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Varieties of nilpotent elements for simple Lie algebras II: Bad primes

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1. Introduction

1.1. Let G be a simple algebraic group over an algebraically closed field k of characteristic $p > 0$ and let $\mathfrak{g}$ be the (restricted) Lie algebra of G with pth power map $[p]$. The maximal ideal spectrum of the cohomology ring of the restricted enveloping algebra $\operatorname{Maxspec}(\mathrm{H}^{2\bullet}(u(\mathfrak{g}), k))$ can be identified with the variety $\mathcal{N}_1(\mathfrak{g}) = \{x \in \mathfrak{g}\colon x^{[p]} = 0\}$. When the characteristic of the field is a good prime, this variety was first described as the closure of a certain Richardson orbit by Carlson, Lin, Nakano and Parshall [4]. Their methods used the techniques developed by Nakano, Parshall and Vella [23] which involved the verification of a conjecture of Jantzen on the support varieties of Weyl modules. More recently, in [30] the authors investigated a more general question. Let ρ be a finite-dimensional representation of $\mathfrak{g}$ which is realized as the derivative of a representation of G. For $r \geqslant 0$, set

$$\mathcal{N}_{r,\rho}(\mathfrak{g}) = \left\{x \in \mathcal{N}(\mathfrak{g})\colon \rho(x)^r = 0\right\},$$

where $\mathcal{N}(\mathfrak{g})$ is the variety of nilpotent elements of $\mathfrak{g}$. The variety $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is a G-invariant subvariety of $\mathcal{N}(\mathfrak{g})$. Since there are finitely many G-orbits on $\mathcal{N}(\mathfrak{g})$, one can ask, how can

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this variety can be expressed as a finite union of orbit closures. The authors determined $\mathcal{N}_{r,\rho}(\mathfrak{g})$ when ρ is either a minimal dimensional irreducible representation or the adjoint representation of $\mathfrak{g}$ and the characteristic of the field k is a good prime relative to the underlying root system of $\mathfrak{g}$. The methods used did not involve the machinery in [23] and one can easily recover all the calculations in [4] by setting $r = p$.

One of the main objectives of this paper is to determine the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$ over fields of bad characteristic. Since $\mathcal{N}_1(\mathfrak{g}) \cong \operatorname{Maxspec}(\mathrm{H}^{2\bullet}(u(\mathfrak{g}), k))$, the determination of $\mathcal{N}_1(\mathfrak{g})$ for bad primes is a key step in the study of support varieties for restricted Lie algebras. We have recently used our determination of $\mathcal{N}_1(\mathfrak{g})$ to calculate support varieties of Weyl modules over fields of bad characteristic [31]. Our results should also be useful in the investigation of support varieties of nonrestricted representations for $\mathfrak{g}$ (see [10, Theorem 4.2]).

We will invoke the general strategy that was established in [30] by providing descriptions of the varieties of both unipotent and nilpotent elements of order r in $\mathfrak{g}$ when the characteristic of the underlying field is a bad prime. Let ρ be a finite-dimensional representation of G and let $\mathcal{U}(G)$ be the variety of unipotent elements in G. One can define a subvariety $\mathcal{U}_{r,\rho}(G)$ of $\mathcal{U}(G)$, which is analogous to $\mathcal{N}_{r,\rho}(\mathfrak{g})$, by letting

$$\mathcal{U}_{r,\rho}(G) = \{x \in \mathcal{U}(G)\colon \ (\rho(x) - 1)^r = 0\}.$$

An analog of the restricted nullcone for the unipotent variety would be $\mathcal{U}_1(G) = \{x \in \mathcal{U}(G)\colon\ x^p = 1\}$. We will refer to this as the *restricted unipotent variety*. Since there are finitely many unipotent classes it is also reasonable to try to describe $\mathcal{U}_{r,\rho}(G)$ as a finite union of closures of such classes.

For good primes, there exists a G-equivariant isomorphism between $\mathcal{U}(G)$ and $\mathcal{N}(\mathfrak{g})$. McNinch [19] has shown that, under mild hypotheses on G, the Bardsley–Richardson isomorphism restricts to give a G-equivariant isomorphism between $\mathcal{U}_1(G)$ and $\mathcal{N}_1(\mathfrak{g})$. In particular, there are bijective correspondences between nilpotent orbits in $\mathcal{N}(\mathfrak{g})$ (respectively $\mathcal{N}_1(\mathfrak{g})$) and unipotent classes in $\mathcal{U}(G)$ (respectively $\mathcal{U}_1(G)$) over fields of good characteristic.

From our computations of $\mathcal{N}_{r,\rho}(\mathfrak{g})$ and $\mathcal{U}_{r,\rho}(G)$, we set $r = p$ to obtain concrete descriptions of the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$ and the restricted unipotent variety $\mathcal{U}_1(G)$ for bad primes. Our results in conjunction with work in [4] demonstrate that these varieties are indeed irreducible for all primes, thus answering an old question posed by Friedlander and Parshall [9, (3.4)].

Furthermore, in this process, we establish a remarkable fact: there is an order preserving bijection between the nilpotent orbits in $\mathcal{N}_1(\mathfrak{g})$ and unipotent classes in $\mathcal{U}_1(G)$ for all primes. This is quite surprising because for bad primes the total number of nilpotent orbits in $\mathcal{N}(\mathfrak{g})$ and unipotent classes in $\mathcal{U}(G)$ are often different (see [5, §5.11]). Computational methods were essential for obtaining these results.

The paper is organized as follows. After the notation for the paper is introduced, we look at $\mathcal{N}_{r,\rho}(\mathfrak{g})$ and $\mathcal{U}_{r,\rho}(G)$ where ρ is the standard representation for classical simple groups G in Section 2. The results in this section will use early work of Großer [11], Hesselink [12] and Spaltenstein [25,26] who determined the nilpotent orbits and unipotent classes and their closure orderings in this setting. In Section 3, we determine $\mathcal{U}_{r,\rho}(G)$

where ρ is the minimal or adjoint representation and $\mathcal{N}_{r,\mathrm{ad}}(\mathfrak{g})$ for the exceptional groups G. The computations for $\mathcal{U}_{r,\rho}(G)$ use the tables in [16,17] which list the Jordan block sizes on the images under ρ of unipotent classes. For the computations of $\mathcal{N}_{r,\mathrm{ad}}(\mathfrak{g})$, we first need explicit orbit representatives for $\mathcal{N}(\mathfrak{g})$. A complete set of representatives was not available in the current literature so we constructed representatives by hand and verified them by using computer calculations. We have made our complete set of nilpotent orbit representatives available in Section 5. Once these representatives were determined, we wrote a program in MAGMA [1,2] to determine Jordan block sizes. This data is given in Section 6 and complements Lawther's previous calculations. Using the information in Sections 2 and 3, explicit descriptions of $\mathcal{N}_1(\mathfrak{g})$ and $\mathcal{U}_1(G)$ are given in Section 4 when p is bad. For exceptional groups, the orbit closure relations in these varieties are given in Section 7.

1.2. Notation

Let G be a simple algebraic group defined over k (see [5, 1.11]). We will not distinguish G in terms of its isogeny class in this paper because this will not have any bearing on our discussion of nilpotent orbits and unipotent classes. Let T be a maximal torus of G. The root system associated to the pair (G, T) is denoted by Φ and identified with a subset of the set of weights $X(T)$. Let Φ^+ be the corresponding set of positive roots and Φ^- be the set of negative roots. The set of simple roots determined by Φ^+ is $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. The Weyl group corresponding to Φ is W. We will use throughout this paper the ordering of simple roots given in [14] following Bourbaki.

In this paper we will always assume that p is a bad prime for Φ. A list of bad primes is provided below.

- Φ of type A_l, no primes;
- Φ of type B_l, C_l, D_l, $p = 2$;
- Φ of type E_6, E_7, F_4, G_2, $p = 2, 3$;
- Φ of type E_8, $p = 2, 3, 5$.

Set $\mathfrak{g} = \operatorname{Lie} G$. Let $\mathcal{N}(\mathfrak{g})$ be the variety of nilpotent elements of $\mathfrak{g}$, which is often called the nullcone. The group G acts on $\mathcal{N}(\mathfrak{g})$ via the adjoint action and $\mathcal{N}(\mathfrak{g})$ has finitely many G-orbits. The nullcone is an irreducible variety of dimension equal to $|\Phi|$. The unipotent variety in G is denoted by $\mathcal{U}(G)$. When k is an algebraically closed field and the characteristic is a good prime the classification and structures of these orbits coincide with the orbit theory for complex simple Lie algebras (see [5,6,15]). If the characteristic of the field is bad then the number of unipotent classes in $\mathcal{U}(G)$ and nilpotent orbits in $\mathcal{N}(\mathfrak{g})$ need not coincide. The unipotent classes were determined in [12,21,22,24–26,29] and the nilpotent orbits were determined in [12,13,26–28]. The definitions of $\mathcal{N}_{r,\rho}(\mathfrak{g})$, $\mathcal{N}_1(\mathfrak{g})$, $\mathcal{U}_{r,\rho}(G)$, and $\mathcal{U}_1(G)$ were provided in Section 1.1.

We should remark that one can identify $\mathcal{N}_1(\mathfrak{g}) = \mathcal{N}_{p,\rho}(\mathfrak{g})$ (respectively $\mathcal{U}_1(G) = \mathcal{U}_{p,\rho}(G)$) if the representation satisfies the condition $\mathcal{N}(\mathfrak{g}) \cap \ker\rho = \{0\}$ (respectively $\mathcal{U}(G) \cap \ker\rho = \{1\}$). These conditions will hold for the representations that we are considering in the paper.

2. Classical groups and Lie algebras

In this section we consider nilpotent and unipotent orbits for $G = Sp(N)$ or $O(N)$ where $\operatorname{char} k = 2$, which is the only bad prime for types B, C and D. The group $O(N)$ is a disconnected algebraic group when N is even. The connected component of the identity in $O(N)$ is denoted $SO(N)$. (This is the convention of Hesselink [12] and Spaltenstein [26], researchers in nilpotent orbits in characteristic two. We have chosen to maintain their notation, rather than the (perhaps more standard) notation $\Omega(N)$ preferred by researchers in finite groups.) Note that this definition of $SO(N)$ differs from the conventional definition as the set of orthogonal matrices of determinant one.

2.1. Nilpotent orbit closures

The nilpotent orbits in $\mathfrak{g}$ have been classified by Hesselink [12]. They are parametrized by pairs consisting of a partition μ of N and an *index function* $\chi : I \to \mathbb{Z}$, where I is the set of (nonzero) parts of μ. The partition gives, as usual, the sizes of the Jordan blocks for any representative of the orbit. The fact that extra data (the index function) is needed to specify an orbit means that, in contrast to the situation in good characteristic (aside from the very even case for type D), two nilpotent elements of $\mathfrak{g}$ having the same Jordan form need not be in the same G-orbit.

There is a one-to-one correspondence between nilpotent orbits and pairs μ, χ satisfying the following conditions. Write $n(m)$ for the multiplicity of $m \in I$ as a part in μ.

(2.1.1) (1) For $m > l$ in I, $\chi(m) \geqslant \chi(l)$ and $m - \chi(m) \geqslant l - \chi(l)$.
- (2) For $G = Sp(N)$ and $m \in I$:
  - (a) $0 \leqslant \chi(m) \leqslant m/2$;
  - (b) $\chi(m) = m/2$ if $n(m)$ is odd.
- (3) For $G = O(N)$ and $m \in I$:
  - (a) $m/2 \leqslant \chi(m) \leqslant m$;
  - (b) $\chi(m) = m$ if $n(m)$ is odd;
  - (c) $\{m \in I\colon\ n(m) \text{ is odd}\} = \{i, i-1\} \cap \mathbb{N}$ for some $i \in \mathbb{Z}$.

