2$ | 5 | 5 | 5 | 5 | 5 | 5 | 5 | |
| $2A_2+A_1$ $(*)$ | 6 | 6 | 6 | 6 | 6 | 6 | 6 | |
| $A_3+A_2+A_1$ | 7–8 | 7–8 | 7–8 | 7–8 | 7–8 | 7–8 | | |
| $A_4+A_2$ | 9–10 | 9–10 | 9–10 | 9–10 | 9–10 | 9–10 | | 5–10 |
| $E_7(a_5)$ | 11–12 | 11–12 | 11–12 | 11–12 | 11–12 | | | 11–14 |
| $A_6$ | 13–14 | 13–14 | 13–14 | 13–14 | | | 7–14 | |
| $E_7(a_4)$ | 15–16 | 15–16 | 15–16 | 15–16 | | | | 15–16 |
| $E_6(a_1)$ | 17–18 | 17–18 | 17–18 | | | | 15–18 | 17–18 |
| $E_7(a_3)$ | 19 | 19 | | | | 11–22 | 19–22 | 19–22 |
| $E_7(a_2)$ | 20–25 | 20–22 | | | 13–26 | | 23–26 | 23–24 |
| $E_7(a_1)$ | 26–27 | | | 17-34 | 27–34 | 23–34 | 27–34 | |
| $E_7$ | 28–35 | 23+ | 19+ | 35+ | 35+ | 35+ | 35+ | 25+ |

Type $F_4$

| Orbits | other | $p=19$ | $p=17$ | $p=13$ | $p=11$ | $p=7$ | $p=5$ |
|---|---|---|---|---|---|---|---|
| 1 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 | 1–2 |
| $\widetilde{A_1}$ (∗) | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| $A_1+\widetilde{A_1}$ (∗) | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| $A_2+\widetilde{A_1}$ (∗) | 5 | 5 | 5 | 5 | 5 | 5 | |
| $\widetilde{A_2}+A_1$ (∗) | 6 | 6 | 6 | 6 | 6 | 6 | |
| $F_4(a_3)$ | 7–10 | 7–10 | 7–10 | 7–10 | 7–10 | | 5–10 |
| $F_4(a_2)$ | 11–14 | 11–14 | 11–14 | 11–12 | | 7–14 | 11–14 |
| $F_4(a_1)$ | 15–22 | 15–18 | 15–16 | | 11–22 | 15–22 | 15–22 |
| $F_4$ | 23+ | 19+ | 17+ | 13+ | 23+ | 23+ | 23+ |

Type $G_2$

| Orbits | other | $p=7$ |
|---|---|---|
| 1 | 1–2 | 1–2 |
| $A_1$ (∗) | 3 | 3 |
| $\widetilde{A_1}$ (∗) | 4 | 4 |
| $G_2(a_1)$ | 5–10 | 5–6 |
| $G_2$ | 11+ | 7+ |

## 5. Applications

*5.1.* Let $G$ be a simple algebraic group scheme over $k$ where the characteristic of $k$ is $p>0$. The maximal ideal spectrum of the cohomology ring $\mathrm{H}^{2\bullet}(G_1,k)$ can be identified with the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$. The following results give a precise description of $\mathcal{N}_1(\mathfrak{g})$ in terms of closures of orbits. For the classical groups, one can use the partition labelling in order to describe the restricted nullcone closures as the closure of certain Richardson orbits. Given $N>0$, express $N=dp+s$ where $0\leqslant s\leqslant p-1$ and recall that $\lambda_{N,p}=(p^d,s)\vdash N$. The first result is an immediate consequence of Theorem 3.2.

**Theorem.** *Let $G$ be a simple classical algebraic group over $k$ where* $\operatorname{char} k=p>0$ *is good.*

(a) *If $\Phi$ is of type $A_l$ then* $\mathcal{N}_1(\mathfrak{g})=\overline{O}_{\lambda_{l+1,p}}$.
(b) *If $\Phi$ is of type $B_l$ then* $\mathcal{N}_1(\mathfrak{g})=\overline{O}_{(\lambda_{2l+1,p})_B}$.
(c) *If $\Phi$ is of type $C_l$ then* $\mathcal{N}_1(\mathfrak{g})=\overline{O}_{(\lambda_{2l,p})_C}$.
(d) *If $\Phi$ is of type $D_l$ then* $\mathcal{N}_1(\mathfrak{g})=\overline{O}_{(\lambda_{2l,p})_D}$.

*5.2.* The following theorem provides a description of the restricted nullcone for exceptional Lie algebras. These results are easily deduced by setting $r=p$ and using the tables in Sections 4.1 or 4.2.

**Theorem.** *Let $G$ be an exceptional algebraic group with $p$ good. Then*

(a) *If $\Phi$ is of type $E_6$ then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = E_6$ ($p \geqslant 13$), $E_6(a_1)$ ($p = 11$), $E_6(a_3)$ ($p = 7$), $A_4 + A_1$ ($p = 5$).*

(b) *If $\Phi$ is of type $E_7$ then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = E_7$ ($p \geqslant 19$), $E_7(a_1)$ ($p = 17$), $E_7(a_2)$ ($p = 13$), $E_7(a_3)$ ($p = 11$), $A_6$ ($p = 7$), $A_4 + A_2$ ($p = 5$).*

(c) *If $\Phi$ is of type $E_8$ then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = E_8$ ($p \geqslant 31$), $E_8(a_1)$ ($p = 29$), $E_8(a_2)$ ($p = 23$), $E_8(a_3)$ ($p = 19$), $E_8(a_4)$ ($p = 17$), $E_8(a_5)$ ($p = 13$), $E_8(a_6)$ ($p = 11$), $A_6 + A_1$ ($p = 7$).*

(d) *If $\Phi$ is of type $F_4$ then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = F_4$ ($p \geqslant 13$), $F_4(a_1)$ ($p = 11$), $F_4(a_2)$ ($p = 7$), $F_4(a_3)$ ($p = 5$).*

(e) *If $\Phi$ is of type $G_2$ then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = G_2$ ($p \geqslant 7$), $G_2(a_1)$ ($p = 5$).*

## 6. VIGRE Algebra Group at the University of Georgia

This project was initiated during the Fall 2003 Semester under the Vertical Integration of Research and Education (VIGRE) Program sponsored by the National Science Foundation (NSF) at the Department of Mathematics at the University of Georgia (UGA). The VIGRE Algebra Group consists of 5 faculty members, 2 postdoctoral fellows and 5 graduate students. The group was led by David J. Benson, Brian D. Boe and Daniel K. Nakano. The email addresses of the members of the group are given below.

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