If $I = \{m_1 > m_2 > \cdots\}$, Hesselink displays the pair μ, χ as a *symbol*

$$\mu_\chi = \left({m_1}^{n(m_1)}_{\chi(m_1)}, {m_2}^{n(m_2)}_{\chi(m_2)}, \ldots\right).$$

The inclusion relations between nilpotent orbit closures (the *Hasse diagram*) have been worked out by Großer [11] for $Sp(N)$, and by Spaltenstein [26] for $O(N)$, who presents these orderings in a unified way, as follows.

If μ is any finite sequence, set $|\mu| = \sum_i \mu_i$. If μ, λ are sequences with $|\mu| = |\lambda|$, extend with zeros if necessary so they have the same length, and write $\mu \trianglelefteq \lambda$ if $\sum_{j \leqslant i} \mu_j \leqslant \sum_{j \leqslant i} \lambda_j$ for all i (a partial order extending the usual dominance order on partitions). If now μ is a partition of N (written as usual with parts nonincreasing), let μ^* be the dual partition of N, with parts $\mu^*_i = \#\{j\colon\ \mu_j \geqslant i\}$. Let $\mathcal{P}$ be the set of pairs (α, β) of partitions such that $|\alpha| + |\beta| = n$; assume α and β have the same length (by extending with zeros as necessary). For $(\alpha,\beta)\in\mathcal{P}$, let $\alpha\int\beta=(\alpha_1,\beta_1,\alpha_2,\beta_2,\ldots)$. For $(\alpha,\beta),(\alpha',\beta')\in\mathcal{P}$, write $(\alpha,\beta)\leqslant(\alpha',\beta')$ if $\alpha\int\beta\trianglelefteq\alpha'\int\beta'$.

Let $\mathcal{H}_X$ denote the set of Hesselink pairs (μ,χ) for type X_n (where $X\in\{B,C,D\}$ is the underlying root system for the group G). There is an injective map $\tau_X:\mathcal{H}_X\to\mathcal{P}$ with $\tau_X(\mu,\chi)=(\alpha,\beta)$ defined as follows.

If $X=B$, there is a unique j for which μ_j^* is odd. Then

$$\alpha_i=\begin{cases}\chi(\mu_{2i})-1, & \text{if } 2i<\mu_j^*,\\ \chi(\mu_{2i}), & \text{if } 2i>\mu_j^*,\end{cases}\qquad \beta_i=\begin{cases}\mu_{2i}-\chi(\mu_{2i})+1, & \text{if } 2i<\mu_j^*,\\ 0, & \text{if } 2i>\mu_j^*.\end{cases}$$

Suppose $X=C$. Define $J=\{i\colon\ \chi(\mu_i)=\mu_i/2\}$, $J_0=\{i\colon\ i\notin J \text{ and } \mu_{\mu_i}^* \text{ is even}\}$, $J_1=\{i\colon\ i\notin J\cup J_0\}$. Then

$$\alpha_i=\begin{cases}\chi(\mu_{2i-1}), & \text{if } 2i-1\in J\cup J_0,\\ \mu_{2i-1}-\chi(\mu_{2i-1}), & \text{if } 2i-1\in J_1,\end{cases}$$
$$\beta_i=\begin{cases}\chi(\mu_{2i}), & \text{if } 2i\in J\cup J_1,\\ \mu_{2i}-\chi(\mu_{2i}), & \text{if } 2i\in J_0.\end{cases}$$

Finally if $X=D$ then

$$\alpha_i=\chi(\mu_{2i}),\qquad \beta_i=\mu_{2i}-\chi(\mu_{2i}).$$

If $\mathcal{O}'$ and $\mathcal{O}$ are G-orbits, write $\mathcal{O}'\leqslant\mathcal{O}$ if $\mathcal{O}'\subseteq\overline{\mathcal{O}}$. Write $\mathcal{O}(\mu,\chi)$ for the nilpotent orbit associated to the pair $(\mu,\chi)\in\mathcal{H}_X$.

Theorem (Großer, Spaltenstein). *Let $G=Sp(N)$ or $O(N)$ be of type $X\in\{B,C,D\}$. Let $(\mu',\chi'),(\mu,\chi)\in\mathcal{H}_X$. Then $\mathcal{O}(\mu',\chi')\leqslant\mathcal{O}(\mu,\chi)$ if and only if $\tau_X(\mu',\chi')\leqslant\tau_X(\mu,\chi)$.*

Remark. When N is even, certain $O(N)$ orbits split into two-equidimensional orbits for $SO(N)$, specifically, those having all parts and multiplicities even and $\chi(\mu_i)=\mu_i/2$ for all i [12, Proposition 5.4]. Whenever $\mathcal{O}\leqslant\mathcal{O}'$ are $O(N)$ orbits consisting of two $SO(N)$ orbits, there exists an $O(N)$ orbit $\mathcal{O}''$ which does not split, satisfying $\mathcal{O}\leqslant\mathcal{O}''\leqslant\mathcal{O}'$ [26, 4.1]. This means that the $SO(N)$ poset may be deduced from the $O(N)$ poset.

2.2. The varieties $\mathcal{N}_{r,\min}(\mathfrak{g})$

In this subsection we characterize, in terms of nilpotent orbit closures, the varieties $\mathcal{N}_{r,\rho}(\mathfrak{g})=\{x\in\mathcal{N}(\mathfrak{g})\colon\ \rho(x)^r=0\}$, when ρ is the minimal irreducible representation. Notice that ρ is faithful, so it certainly satisfies the condition $\mathcal{N}(\mathfrak{g})\cap\ker\rho=\{0\}$ mentioned at the end of Section 1.2.

Theorem. *Let $G=Sp(N)$ or $O(N)$ be of type X_n where $X\in\{B,C,D\}$. Let $\mathfrak{g}=Lie(G)$ and let ρ be the minimal irreducible representation of $\mathfrak{g}$. Then for $r\in\mathbb{N}$, $\mathcal{N}_{r,\rho}(\mathfrak{g})=\overline{\mathcal{O}(\mu,\chi)}$ where μ_χ is given as follows.*

(1) *If* $X = B$ *write* $n = dr + s$ *with* $d, s \in \mathbb{Z}$ *and* $0 \leqslant s < r$.

$$\mu_\chi = \begin{cases} (r_r^{2d}(s+1)_{s+1}s_s), & \text{if } r \leqslant n, \\ ((n+1)_{n+1}n_n), & \text{if } r > n. \end{cases}$$

(2) *If* $X = C$ *write* $2n = dr + s$ *with* $d, s \in \mathbb{Z}$ *and* $0 \leqslant s < r$. *For* $r \leqslant n$,

$$\mu_\chi = \begin{cases} (r^d_{\lfloor r/2 \rfloor} s_{s/2}), & \text{if } r \text{ or } d \text{ is even}, \\ (r^{d-1}_{(r-1)/2}(r-1)_{(r-1)/2}(s+1)_{(s+1)/2}), & \text{otherwise}, \end{cases}$$

while for $r > n$,

$$\mu_\chi = \begin{cases} (r_{r/2}(2n-r)_{(2n-r)/2}), & \text{if } r \text{ is even}, \\ ((r-1)_{(r-1)/2}(2n-(r-1))_{(2n-(r-1))/2}), & \text{if } r \text{ is odd and } r > n+1, \\ ((r-1)^2_{(r-1)/2}), & \text{if } r \text{ is odd and } r = n+1. \end{cases}$$

(3) *If* $X = D$ *write* $n = dr + s$ *with* $d, s \in \mathbb{Z}$ *and* $0 \leqslant s < r$.

$$\mu_\chi = \begin{cases} (r_r^{2d} s_s^2), & \text{if } r \leqslant n, \\ (n_n^2), & \text{if } r > n. \end{cases}$$

Proof. We will show in each case that if $\mu'_{\chi'}$ is a nilpotent orbit parameter with all parts less than or equal to r, then

$$\mathcal{O}(\mu', \chi') \leqslant \mathcal{O}(\mu, \chi).$$

Let

$$\mu'_{\chi'} = \left(a(1)^{c(1)}_{b(1)}, a(2)^{c(2)}_{b(2)}, \ldots, a(k)^{c(k)}_{b(k)}\right)$$

with $a(1) \leqslant r$ and corresponding Spaltenstein pair $\alpha' = (\alpha'_1, \ldots, \alpha'_l)$, $\beta' = (\beta'_1, \ldots, \beta'_l)$; let α, β be the Spaltenstein pair associated to μ_χ.

Recall the notation $\alpha \int \beta$ for the (unordered) partition of n given by $(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots)$. Write $\alpha \int_m \beta$ for $\sum_{i=1}^{m-1}(\alpha_i + \beta_i) + \alpha_m$ and $\alpha \int^m \beta$ for $\sum_{i=1}^{m}(\alpha_i + \beta_i)$.

Type B_n. Suppose first that $r \leqslant n$. It is clear that $\mu_\chi = (r_r^{2d}(s+1)_{s+1}s_s)$ is a nilpotent orbit symbol, with $j = s+1$ and $\mu^*_j = 2d+1$ (note that $s+1 \neq 0$). Thus, its corresponding Spaltenstein pair is given by

$$\alpha_i = \begin{cases} r-1, & \text{if } i \leqslant d, \\ s, & \text{if } i = d+1, \end{cases} \qquad \beta_i = \begin{cases} 1, & \text{if } i \leqslant d, \\ 0, & \text{if } i = d+1. \end{cases}$$

Since $a(1) \leqslant r$ it is clear that $\alpha'_1 \leqslant r-1$ and thus (since α is nonincreasing) $\alpha'_i \leqslant r-1$ for all i. Also, $\alpha'_i + \beta'_i \leqslant \mu'_{2i} \leqslant r$ for all i. It is then apparent that

$$\alpha' \textstyle\int_m \beta' \leqslant \alpha \int_m \beta \quad \text{and} \quad \alpha' \int^m \beta' \leqslant \alpha \int^m \beta \tag{2.2.1}$$

for $1 \leqslant m \leqslant d$. If $s = 0$ then $\alpha \int^d \beta = n$. If $s \neq 0$ then $\alpha \int_{d+1} \beta = n$. We know $\alpha' \int \beta'$ is a partition of n, so we see that (2.2.1) holds for all m. It follows that $\tau_B(\mu', \chi') \leqslant \tau_B(\mu, \chi)$, which completes the proof for the case $r \leqslant n$.

Now assume $r > n$. Then $((n+1)_{n+1} n_n)$ is a nilpotent orbit symbol, with $\mu^*_j = 1$. Consequently, $\alpha_1 = n$. Thus our result is immediate.

Type D_n. Again, begin with the case $r \leqslant n$. Observe that $(r_r^{2d} s_s^2)$ is a nilpotent orbit symbol with its corresponding Spaltenstein pair given by

$$\alpha_i = \begin{cases} r, & \text{if } i \leqslant d, \\ s, & \text{if } i = d+1, \end{cases} \qquad \beta_i = 0 \quad \text{for all } i.$$

We have $\alpha'_i = \chi'(\mu'_{2i}) \leqslant r$ and $\alpha'_i + \beta'_i = \mu'_{2i} \leqslant r$ whenever $1 \leqslant i \leqslant l$. Thus (2.2.1) holds for $1 \leqslant m \leqslant d$. Finally,

$$\alpha \textstyle\int_{d+1} \beta = n \quad \text{while} \quad \alpha' \int_m \beta' \leqslant n \quad \text{and} \quad \alpha' \int^m \beta' \leqslant n$$

for all m. Consequently, $\tau_D(\mu', \chi') \leqslant \tau_D(\mu, \chi)$, as claimed.

The case $r > n$ follows readily since the nilpotent orbit symbol (n_n^2) has $\alpha_1 = n$.

Type C_n. As before, start with $r \leqslant n$. One sees that each proposed μ_χ is indeed a nilpotent orbit symbol. The corresponding sequences $\alpha \int \beta$ are given as follows:

$$\alpha \textstyle\int \beta = \begin{cases} \left(\frac{r}{2}, \frac{r}{2}, \ldots, \frac{r}{2}, \frac{s}{2}\right), & \text{if } r \text{ is even}, \\ \left(\frac{r-1}{2}, \frac{r+1}{2}, \frac{r-1}{2}, \ldots, \frac{r+1}{2}, \frac{s}{2}\right), & \text{if } r \text{ is odd and } d \text{ is even}, \\ \left(\frac{r-1}{2}, \frac{r+1}{2}, \frac{r-1}{2}, \ldots, \frac{r-1}{2}, \frac{s+1}{2}\right), & \text{if } r \text{ is odd and } d \text{ is odd}. \end{cases}$$

For each case, if some nonzero α_q or β_q is less than $\lfloor r/2 \rfloor$ then exactly one of the following occurs:

(1) $\alpha_q = s/2$ and d is even with $q = (d+2)/2$;
(2) $\beta_q = s/2$ and d is odd while r is even and $q = (d+1)/2$;
(3) $\beta_q = (s+1)/2$ and d, r are odd with $q = (d+1)/2$.

In each case α_q (respectively β_q) is the last nonzero entry in $\alpha \int \beta$ and thus $\alpha \int_q \beta = n$ (respectively $\alpha \int^q \beta = n$). For the purposes of proving (2.2.1) we may therefore assume that each α_i and β_i which occurs as a summand on the right-hand sides of the inequalities in (2.2.1) is $\geqslant \lfloor r/2 \rfloor$.

Fix $t \in \{1, 2, \ldots, k\}$ and set $S := \sum_{j<t} c(j)$. We will show that any partial sum of consecutive terms in $\alpha' \int \beta'$ involving only terms α'_i with $\mu'_{2i-1} = a(t)$ and β'_i with $\mu'_{2i} = a(t)$, and beginning with $\alpha'_{(S+2)/2}$ if S is even (respectively $\beta'_{(S+1)/2}$ if S is odd), is less than or equal to the corresponding partial sum of terms in $\alpha \int \beta$. Putting together several such partial sums will verify (2.2.1) for all m.

If $c(t)$ is odd then all the terms α'_i, β'_i in the partial sum equal $b(t)$ which is at most $\lfloor r/2 \rfloor$, so we get what we want. Now assume $c(t)$ is even. First suppose S is even. Thus the terms involved are of the form $\alpha'_i = b(t)$ and $\beta'_i = a(t) - b(t)$. Thus $\alpha'_i \leqslant \lfloor r/2 \rfloor \leqslant \alpha_i$, while $\alpha'_i + \beta'_i \leqslant r = \alpha_i + \beta_i$. Since in our alternating sum we start with an α'_i this case is proved. Now suppose S is odd. Thus we get $\beta'_i = b(t)$ and $\alpha'_i = a(t) - b(t)$. Thus $\beta'_i \leqslant \lfloor r/2 \rfloor \leqslant \beta_i$, while $\beta'_i + \alpha'_{i+1} \leqslant r = \beta_i + \alpha_{i+1}$. Since in our alternating sum we start with a β'_i we have verified this case.

Finally assume $r > n$. Then in each case α_1 is as large as possible for orbit symbols with parts of size at most r, while $\alpha_1 + \beta_1 = n$. This completes the proof. □

2.3. *Orbit closures in the restricted nullcone*

Our primary interest is in the restricted nullcone, $\mathcal{N}_1(\mathfrak{g}) = \{x \in \mathfrak{g}\colon\ x^2 = 0\}$ (since $\operatorname{char} k = 2$): in [31] we investigate support varieties of Weyl modules, which are G-invariant subvarieties of $\mathcal{N}_1(\mathfrak{g})$. For this reason, we will focus on the Hasse diagram of nilpotent orbits contained in $\mathcal{N}_1(\mathfrak{g})$. We give below the closure order relations between these orbits, in terms of their Hesselink parameters. Although this can be deduced from the Großer–Spaltenstein theorem above, we give an elementary, self-contained proof, which may be of some independent interest.

We will constantly use the elementary fact that f is a continuous function if and only if $f(\overline{A}) \subset \overline{f(A)}$ for any subset A of the domain of f. For example, if $\mathcal{O}$ is any set of matrices x satisfying $x^2 = 0$, then $y^2 = 0$ for all $y \in \overline{\mathcal{O}}$ in the Zariski topology. In particular, for $\mathcal{N}_1(\mathfrak{g})$, it suffices to consider partitions all of whose parts are at most 2. As another application of the fact, suppose A is an infinite subset of the field k (recall k is algebraically closed so k is infinite). If $f : A \to B$ is a continuous function, then $f(t) \in \overline{B}$ for all $t \in k$, since the only closed subsets of k are either finite or all of k.

Theorem. *Assume* $(2^s_\gamma, 1^{N-2s}_\eta)$ *and* $(2^r_\delta, 1^{N-2r}_\eta)$ *are symbols parametrizing nilpotent orbits for* G (*where* $s, r, \gamma, \delta, \eta \in \mathbb{N} \cup \{0\}$, *and* η *must be 0* (*respectively* 1) *if* $G = Sp(N)$ (*respectively* $O(N)$)). *Then* $\mathcal{O}(2^s_\gamma, 1^{N-2s}_\eta) \leqslant \mathcal{O}(2^r_\delta, 1^{N-2r}_\eta)$ *if and only if* $s \leqslant r$ *and* $\gamma \leqslant \delta$.

Before giving the proof, we need to give more details of how the index function arises.

We first treat the case $G = Sp(N)$. Following Hesselink, let V be an N-dimensional vector space over k equipped with a nondegenerate bilinear form β satisfying $\beta(v, v) = 0$ for all $v \in V$; then N must be even. G is the group of automorphisms of V leaving β invariant. Fix $x \in \mathfrak{g}$ with x nilpotent. For each $i \in \mathbb{N} \cup \{0\}$ define a quadratic form $\alpha_i : V \to k$ by $\alpha_i(v) = \beta(x^{i+1}v, x^i v)$. The index function $\chi : \mathbb{N} \to \mathbb{Z}$ is defined by

$$\chi(m) = \min\{i \geqslant 0\colon\ \alpha_i|_{\ker(x^m)} \equiv 0\}. \tag{2.3.1}$$

In particular, if $x^2 = 0$ then $\alpha_1 \equiv 0$, and

$$\chi(2) = \begin{cases} 0, & \text{if } \alpha_0 \equiv 0, \\ 1, & \text{otherwise.} \end{cases} \tag{2.3.2}$$

We take the matrix of β to be

$$J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}$$

so that $x \in \mathfrak{g} \Leftrightarrow x^t J = Jx \Leftrightarrow Jx^t J = x$. The matrix $x' := Jx^t J$ is obtained from x by transposing across the antidiagonal, so $x \in \mathfrak{g}$ if and only if x is symmetric about the antidiagonal.

Proof of Theorem (*Symplectic case*). Assume $G = Sp(N)$, where $N = 2n$ is even. Note that $x \in \mathfrak{g}$ has Jordan form corresponding to the partition $(2^r, 1^{N-2r})$ if and only if $x^2 = 0$ and $\operatorname{rk} x = r$. We will take x to have the block form

$$x = x(C) = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} \tag{2.3.3}$$

where $C = (c_{ij})$ is $n \times n$ and $C' = C$. Then $x^2 = 0$, and it is an easy calculation to show that $\alpha_0(v) = c_{1n} v_{2n}^2 + c_{2,n-1} v_{2n-1}^2 + \cdots + c_{n1} v_{n+1}^2$. By (2.3.2), $\chi(2) = 0$ if and only if all antidiagonal entries of C are 0. If $A_1, \ldots, A_q$ are square matrices whose dimensions sum to n, write

$$\operatorname{anti}(A_1, \ldots, A_q) = \begin{pmatrix} 0 & & & A_1 \\ & & A_2 & \\ & \cdot^{\cdot^{\cdot}} & & \\ A_q & & & 0 \end{pmatrix}. \tag{2.3.4}$$

For $t \in k$ put $I_2(t) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$.

Taking $C = C(t) = \operatorname{anti}(t, 1^{r-1}, 0^{n-r})$, then $x(C(t))$ corresponds to the symbol $(2_1^r, 1_0^{N-2r})$ if $t \neq 0$, and to $(2_1^{r-1}, 1_0^{N-2(r-1)})$ if $t = 0$. Thus

$$\mathcal{O}\left(2_1^{r-1}, 1_0^{N-2(r-1)}\right) \subset \overline{\mathcal{O}\left(2_1^r, 1_0^{N-2r}\right)}.$$

Recall that the symbol $(2_0^r, 1_0^{N-2r})$ only occurs when r is even. Suppose $r = 2l \geqslant 2$. Take $C = C(t) = \operatorname{anti}(I_2(t), I_2^{l-1}, 0^{n-2l})$. Then $\operatorname{rk} x(C(t)) = r$ for all $t \in k$, and $\chi(2) = 0$ if and only if $t = 0$. This shows

$$\mathcal{O}\left(2_0^r, 1_0^{N-2r}\right) \subset \overline{\mathcal{O}\left(2_1^r, 1_0^{N-2r}\right)}.$$

Similarly, taking $C = C(t) = \mathrm{anti}(tI_2, I_2^{l-1}, 0^{n-2l})$ gives

$$\mathcal{O}\big(2_0^{r-2}, 1_0^{N-2(r-2)}\big) \subset \overline{\mathcal{O}\big(2_0^r, 1_0^{N-2r}\big)}.$$

Finally, if $\beta(xv, v) = 0$ for all $v \in V$ and all $x \in B \subset M_N(k)$, and if $y \in \overline{B}$, then $\beta(yv, v) = 0$ for all $v \in V$, by continuity. This shows that $\chi(2)$ cannot increase on closures, and completes the proof of the theorem for $Sp(N)$. □

For the orthogonal case, let V be an N-dimensional vector space over k, equipped with a quadratic form α and a bilinear form β satisfying $\beta(v, w) = \alpha(v+w) - \alpha(v) - \alpha(w)$ for all $v, w \in V$. Then $O(N)$ is the algebraic group of automorphisms of V leaving α invariant. Fix $x \in \mathfrak{g}$ with x nilpotent. For each $i \in \mathbb{N} \cup \{0\}$ define a quadratic form $\alpha_i : V \to k$ by $\alpha_i(v) = \alpha(x^i v)$. The index function χ is defined as in (2.3.1). When $x^2 = 0$, $\chi(2)$ is given by

$$\chi(2) = \begin{cases} 1, & \text{if } \alpha_1 \equiv 0, \\ 2, & \text{otherwise.} \end{cases} \tag{2.3.5}$$

We may take $\alpha(v) = v_1 v_N + v_2 v_{N-1} + \cdots$, so that the matrix of β is J when N is even, and is $J_0 = J - E_{n+1,n+1}$ when $N = 2n + 1$ is odd (E_{ij} the matrix units). A matrix x belongs to $\mathfrak{g}$ if and only if $\beta(xv, v) = 0$ for all $v \in V$ and $\mathrm{tr}\, x = 0$. When N is even the first condition is equivalent to x being symmetric about the antidiagonal, with antidiagonal elements all zero. If N is odd, say $N = 2n + 1$, then x has the block form

$$x = \begin{pmatrix} A & 0 & C \\ w & 0 & u \\ B & 0 & A' \end{pmatrix} \tag{2.3.6}$$

where A, B, C are $n \times n$, w and u are $1 \times n$, and B and C are symmetric about the antidiagonal with antidiagonal elements all zero.

Proof of Theorem (*Orthogonal case*). Assume $G = O(N)$, first with $N = 2n$ even. Then the symbol $(2_\delta^r, 1_1^{N-2r})$ only occurs when r is even, because of condition (3)(c) in the definition of symbols. So assume $r = 2l \geqslant 2$.

Taking $x = x(C)$ as in (2.3.3), where $C = C(t) = \mathrm{anti}(tI_2, I_2^{l-1}, 0^{n-2l})$, shows

$$\mathcal{O}\big(2_1^{r-2}, 1_1^{N-2(r-2)}\big) \subset \overline{\mathcal{O}\big(2_1^r, 1_1^{N-2r}\big)}.$$

In order to get $\alpha_0 \not\equiv 0$, we must have some nonzero entries in x outside the upper-right $n \times n$ block. Consider $x \in \mathfrak{g}$ having the form

$$x = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} + a(E_{n-1,n} + E_{n+1,n+2})$$

where $C = \mathrm{anti}(C_1, bI_2)$ with C_1 an $(n-2)\times(n-2)$ matrix, and $a, b \in k$. Then $x^2 = 0$ and $\alpha_0(v) = abv_{n+2}^2$. Also $\mathrm{rk}\, x = 2 + \mathrm{rk}\, C_1$ provided a or b is nonzero.

Assume $r \geqslant 4$ so that $l \geqslant 2$. Take $C_1 = \mathrm{anti}(tI_2, I_2^{l-2}, 0^{n-2l})$. With $a = b = 1$, and letting t be either nonzero or 0, we obtain

$$\mathcal{O}\big(2_2^{r-2}, 1_1^{N-2(r-2)}\big) \subset \overline{\mathcal{O}\big(2_2^r, 1_1^{N-2r}\big)}.$$

Similarly, with $a = 0, b = 1$ we obtain

$$\mathcal{O}\big(2_1^{r-2}, 1_1^{N-2(r-2)}\big) \subset \overline{\mathcal{O}\big(2_1^r, 1_1^{N-2r}\big)}.$$

Fixing $b = t = 1$ and letting a be either nonzero or 0, we obtain

$$\mathcal{O}\big(2_1^r, 1_1^{N-2r}\big) \subset \overline{\mathcal{O}\big(2_2^r, 1_1^{N-2r}\big)}.$$

When $r = 2$, the first two orbit closure inclusions of this paragraph are trivial, and the last one is obtained in the same way using $C_1 = 0$.

Next assume $N = 2n + 1$ is odd. Let x have the block form (2.3.6), where $A = B = 0$ and $w = 0$. Then $x^2 = 0$, and $\alpha_0 \equiv 0$ if and only if $u = 0$. Let $r = 2l \geqslant 2$.

Taking $C = \mathrm{anti}(I_2^l, 0^{n-2l})$, $u = (0^{n-1}, t)$ gives

$$\mathcal{O}\big(2_1^r, 1_1^{N-2r}\big) \subset \overline{\mathcal{O}\big(2_2^r, 1_1^{N-2r}\big)}.$$

Taking $C = \mathrm{anti}(tI_2, I_2^{l-1}, 0^{n-2l})$, $u = 0$ gives

$$\mathcal{O}\big(2_1^{r-2}, 1_1^{N-2(r-2)}\big) \subset \overline{\mathcal{O}\big(2_1^r, 1_1^{N-2r}\big)}.$$

Taking $C = \mathrm{anti}(tI_2, I_2^{l-1}, 0^{n-2l})$, $u = (0^{n-1}, 1)$ gives

$$\mathcal{O}\big(2_2^{r-1}, 1_1^{N-2(r-1)}\big) \subset \overline{\mathcal{O}\big(2_2^r, 1_1^{N-2r}\big)}.$$

Taking $C = \mathrm{anti}(I_2^l, 0^{n-2l})$, $u = (t, 0^{n-2}, 1)$ gives

$$\mathcal{O}\big(2_2^r, 1_1^{N-2r}\big) \subset \overline{\mathcal{O}\big(2_2^{r+1}, 1_1^{N-2(r+1)}\big)}.$$

Finally, for N either even or odd, since for fixed $v \in V$, $\alpha(xv)$ is a polynomial in the entries of x, it follows that if $\alpha(xv) = 0$ for all $x \in B \subset M_N(k)$, then $\alpha(yv) = 0$ for $y \in \overline{B}$. This shows that $\chi(2)$ cannot increase on closures, and completes the proof for $O(N)$. $\square$

2.4. Unipotent orbits

In this subsection we assume G is connected; i.e., $G = SO(N)$ in type D. (Recall our convention that $SO(N)$ is the identity component of $O(N)$.) The classification of unipotent classes in characteristic two is provided in [7,8,12,32]. We will use the parametrization given by Hesselink [12]. The classification is by pairs (μ, χ), analogous to the nilpotent case. The parameters for unipotent orbits in G, when G is connected, are as follows. Recall that I denotes the set of parts appearing in the partition μ.

(2.4.1) (1) For $m > l$ in I, $\chi(m) \geqslant \chi(l)$ and $m - \chi(m) \geqslant l - \chi(l)$.
(2) For $G = Sp(N)$ and $m \in I$:
(a) $(m-2)/2 \leqslant \chi(m) \leqslant m/2$;
(b) $\chi(m) = m/2$ if $n(m)$ is odd.
(3) For $G = SO(N)$ and $m \in I$:
(a) $m/2 \leqslant \chi(m) \leqslant (m+2)/2$;
(b) $\chi(m) = (m+2)/2$ if $m \geqslant 2$ and $n(m)$ is odd;
(c) $\sum_{m \in I} n(m)$ is even if N is even.

For each pair (μ, χ) satisfying the above conditions, let $\mathcal{C}(\mu, \chi)$ be the corresponding unipotent class in $\mathcal{U}(G)$.

Recall the restricted unipotent variety

$$\mathcal{U}_1(G) = \{u \in \mathcal{U}(G):\ (u-1)^2 = 0\} = \{u \in \mathcal{U}(G):\ u^2 = 1\}.$$

One of our main results is that there is an order-preserving bijection between the G-orbits in $\mathcal{N}_1(\mathfrak{g})$ and in $\mathcal{U}_1(G)$.

Theorem. *Let $G = Sp(N)$ or $SO(N)$ with $\operatorname{char} k = 2$. There is an order-preserving bijection between the nilpotent orbits in $\mathcal{N}_1(\mathfrak{g})$ and the unipotent classes in $\mathcal{U}_1(G)$.*

Proof. Since the elements $x \in \mathcal{N}_1(\mathfrak{g})$ (respectively $u \in \mathcal{U}_1(G)$) satisfy $x^2 = 0$ (respectively $(u-1)^2 = 0$), the associated partitions μ are precisely those having all parts at most 2. We will show that the same symbols arise in the nilpotent and unipotent case, provided all parts are at most 2.

First, the condition (1) is identical in (2.1.1) and (2.4.1). For $G = Sp(N)$, condition (2)(a) is equivalent to $\chi(2) = 0$ or 1, $\chi(1) = 0$ in either case, and the conditions (2)(b) are identical. Assume $G = SO(N)$. In both (2.1.1) and (2.4.1), condition (3)(a) becomes $\chi(2) = 1$ or 2, $\chi(1) = 1$. In the presence of (3)(a), each condition (3)(b) reduces to $\chi(2) = 2$ if $n(2)$ is odd. And in each case, condition (3)(c) only rules out the possibility that $n(2)$ is odd and $n(1)$ is even. Finally, the parameters for the nilpotent and unipotent $O(N)$ orbits which split into two $SO(N)$ orbits are the same.

To show that the bijection is order-preserving, it suffices to consider $O(N)$ orbits in $SO(N)$ (respectively $\mathfrak{so}(N)$), by Remark 2.1. The closure order relations between nilpotent orbits in $\mathcal{N}_1(\mathfrak{g})$ are given in Theorem 2.3. The closure order relations between unipotent classes for classical groups in characteristic 2 were worked out by Spaltenstein [25]. He parametrizes the unipotent classes by pairs (λ, ε), where λ is a partition as usual, and $\varepsilon : \mathbb{N} \to \{-1, 0, 1\} \subset \mathbb{Z}$ takes the place of Hesselink's index function χ. Specifically, letting $n(i) \in \mathbb{Z}_+$ be the multiplicity of i in the partition λ,

(1) $\varepsilon(i) = -1$ if i is odd or if $n(i) = 0$,
(2) $\varepsilon(i) = 1$ if i is even and $n(i)$ is odd,
(3) $\varepsilon(i) = 1$ if i is even, $n(i)$ is even, and $\chi(i)$ is the larger of its two possible values (see (2)(a) and (3)(a) above),
(4) $\varepsilon(i) = 0$ if i is even, $n(i)$ is even, and $\chi(i)$ is the smaller of its two possible values (see (2)(a) and (3)(a) above).

Given a partition λ of N, let λ^* be the dual partition of N defined by $\lambda_i^* = \#\{j\colon \lambda_j \geqslant i\}$. Order the pairs (λ, ε) by $(\lambda, \varepsilon) \leqslant (\mu, \phi)$ if and only if, for each i,

(1) $\sum_{j \leqslant i} \lambda_j^* \geqslant \sum_{j \leqslant i} \mu_j^*$ (i.e., $\lambda \trianglelefteq \mu$ in the usual dominance ordering),
(2) $\sum_{j \leqslant i} \lambda_j^* - \max(\varepsilon(i), 0) \geqslant \sum_{j \leqslant i} \mu_j^* - \max(\phi(i), 0)$, and
(3) if $\sum_{j \leqslant i} \lambda_j^* = \sum_{j \leqslant i} \mu_j^*$ and $\lambda_{i+1}^* - \mu_{i+1}^*$ is odd, then $\phi(i) \neq 0$.

If $\mathcal{C}_{\lambda,\varepsilon}$ is the unipotent class parametrized by (λ, ε), then $\mathcal{C}_{\lambda,\varepsilon} \leqslant \mathcal{C}_{\mu,\phi}$ if and only if $(\lambda, \varepsilon) \leqslant (\mu, \phi)$ [25, Theorem 8.2].

Transferring the ordering on Spaltenstein pairs (λ, ε) into the corresponding ordering on Hesselink symbols, we obtain the following. Observe that if $\lambda = (2^s, 1^{N-2s})$, then $\lambda^* = (N - s, s)$. Condition (1) for $(2_\gamma^s, 1_\eta^{N-2s}) \leqslant (2_\delta^r, 1_\eta^{N-2r})$ becomes $s \leqslant r$. Condition (2) (with $i = 2$) becomes $\gamma \leqslant \delta$. The remaining conditions are vacuous. These are the same conditions as in Theorem 2.3. Thus the ordering on unipotent classes in $\mathcal{U}_1(G)$ corresponds, under our bijection, to the ordering on nilpotent orbits in $\mathcal{N}_1(\mathfrak{g})$. □

3. Exceptional groups and Lie algebras

Let G be an exceptional simple algebraic group. For good primes $\mathcal{N}_{r,\rho}(\mathfrak{g})$ was computed in [30] for minimal dimensional representations or the adjoint representation. When ρ is the adjoint representation Lawther [18] showed that the Jordan block sizes for unipotent elements and corresponding nilpotent element coincide. Therefore, the tables in [30, §4.2] can be used to determine $\mathcal{U}_{r,\mathrm{ad}}(G)$ for good primes. For minimal dimensional representations Lawther has also shown that the Jordan block sizes coincide except for one case when $p = 5$ for the regular element in E_7. The partition for the unipotent class is $(24, 22, 10)$ while the partition for the nilpotent element is $(23^2, 10)$. In order to obtain $\mathcal{U}_{r,\min}(G)$ in this situation one can use the table given in [30, §4.1] by replacing 22 (respectively 23+) by 22–23 (respectively 24+) in the $p = 5$ column.

When the prime p is bad the tables below describe $\mathcal{U}_{r,\rho}(G)$ when ρ is a minimal dimensional representation or the adjoint representation. The variety of nilpotent elements $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is also provided when ρ is the adjoint representation. It can be verified by using a base change argument that these representations satisfy the hypotheses mentioned at the end of Section 1.2. Note that the adjoint representation for E_8 coincides with the minimal dimensional (nontrivial) irreducible representation.

The methods of computation and conventions used in the tables are the same as the ones given in [30, §4]. For unipotent classes, the Jordan block sizes for the images of ρ on a unipotent class are given in [16,17]. For nilpotent orbits, we determined the Jordan block sizes of the image of the orbit representatives under the adjoint representation. This involved using the built-in functions in the MAGMA package (i.e., "StructureConstants," "JordanForm") [1,2] to construct Chevalley bases for exceptional Lie algebras, and compute Jordan block sizes.

In more detail, the first step entailed constructing a set of orbit representatives in characteristic zero. Once we had such a set of representatives they were tested on the computer by showing that each one gave rise to a different Jordan block partition. Next, additional orbit representatives for bad primes were obtained using information in the literature [13,28, 29]. These were also verified by using computer calculations. Finally, we used MAGMA to deduce the Jordan block sizes of the image of the adjoint representation on the representatives. The Jordan block sizes are provided in Section 6.

In the tables the first column consists of a list of orbits via the Bala–Carter labeling. The cell entries provide the values of r for which $\mathcal{U}_{r,\rho}(G)$ (or $\mathcal{N}_{r,\rho}(\mathfrak{g})$) is the union of the closures of the orbits listed in that particular row. For example, for the group E_6,

$$\mathcal{U}_{7,\min}(G) = \overline{\mathcal{C}(A_4 + A_1)} \cup \overline{\mathcal{C}(D_4)} \quad \text{when } p = 3, \quad \text{and}$$
$$\mathcal{N}_{3,\mathrm{ad}}(\mathfrak{g}) = \overline{\mathcal{O}(3A_1)} \quad \text{when } p = 2.$$

(As mentioned in Section 7, the closure relations between nilpotent orbits for E_7 and E_8 have not been completely determined. Thus in those cases, when two or more orbits are listed in a row, it may happen that one of the orbits lies in the closure of the other.)

Type E_6

Orbits	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{N}_{r,\rho}(\mathfrak{g})$	$\mathcal{N}_{r,\rho}(\mathfrak{g})$
	min	min	adj	adj	adj	adj
	$p=3$	$p=2$	$p=3$	$p=2$	$p=3$	$p=2$
1	1	1	1–2	1	1–2	1
A_1	2					
$3A_1$		2		2–3		2–3
A_2		3				
$2A_2 + A_1$	3–4		3–6		3–6	
$D_4(a_1)$	5–6	4–5	7–8	4–7	7–8	
D_4		6				4–7
$A_4 + A_1$		7				
$A_4 + A_1, D_4$	7					
$D_5(a_1)$	8					
D_5		8–12		8–15		8–15
$E_6(a_1)$	9–14	13–15	9–18			
E_6	15+	16+	19+	16+	9+	16+

Type E_7

Orbits	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{N}_{r,\rho}(\mathfrak{g})$	$\mathcal{N}_{r,\rho}(\mathfrak{g})$
	min	min	adj	adj	adj	adj
	$p=3$	$p=2$	$p=3$	$p=2$	$p=3$	$p=2$
1	1	1	1–2	1	1–2	1
A_1	2					
$4A_1$		2		2–3		2–3
$A_2, 4A_1$		3				
$2A_2+A_1$	3–4		3–6		3–6	
$D_4(a_1)$	5					
$A_3+A_2+A_1$	6	4–5	7–8	4–7	7–8	
D_4+A_1		6				
D_4+A_1, $(A_3+A_2)^{(2)}$						4–7
A_4+A_2, D_4	7					
A_4+A_2, D_4+A_1		7				
$D_5(a_1)+A_1, A_5+A_1$	8					
$E_7(a_4)$		8–12		8–15		
$E_6(a_1)$	9–14	13	9–18			
D_6, $(A_6)^{(2)}$						8–15
$E_7(a_3)$		14–15				
E_6	15				9–18	
$E_6, E_7(a_3)$	16–17					
$E_7(a_2)$	18–19		19–26		19–26	
$E_7(a_1)$	20–25	16–25		16–31		
E_7	26+	26+	27+	32+	27+	16+

Type F_4

Orbits	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{N}_{r,\rho}(\mathfrak{g})$	$\mathcal{N}_{r,\rho}(\mathfrak{g})$
	min	min	adj	adj	adj	adj
	$p=3$	$p=2$	$p=3$	$p=2$	$p=3$	$p=2$
1	1	1	1–2	1	1–2	1
A_1	2					
$A_1+\widetilde{A}_1$		2		2–3		2–3
A_2		3				
$A_1+\widetilde{A}_2$	3–4		3–6		3–6	
$F_4(a_3)$	5–6	4–5	7–8	4–7	7–8	
B_3	7–8	6–7				4–7
$F_4(a_1)$	9–14	8–15	9–18	8–15		8–15
F_4	15+	16+	19+	16+	9+	16+

Type E_8

Orbits	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{N}_{r,\rho}(\mathfrak{g})$	$\mathcal{N}_{r,\rho}(\mathfrak{g})$	$\mathcal{N}_{r,\rho}(\mathfrak{g})$
	min/adj	min/adj	min/adj	min/adj	min/adj	min/adj
	$p=5$	$p=3$	$p=2$	$p=5$	$p=3$	$p=2$
1	1–2	1–2	1	1–2	1–2	1
$2A_1$	3			3		
$4A_1$	4		2–3	4		2–3
$2A_2+2A_1$		3–6			3–6	
$D_4(a_1)+A_2$		7			7	
$2A_3$		8	4–7		8	
D_4+A_2, $(A_3+A_2)^{(2)}$						4–7
A_4+A_3	5–10			5–10		
$E_8(a_7)$	11–12			11–12		
A_6+A_1	13–14			13–14		
$D_7(a_2)$	15			15		
A_7	16		8–15	16		
D_7, $(A_6)^{(2)}$, $(D_5+A_2)^{(2)}$						8–15
$E_8(b_6)$	17–18	9–18		17–18		
E_6+A_1, $(A_7)^{(3)}$					9–18	
$E_8(a_6)$	19–22			19–22		
$E_8(b_5)$		19–22			19–22	
$E_8(a_5)$	23–24	23–26		23–24	23–26	
$E_8(a_4)$			16–31			
$E_8(a_1)$	25–50	27–54				
E_8	51+	55+	32+	25+	27+	16+

Type G_2

Orbits	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{U}_{r,\rho}(G)$	$\mathcal{N}_{r,\rho}(\mathfrak{g})$	$\mathcal{N}_{r,\rho}(\mathfrak{g})$
	min	min	adj	adj	adj	adj
	$p=3$	$p=2$	$p=3$	$p=2$	$p=3$	$p=2$
1	1	1	1–2	1	1–2	1
A_1	2					
$\widetilde{A}_1$		2		2–3		2–3
$G_2(a_1)$	3–6	3–5	3–8	4–7	3–8	
G_2	7+	6+	9+	8+	9+	4+

4. Applications

4.1. Restricted unipotent varieties

The following two theorems provide a description of the restricted unipotent variety for simple connected algebraic groups when the characteristic of the field is a bad prime. These results are easily deduced by setting $r=p$ for $\mathcal{U}_{r,\rho}(G)$ and using the results in Sections 2 and 3.

Theorem.

(A) *Let G be a simple classical connected algebraic group over k where* $\operatorname{char} k = 2$.
  (i) *If Φ is of type B_l then* $\mathcal{U}_1(G) = \overline{\mathcal{C}(2_2^l, 1_1)}$.
  (ii) *If Φ is of type C_l then* $\mathcal{U}_1(G) = \overline{\mathcal{C}(2_1^l)}$.
  (iii) *If Φ is of type D_l then* $\mathcal{U}_1(G) = \overline{\mathcal{C}(2_2^l)}$.

(B) *Let G be an exceptional algebraic group with p a bad prime.*
  (i) *If Φ is of type E_6 then* $\mathcal{U}_1(G) = \overline{\mathcal{C}(X)}$ *where* $X = 2A_2 + A_1$ $(p = 3)$, $3A_1$ $(p = 2)$.
  (ii) *If Φ is of type E_7 then* $\mathcal{U}_1(G) = \overline{\mathcal{C}(X)}$ *where* $X = 2A_2 + A_1$ $(p = 3)$, $4A_1$ $(p = 2)$.
  (iii) *If Φ is of type E_8 then* $\mathcal{U}_1(G) = \overline{\mathcal{C}(X)}$ *where* $X = A_4 + A_3$ $(p = 5)$, $2A_2 + 2A_1$ $(p = 3)$, $4A_1$ $(p = 2)$.
  (iv) *If Φ is of type F_4 then* $\mathcal{U}_1(G) = \overline{\mathcal{C}(X)}$ *where* $X = A_1 + \widetilde{A_2}$ $(p = 3)$, $A_1 + \widetilde{A_1}$ $(p = 2)$.
  (v) *If Φ is of type G_2 then* $\mathcal{U}_1(G) = \overline{\mathcal{C}(X)}$ *where* $X = G_2(a_1)$ $(p = 3)$, $\widetilde{A_1}$ $(p = 2)$.

4.2. Restricted nullcones

In a similar way using our computations of $\mathcal{N}_{r,\rho}(\mathfrak{g})$, we can determine $\mathcal{N}_1(\mathfrak{g})$ over fields of bad characteristic.

Theorem.

(A) *Let G be a simple classical connected algebraic group over k where* $\operatorname{char} k = 2$.
  (i) *If Φ is of type B_l then* $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(2_2^l, 1_1)}$.
  (ii) *If Φ is of type C_l then* $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(2_1^l)}$.
  (iii) *If Φ is of type D_l then* $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(2_2^l)}$.

(B) *Let G be an exceptional algebraic group with p a bad prime.*
  (i) *If Φ is of type E_6 then* $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ *where* $X = 2A_2 + A_1$ $(p = 3)$, $3A_1$ $(p = 2)$.
  (ii) *If Φ is of type E_7 then* $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ *where* $X = 2A_2 + A_1$ $(p = 3)$, $4A_1$ $(p = 2)$.
  (iii) *If Φ is of type E_8 then* $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ *where* $X = A_4 + A_3$ $(p = 5)$, $2A_2 + 2A_1$ $(p = 3)$, $4A_1$ $(p = 2)$.
  (iv) *If Φ is of type F_4 then* $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ *where* $X = A_1 + \widetilde{A_2}$ $(p = 3)$, $A_1 + \widetilde{A_1}$ $(p = 2)$.
  (v) *If Φ is of type G_2 then* $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ *where* $X = G_2(a_1)$ $(p = 3)$, $\widetilde{A_1}$ $(p = 2)$.

4.3. The previous two theorems show that $\mathcal{U}_1(G)$ and $\mathcal{N}_1(\mathfrak{g})$ "coincide" for G exceptional. The G-orbits in these two varieties also coincide (see Section 7 for Hasse diagrams). Combining this with our results for classical groups, we can now state the following result.

Theorem. *Let G be a connected reductive algebraic group over an algebraically closed field of characteristic $p > 0$. There exists a one-to-one correspondence between unipotent* *orbits in $\mathcal{U}_1(G)$ and nilpotent orbits in $\mathcal{N}_1(\mathfrak{g})$ which respects the closure ordering of orbits. This correspondence also respects dimensions of unipotent classes and nilpotent orbits.*

Proof. For good primes, this is established via the computations in [30]. (This can also be deduced from [19, Theorem 35].) For bad primes, the correspondence is given by Theorems 4.1(A), 4.2(A) for the classical groups and Theorems 4.1(B), 4.2(B) for the exceptional groups. The inclusion relations are checked in Theorem 2.4 for the classical groups and case by case in Section 7 for the exceptional groups. Furthermore, one can verify directly that the orbit correspondence respects dimensions of unipotent classes and nilpotent orbits. □

We should also remark that under the correspondence in the preceding theorem, the Jordan block sizes for the unipotent classes in $\mathcal{U}_1(G)$ and the nilpotent orbits in $\mathcal{N}_1(\mathfrak{g})$ coincide for the adjoint representation when G is an exceptional group. This can be verified by using [16,20] and the tables in Section 6. In general for $\mathcal{U}$ and $\mathcal{N}$ the Jordan block sizes need not coincide. For example when $\Phi = G_2,\ p = 3$, the partition for the regular unipotent class $\mathcal{C}(G_2)$ for the adjoint representation is $(11, 3)$ whereas the partition for the corresponding regular nilpotent orbit $\mathcal{O}(G_2)$ is $(9, 3, 2)$. This leads one to conjecture that there might be a G-equivariant isomorphism between $\mathcal{U}_1(G)$ and $\mathcal{N}_1(\mathfrak{g})$ for all primes. For good primes such an "exponential" map has been constructed in [3]; and as remarked in the introduction, [19] shows that the Bardsley–Richardson map restricts to give a G-equivariant isomorphism in the opposite direction. An interesting problem would be to determine if such a map exists for bad primes.

In [23], it was shown that $\mathcal{N}_1(\mathfrak{g})$ is irreducible and the closure of a Richardson orbit when the characteristic of the field is good. Our results also show that $\mathcal{N}_1(\mathfrak{g})$ is an irreducible variety even for bad primes. However, it is not true that $\mathcal{N}_1(\mathfrak{g})$ is the closure of a Richardson orbit (e.g., when Φ is of type G_2 and $p = 2$).

5. Orbit representatives

In this section, we provide orbit representatives for exceptional Lie algebras over algebraically closed fields of all characteristics. Throughout, the conventions in [16] will be used. If X is the Bala–Carter label for the orbit in characteristic zero then $X^{(p)}$ will denote the new orbit that arises. For the unipotent classes such representatives can be found in [16, 21,22,24,29]. After asking several experts in the field we were amazed to learn that such explicit representatives for nilpotent orbits were not available in the literature. Since we were interested in using such representatives for computation purposes, we wanted to have the orbits represented in the simplest form possible. For this purpose we have attempted to choose representatives which are consistent with the Bala–Carter labeling whenever possible. Several sources we used to construct representatives include [13,16,22,28,29]. We hope that making such a list available will be useful for future computational purposes.

We should make the reader aware of several places in the literature which are pertinent to our results. First, in [6, p. 78], the algorithm for providing the orbit representative for $D_4(a_1)$ in D_4 is slightly misleading. In the case of types B and D one may not be able to simply take a sum of root vectors in $\mathcal{C}^+$. If one does this for the example given the Jordan block size for this representative is $(5, 1^3)$ instead of $(5, 3)$. Rather, one needs to take a generic sum of root vectors in $\mathcal{C}^+$. The orbit representative for $D_4(a_1)$ that needs to be taken is obtained by deleting one of the summands. Second, in [22] for the orbit $E_8(a_7)$ (in Mizuno's notation $2A_4$), the root α_{47} does not exist as a root of E_8. This root is used as part of his description of the unipotent class representing this orbit. Finally, the representatives for the nilpotent orbits $(A_3 + A_2)^{(2)}$ in E_7 and E_8 and $(D_4 + A_2)^{(2)}$ in E_8 are not explicitly given in [13]. We constructed these representatives by looking at the orbits given by $\mathcal{O}(4_3^2, 2_2^2)$ in D_6 (embedded into E_7 and E_8) and $\mathcal{O}(5_4^2, 2_2^2)$ in D_7 (embedded into E_8).

Type E_6

Orbit	Representative
E_6	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_6}$
$E_6(a_1)$	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_2+\alpha_4} + x_{\alpha_3+\alpha_4} + x_{\alpha_6}$
D_5	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_6}$
$E_6(a_3)$	$x_{\alpha_3+\alpha_4+\alpha_5} + x_{\alpha_2+\alpha_4} + x_{\alpha_1} + x_{\alpha_1+\alpha_3} + x_{\alpha_6} + x_{\alpha_5+\alpha_6}$
$D_5(a_1)$	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_3+\alpha_4} + x_{\alpha_4+\alpha_5}$
A_5	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_6}$
$A_4 + A_1$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_6}$
D_4	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5}$
A_4	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5}$
$D_4(a_1)$	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_3+\alpha_4} + x_{\alpha_4+\alpha_5}$
$A_3 + A_1$	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_6}$
$2A_2 + A_1$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_6}$
A_3	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4}$
$A_2 + 2A_1$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_6}$
$2A_2$	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_6}$
$A_2 + A_1$	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_5}$
A_2	$x_{\alpha_1} + x_{\alpha_3}$
$3A_1$	$x_{\alpha_1} + x_{\alpha_4} + x_{\alpha_6}$
$2A_1$	$x_{\alpha_1} + x_{\alpha_4}$
A_1	x_{α_1}
$\{0\}$	0

Type G_2

Orbit	Representative $p \geqslant 3$	Representative $p = 2$
G_2	$x_{\alpha_1} + x_{\alpha_2}$	$x_{\alpha_1} + x_{\alpha_2}$
$G_2(a_1)$	$x_{\alpha_2} + x_{2\alpha_1+\alpha_2}$	$x_{\alpha_1} + x_{2\alpha_1+\alpha_2}$
$(\widetilde{A_1})^{(3)}$	$x_{\alpha_2} + x_{\alpha_1+\alpha_2}$	
$\widetilde{A_1}$	x_{α_1}	x_{α_1}
A_1	x_{α_2}	x_{α_2}
$\{0\}$	0	0

Type E_7

Orbit	Representative
E_7	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_7}$
$E_7(a_1)$	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_3+\alpha_4} + x_{\alpha_2+\alpha_4} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_7}$
$E_7(a_2)$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_2+\alpha_4} + x_{\alpha_4+\alpha_5} + x_{\alpha_5+\alpha_6} + x_{\alpha_6+\alpha_7}$
$E_7(a_3)$	$x_{\alpha_1} + x_{\alpha_2+\alpha_4} + x_{\alpha_3+\alpha_4} + x_{\alpha_2+\alpha_4+\alpha_5} + x_{\alpha_2+\alpha_3+\alpha_4+\alpha_5} + x_{\alpha_5+\alpha_6} + x_{\alpha_7}$
E_6	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_6}$
$E_6(a_1)$	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_2+\alpha_4} + x_{\alpha_3+\alpha_4} + x_{\alpha_6}$
D_6	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_7}$
$E_7(a_4)$	$x_{\alpha_1} + x_{\alpha_2+\alpha_3+\alpha_4} + x_{\alpha_3+\alpha_4+\alpha_5} + x_{\alpha_2+\alpha_4+\alpha_5} + x_{\alpha_4+\alpha_5+\alpha_6} + x_{\alpha_3+\alpha_4+\alpha_5+\alpha_6} + x_{\alpha_6+\alpha_7}$
$D_6(a_1)$	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_7} + x_{\alpha_3+\alpha_4} + x_{\alpha_4+\alpha_5}$
$D_5 + A_1$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_7}$
$(A_6)^{(2)}$	$x_{\alpha_1+\alpha_3+\alpha_4} + x_{\alpha_3+\alpha_4+\alpha_5} + x_{\alpha_2+\alpha_3+\alpha_4} + x_{\alpha_2+\alpha_4+\alpha_5} + x_{\alpha_5+\alpha_6} + x_{\alpha_6+\alpha_7} + x_{\alpha_4+\alpha_5+\alpha_6+\alpha_7}$
A_6	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_7}$
$E_7(a_5)$	$x_{\alpha_1+\alpha_3+\alpha_4} + x_{\alpha_2+\alpha_3+\alpha_4} + x_{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6} + x_{\alpha_2+\alpha_4+\alpha_5} + x_{\alpha_4+\alpha_5+\alpha_6}$ $+ x_{\alpha_5+\alpha_6+\alpha_7} + x_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}$
D_5	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5}$
$E_6(a_3)$	$x_{\alpha_3+\alpha_4+\alpha_5} + x_{\alpha_2+\alpha_4} + x_{\alpha_1} + x_{\alpha_1+\alpha_3} + x_{\alpha_6} + x_{\alpha_5+\alpha_6}$
$D_6(a_2)$	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_7} + x_{\alpha_3+\alpha_4} + x_{\alpha_4+\alpha_5} + x_{\alpha_5+\alpha_6} + x_{\alpha_6+\alpha_7}$
$D_5(a_1) + A_1$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_7} + x_{\alpha_3+\alpha_4} + x_{\alpha_4+\alpha_5}$
$A_5 + A_1$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_7}$
$(A_5)'$	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_6}$
$A_4 + A_2$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_6} + x_{\alpha_7}$
$D_5(a_1)$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_3+\alpha_4} + x_{\alpha_4+\alpha_5}$
$A_4 + A_1$	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_7}$
$D_4 + A_1$	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_7}$
$(A_5)''$	$x_{\alpha_2} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_7}$
$A_3 + A_2 + A_1$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_7}$
A_4	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5}$
$(A_3 + A_2)^{(2)}$	$x_{\alpha_7} + x_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7} + x_{\alpha_5} + x_{\alpha_2+\alpha_4+\alpha_5} + x_{\alpha_4} + x_{\alpha_3}$
$A_3 + A_2$	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_6} + x_{\alpha_7}$
$D_4(a_1) + A_1$	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_3+\alpha_4} + x_{\alpha_4+\alpha_5} + x_{\alpha_7}$
D_4	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5}$
$A_3 + 2A_1$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_7}$
$D_4(a_1)$	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_3+\alpha_4} + x_{\alpha_4+\alpha_5}$
$(A_3 + A_1)'$	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_6}$
$2A_2 + A_1$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_6}$
$(A_3 + A_1)''$	$x_{\alpha_2} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_7}$
$A_2 + 3A_1$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_7}$
$2A_2$	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_6} + x_{\alpha_7}$
A_3	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4}$
$A_2 + 2A_1$	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_7}$
$A_2 + A_1$	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_5}$
$4A_1$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_5} + x_{\alpha_7}$
A_2	$x_{\alpha_1} + x_{\alpha_3}$
$(3A_1)'$	$x_{\alpha_1} + x_{\alpha_4} + x_{\alpha_6}$
$(3A_1)''$	$x_{\alpha_2} + x_{\alpha_5} + x_{\alpha_7}$
$2A_1$	$x_{\alpha_1} + x_{\alpha_4}$
A_1	x_{α_1}
$\{0\}$	0

Type E_8 (I)

Orbit	Representative
E_8	$x_{\alpha_1}+x_{\alpha_2}+x_{\alpha_3}+x_{\alpha_4}+x_{\alpha_5}+x_{\alpha_6}+x_{\alpha_7}+x_{\alpha_8}$
$E_8(a_1)$	$x_{\alpha_1}+x_{\alpha_2}+x_{\alpha_2+\alpha_4}+x_{\alpha_3+\alpha_4}+x_{\alpha_5}+x_{\alpha_6}+x_{\alpha_7}+x_{\alpha_8}$
$E_8(a_2)$	$x_{\alpha_1}+x_{\alpha_2}+x_{\alpha_3}+x_{\alpha_2+\alpha_4}+x_{\alpha_4+\alpha_5}+x_{\alpha_5}+x_{\alpha_6}+x_{\alpha_6+\alpha_7}+x_{\alpha_8}$
$E_8(a_3)$	$x_{\alpha_1+\alpha_3}+x_{\alpha_2+\alpha_4}+x_{\alpha_3+\alpha_4}+x_{\alpha_4+\alpha_5}+x_{\alpha_3+\alpha_4+\alpha_5}+x_{\alpha_5}+x_{\alpha_6}+x_{\alpha_7}+x_{\alpha_8}$
$E_8(a_4)$	$x_{\alpha_1+\alpha_3}+x_{\alpha_2+\alpha_4}+x_{\alpha_3+\alpha_4}+x_{\alpha_4+\alpha_5}+x_{\alpha_3+\alpha_4+\alpha_5}+x_{\alpha_5}+x_{\alpha_6}+x_{\alpha_6+\alpha_7}+x_{\alpha_7+\alpha_8}$
E_7	$x_{\alpha_1}+x_{\alpha_2}+x_{\alpha_3}+x_{\alpha_4}+x_{\alpha_5}+x_{\alpha_6}+x_{\alpha_7}$
$E_8(b_4)$	$x_{\alpha_1+\alpha_3}+x_{\alpha_2+\alpha_3+\alpha_4}+x_{\alpha_3+\alpha_4+\alpha_5}+x_{\alpha_2+\alpha_4+\alpha_5}+x_{\alpha_4+\alpha_5+\alpha_6}+x_{\alpha_5+\alpha_6+\alpha_7}+x_{\alpha_7}+x_{\alpha_8}$
$E_8(a_5)$	$x_{\alpha_1+\alpha_3}+x_{\alpha_2+\alpha_3+\alpha_4}+x_{\alpha_3+\alpha_4+\alpha_5}+x_{\alpha_2+\alpha_4+\alpha_5}+x_{\alpha_4+\alpha_5+\alpha_6}+x_{\alpha_2+\alpha_4+\alpha_5+\alpha_6}$ $+x_{\alpha_6+\alpha_7}+x_{\alpha_7+\alpha_8}$
$E_7(a_1)$	$x_{\alpha_1}+x_{\alpha_3+\alpha_4}+x_{\alpha_2+\alpha_4}+x_{\alpha_5}+x_{\alpha_6}+x_{\alpha_7}$
$E_8(b_5)$	$x_{\alpha_1+\alpha_3+\alpha_4}+x_{\alpha_2+\alpha_3+\alpha_4}+x_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}+x_{\alpha_2+\alpha_4+\alpha_5}+x_{\alpha_4+\alpha_5+\alpha_6}$ $+x_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6}+x_{\alpha_5+\alpha_6+\alpha_7}+x_{\alpha_8}$
$(D_7)^{(2)}$	$x_{\alpha_2+\alpha_3+\alpha_4}+x_{\alpha_4+\alpha_5+\alpha_6}+x_{\alpha_5+\alpha_6+\alpha_7}+x_{\alpha_3+\alpha_4+\alpha_5}+x_{\alpha_2+\alpha_4+\alpha_5}$ $+x_{\alpha_6+\alpha_7+\alpha_8}+x_{\alpha_1}+x_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6}$
D_7	$x_{\alpha_2}+x_{\alpha_3}+x_{\alpha_4}+x_{\alpha_5}+x_{\alpha_6}+x_{\alpha_7}+x_{\alpha_8}$
$E_8(a_6)$	$x_{\alpha_1+\alpha_3+\alpha_4}+x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}+x_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}+x_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}+x_{\alpha_4+\alpha_5+\alpha_6}$ $+x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6}+x_{\alpha_6+\alpha_7}+x_{\alpha_7+\alpha_8}$
$E_7(a_2)$	$x_{\alpha_1}+x_{\alpha_2}+x_{\alpha_3}+x_{\alpha_2+\alpha_4}+x_{\alpha_4+\alpha_5}+x_{\alpha_5+\alpha_6}+x_{\alpha_6+\alpha_7}$
E_6+A_1	$x_{\alpha_1}+x_{\alpha_2}+x_{\alpha_3}+x_{\alpha_4}+x_{\alpha_5}+x_{\alpha_6}+x_{\alpha_8}$
$(D_7(a_1))^{(2)}$	$x_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}+x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}+x_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5}+x_{\alpha_7+\alpha_8}+x_{\alpha_5+\alpha_6}$ $+x_{\alpha_6+\alpha_7}+x_{\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8}+x_{\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6}$
$D_7(a_1)$	$x_{\alpha_2}+x_{\alpha_3}+x_{\alpha_5}+x_{\alpha_6}+x_{\alpha_7}+x_{\alpha_8}+x_{\alpha_3+\alpha_4}+x_{\alpha_4+\alpha_5}$
$E_8(b_6)$	$x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}+x_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}+x_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}+x_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}$ $+x_{\alpha_2+\alpha_4+\alpha_5+\alpha_6}+x_{\alpha_4+\alpha_5+\alpha_6+\alpha_7}+x_{\alpha_7+\alpha_8}+x_{\alpha_6+\alpha_7+\alpha_8}$
$E_7(a_3)$	$x_{\alpha_1}+x_{\alpha_2+\alpha_4}+x_{\alpha_3+\alpha_4}+x_{\alpha_2+\alpha_4+\alpha_5}+x_{\alpha_2+\alpha_3+\alpha_4}+x_{\alpha_5}+x_{\alpha_5+\alpha_6}+x_{\alpha_7}$
$E_6(a_1)+A_1$	$x_{\alpha_1}+x_{\alpha_3}+x_{\alpha_5}+x_{\alpha_2+\alpha_4}+x_{\alpha_3+\alpha_4}+x_{\alpha_6}+x_{\alpha_8}$
$(A_7)^{(3)}$	$x_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}+x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}+x_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5}+x_{\alpha_5+\alpha_6+\alpha_7}+x_{\alpha_6+\alpha_7+\alpha_8}$ $+x_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}+x_{\alpha_2+\alpha_4+\alpha_5+\alpha_6}+x_{\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8}$
A_7	$x_{\alpha_1}+x_{\alpha_3}+x_{\alpha_4}+x_{\alpha_5}+x_{\alpha_6}+x_{\alpha_7}+x_{\alpha_8}$
$D_7(a_2)$	$x_{\alpha_2}+x_{\alpha_3}+x_{\alpha_5}+x_{\alpha_7}+x_{\alpha_8}+x_{\alpha_3+\alpha_4}+x_{\alpha_4+\alpha_5}+x_{\alpha_5+\alpha_6}+x_{\alpha_6+\alpha_7}$
E_6	$x_{\alpha_1}+x_{\alpha_2}+x_{\alpha_3}+x_{\alpha_4}+x_{\alpha_5}+x_{\alpha_6}$
D_6	$x_{\alpha_2}+x_{\alpha_3}+x_{\alpha_4}+x_{\alpha_5}+x_{\alpha_6}+x_{\alpha_7}$
$(D_5+A_2)^{(2)}$	$x_{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6}+x_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}+x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}+x_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6}$ $+x_{\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7}+x_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5}+x_{\alpha_7+\alpha_8}+x_{\alpha_6+\alpha_7+\alpha_8}$
D_5+A_2	$x_{\alpha_1}+x_{\alpha_2}+x_{\alpha_3}+x_{\alpha_4}+x_{\alpha_5}+x_{\alpha_7}+x_{\alpha_8}$
$E_6(a_1)$	$x_{\alpha_1}+x_{\alpha_3}+x_{\alpha_5}+x_{\alpha_2+\alpha_4}+x_{\alpha_3+\alpha_4}+x_{\alpha_6}$
$E_7(a_4)$	$x_{\alpha_1}+x_{\alpha_2+\alpha_3+\alpha_4}+x_{\alpha_3+\alpha_4+\alpha_5}+x_{\alpha_2+\alpha_4+\alpha_5}+x_{\alpha_4+\alpha_5+\alpha_6}+x_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}+x_{\alpha_6+\alpha_7}$
A_6+A_1	$x_{\alpha_1}+x_{\alpha_3}+x_{\alpha_4}+x_{\alpha_5}+x_{\alpha_6}+x_{\alpha_7}+x_{\alpha_8}$
$D_6(a_1)$	$x_{\alpha_2}+x_{\alpha_3}+x_{\alpha_5}+x_{\alpha_6}+x_{\alpha_7}+x_{\alpha_3+\alpha_4}+x_{\alpha_4+\alpha_5}$
$(A_6)^{(2)}$	$x_{\alpha_1+\alpha_2+\alpha_3}+x_{\alpha_3+\alpha_4+\alpha_5}+x_{\alpha_2+\alpha_3+\alpha_4}+x_{\alpha_2+\alpha_4+\alpha_5}+x_{\alpha_5+\alpha_6}$ $+x_{\alpha_6+\alpha_7}+x_{\alpha_4+\alpha_5+\alpha_6+\alpha_7}$
A_6	$x_{\alpha_1}+x_{\alpha_3}+x_{\alpha_4}+x_{\alpha_5}+x_{\alpha_6}+x_{\alpha_7}$

Type E_8 (II)

Orbit	Representative
$E_8(a_7)$	$x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5} + x_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6} + x_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6}$ $+ x_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6} + x_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7} + x_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7}$ $+ x_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8} + x_{\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8}$
D_5+A_1	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_6} + x_{\alpha_7}$
$E_7(a_5)$	$x_{\alpha_1+\alpha_3+\alpha_4} + x_{\alpha_2+\alpha_3+\alpha_4} + x_{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6} + x_{\alpha_2+\alpha_4+\alpha_5} + x_{\alpha_4+\alpha_5+\alpha_6}$ $+ x_{\alpha_5+\alpha_6+\alpha_7} + x_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}$
$E_6(a_3)+A_1$	$x_{\alpha_3+\alpha_4+\alpha_5} + x_{\alpha_2+\alpha_4} + x_{\alpha_1} + x_{\alpha_1+\alpha_3} + x_{\alpha_6} + x_{\alpha_5+\alpha_6} + x_{\alpha_8}$
$D_6(a_2)$	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_7} + x_{\alpha_3+\alpha_4} + x_{\alpha_4+\alpha_5} + x_{\alpha_5+\alpha_6} + x_{\alpha_6+\alpha_7}$
$D_5(a_1)+A_2$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_3+\alpha_4} + x_{\alpha_4+\alpha_5} + x_{\alpha_7} + x_{\alpha_8}$
A_5+A_1	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_8}$
A_4+A_3	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_6} + x_{\alpha_7} + x_{\alpha_8}$
D_5	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5}$
$E_6(a_3)$	$x_{\alpha_3+\alpha_4+\alpha_5} + x_{\alpha_2+\alpha_4} + x_{\alpha_1} + x_{\alpha_1+\alpha_3} + x_{\alpha_6} + x_{\alpha_5+\alpha_6}$
$(D_4+A_2)^{(2)}$	$x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_8} + x_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8} + x_{\alpha_2+\alpha_4+\alpha_5}$
D_4+A_2	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_7} + x_{\alpha_8}$
$A_4+A_2+A_1$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_7} + x_{\alpha_8}$
$D_5(a_1)+A_1$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_3+\alpha_4} + x_{\alpha_4+\alpha_5} + x_{\alpha_7}$
A_5	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_6}$
A_4+A_2	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_7} + x_{\alpha_8}$
A_4+2A_1	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_7} + x_{\alpha_8}$
$D_5(a_1)$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_3+\alpha_4} + x_{\alpha_4+\alpha_5}$
$2A_3$	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_6} + x_{\alpha_7} + x_{\alpha_8}$
A_4+A_1	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_7}$
$D_4(a_1)+A_2$	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_3+\alpha_4} + x_{\alpha_4+\alpha_5} + x_{\alpha_7} + x_{\alpha_8}$
D_4+A_1	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_7}$
$A_3+A_2+A_1$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_7}$
A_4	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5}$
$(A_3+A_2)^{(2)}$	$x_{\alpha_7} + x_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7} + x_{\alpha_5} + x_{\alpha_2+\alpha_4+\alpha_5} + x_{\alpha_4} + x_{\alpha_3}$
A_3+A_2	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_6} + x_{\alpha_7}$
$D_4(a_1)+A_1$	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_3+\alpha_4} + x_{\alpha_4+\alpha_5} + x_{\alpha_7}$
A_3+2A_1	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_6} + x_{\alpha_8}$
$2A_2+2A_1$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_8}$
D_4	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5}$
$D_4(a_1)$	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_3+\alpha_4} + x_{\alpha_4+\alpha_5}$
A_3+A_1	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_6}$
$2A_2+A_1$	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_6} + x_{\alpha_8}$
$2A_2$	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_6}$
A_2+3A_1	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_7}$
A_3	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4}$
A_2+2A_1	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_5} + x_{\alpha_7}$
A_2+A_1	$x_{\alpha_1} + x_{\alpha_4} + x_{\alpha_5}$
$4A_1$	$x_{\alpha_1} + x_{\alpha_4} + x_{\alpha_6} + x_{\alpha_8}$
A_2	$x_{\alpha_1} + x_{\alpha_3}$
$3A_1$	$x_{\alpha_1} + x_{\alpha_4} + x_{\alpha_6}$
$2A_1$	$x_{\alpha_1} + x_{\alpha_4}$
A_1	x_{α_1}
$\{0\}$	0

Type F_4

Orbit	Representative
F_4	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4}$
$F_4(a_1)$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_2+\alpha_3} + x_{\alpha_3+\alpha_4}$
$F_4(a_2)$	$x_{\alpha_1+\alpha_2} + x_{\alpha_2+2\alpha_3} + x_{\alpha_4} + x_{\alpha_3+\alpha_4}$
$(C_3)^{(2)}$	$x_{\alpha_4} + x_{\alpha_1+\alpha_2+\alpha_3} + x_{\alpha_2+2\alpha_3} + x_{\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4}$
C_3	$x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4}$
B_3	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3}$
$F_4(a_3)$	$x_{\alpha_2} + x_{\alpha_1+\alpha_2} + x_{\alpha_2+2\alpha_3} + x_{\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4}$
$C_3(a_1)^{(2)}$	$x_{\alpha_1+\alpha_2} + x_{\alpha_2+2\alpha_3+\alpha_4} + x_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4} + x_{\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4}$
$C_3(a_1)$	$x_{\alpha_2} + x_{\alpha_4} + x_{\alpha_2+2\alpha_3}$
$(\widetilde{A_2} + A_1)^{(2)}$	$x_{\alpha_1+\alpha_2+\alpha_3} + x_{\alpha_2+2\alpha_3+\alpha_4} + x_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4} + x_{\alpha_1+2\alpha_2+2\alpha_3}$
$\widetilde{A_2} + A_1$	$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4}$
$(B_2)^{(2)}$	$x_{\alpha_1+\alpha_2} + x_{\alpha_1+\alpha_2+2\alpha_3} + x_{\alpha_2+2\alpha_3+2\alpha_4}$
B_2	$x_{\alpha_2} + x_{\alpha_3}$
$A_2 + \widetilde{A_1}$	$x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_4}$
$\widetilde{A_2}$	$x_{\alpha_3} + x_{\alpha_4}$
$(A_2)^{(2)}$	$x_{\alpha_1+2\alpha_2+2\alpha_3} + x_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4}$
A_2	$x_{\alpha_1} + x_{\alpha_2}$
$(\widetilde{A_1})^{(2)}$	$x_{\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4} + x_{2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4}$
$A_1 + \widetilde{A_1}$	$x_{\alpha_1} + x_{\alpha_3}$
$\widetilde{A_1}$	x_{α_3}
A_1	x_{α_1}
$\{0\}$	0

6. Jordan blocks

We provide below the Jordan block sizes of $\mathrm{ad}(x)$ for x a nilpotent element in an exceptional Lie algebra $\mathfrak{g}$ when p is a bad prime. Lawther [16,17] computed this information for unipotent elements for the minimal and adjoint representations for exceptional groups for all primes.

For good primes the nilpotent orbits and the unipotent classes are in bijective correspondence. Under this bijection, Lawther has also shown that the Jordan block sizes for the unipotent classes and the nilpotent orbits coincide for the adjoint representation. Consequently, our computations in conjunction with Lawther's results complete the determination of the Jordan block sizes on the adjoint representation for exceptional Lie algebras for all primes. Furthermore, our work shows that the Jordan block sizes for corresponding unipotent and nilpotent classes need not coincide in bad characteristic.

Type E_6

Orbit	Prime	Partition
E_6	3	$(9^6, 8^3)$
	2	$(16^4, 4^2, 3^2)$
$E_6(a_1)$	3	$(9^6, 8^3)$
	2	$(16^2, 11^2, 8^2, 4^2)$
D_5	3	$(9^5, 8^4, 1)$
	2	$(8^8, 4^2, 3^2)$
$E_6(a_3)$	3	$(9^3, 8^2, 6^4, 3^3, 2)$
	2	$(8^8, 4^2, 3^2)$
$D_5(a_1)$	3	$(9^3, 8^2, 6^4, 3^2, 2^2, 1)$
	2	$(8^2, 7^4, 5^2, 4^2, 3^4, 1^4)$
A_5	3	$(9^3, 7^4, 6^2, 3^2, 2, 1^3)$
	2	$(8^8, 2^6, 1^2)$
$A_4 + A_1$	3	$(9, 8^2, 7, 6^2, 5^3, 3^6, 1)$
	2	$(8^4, 6^2, 5^2, 4^4, 2^4)$
D_4	3	$(9^2, 7^7, 3, 1^8)$
	2	$(4^{18}, 3^2)$
A_4	3	$(9, 7^5, 5^3, 3^5, 1^4)$
	2	$(8^2, 7^4, 5^2, 4^2, 3^4, 1^4)$
$D_4(a_1)$	3	$(7, 6^2, 5^6, 3^9, 1^2)$
	2	$(4^{12}, 3^8, 1^6)$
$A_3 + A_1$	3	$(7, 6^2, 5^3, 4^6, 3^4, 2^2, 1^4)$
	2	$(4^{16}, 2^6, 1^2)$
$2A_2 + A_1$	3	$(3^{24}, 2^3)$
	2	$(4^{16}, 2^6, 1^2)$
A_3	3	$(7, 5^5, 4^8, 3, 1^{11})$
	2	$(4^{12}, 3^8, 1^6)$
$A_2 + 2A_1$	3	$(3^{23}, 2^4, 1)$
	2	$(4^{10}, 3^2, 2^{16})$
$2A_2$	3	$(3^{23}, 2, 1^7)$
	2	$(4^{16}, 1^{14})$
$A_2 + A_1$	3	$(3^{22}, 2^2, 1^8)$
	2	$(4^8, 3^6, 2^{10}, 1^8)$
A_2	3	$(3^{21}, 1^{15})$
	2	$(4^2, 3^{18}, 1^{16})$
$3A_1$	3	$(3^{13}, 2^{14}, 1^{11})$
	2	$(2^{38}, 1^2)$
$2A_1$	3	$(3^8, 2^{16}, 1^{22})$
	2	$(2^{32}, 1^{14})$
A_1	3	$(3, 2^{20}, 1^{35})$
	2	$(2^{22}, 1^{34})$
$\{0\}$	3, 2	(1^{78})

Type E_7

Orbit	Prime	Partition
E_7	3	$(27^3, 9^4, 8^2)$
	2	$(16^6, 15^2, 2, 1^5)$
$E_7(a_1)$	3	$(27, 19, 17^3, 9^4)$
	2	$(16^6, 15^2, 2, 1^5)$
$E_7(a_2)$	3	$(19, 17^3, 9^7)$
	2	$(16^4, 13^4, 4^2, 3^2, 2, 1)$
$E_7(a_3)$	3	$(19, 17, 15^2, 11^2, 9^3, 6^2, 3^2)$
	2	$(16^2, 15^2, 9^4, 8^4, 2, 1)$
E_6	3	$(9^{13}, 8^2)$
	2	$(16^4, 13^4, 4^2, 3^2, 1^3)$
$E_6(a_1)$	3	$(9^{13}, 8^2)$
	2	$(16^2, 13^2, 11^2, 9^2, 8^2, 5^2, 4^2, 1)$
D_6	3	$(19, 16^2, 15, 11, 10^2, 9^2, 6^2, 3, 1^3)$
	2	$(8^{14}, 7^2, 2, 1^5)$
$E_7(a_4)$	3	$(9^{11}, 8^2, 6^2, 3^2)$
	2	$(8^{14}, 7^2, 2, 1^5)$
$D_6(a_1)$	3	$(9^{11}, 7^4, 3, 1^3)$
	2	$(8^8, 7^4, 6^4, 2^6, 1^5)$
D_5+A_1	3	$(9^{11}, 7^4, 3, 1^3)$
	2	$(8^{12}, 5^4, 4^2, 3^2, 2, 1)$
$(A_6)^{(2)}$	2	$(8^{14}, 7^2, 2, 1^5)$
A_6	3	$(9^{11}, 7^4, 3, 1^3)$
	2	$(8^{14}, 7^2, 1^7)$
$E_7(a_5)$	3	$(9^9, 7, 6^2, 5^3, 3^6)$
	2	$(8^{12}, 5^4, 4^2, 3^2, 2, 1)$
D_5	3	$(9^7, 8^8, 1^6)$
	2	$(8^{12}, 5^4, 4^2, 3^2, 1^3)$
$E_6(a_3)$	3	$(9^5, 8^2, 6^8, 3^8)$
	2	$(8^{12}, 5^4, 4^2, 3^2, 1^3)$
$D_6(a_2)$	3	$(9^9, 3^{15}, 1^7)$
	2	$(8^8, 7^4, 6^4, 2^7, 1^3)$
$D_5(a_1)+A_1$	3	$(9^5, 7^4, 6^6, 3^7, 1^3)$
	2	$(8^4, 7^2, 6^4, 5^4, 4^6, 3^2, 2^6, 1)$
A_5+A_1	3	$(9^9, 3^{16}, 2^2)$
	2	$(8^8, 7^4, 6^4, 2^7, 1^3)$
$(A_5)'$	3	$(9^5, 7^4, 6^6, 3^7, 1^3)$
	2	$(8^8, 7^4, 6^4, 2^6, 1^5)$
A_4+A_2	3	$(9^3, 7, 6^8, 5^3, 3^{12})$
	2	$(8^6, 7^2, 5^4, 4^{12}, 1^3)$
$D_5(a_1)$	3	$(9^3, 8^4, 7^2, 6^6, 3^4, 2^4, 1^4)$
	2	$(8^2, 7^6, 5^8, 4^2, 3^6, 1^9)$
A_4+A_1	3	$(9, 8^2, 7^3, 6^4, 5^5, 4^2, 3^8, 2^2, 1^2)$
	2	$(8^4, 7^2, 6^4, 5^4, 4^6, 3^2, 2^6, 1)$
D_4+A_1	3	$(9^2, 8^4, 7^5, 6^4, 3^2, 2^4, 1^{10})$
	2	$(4^{30}, 3^2, 2, 1^5)$

Orbit	Prime	Partition
$(A_5)''$	3	$(9^9, 3^{15}, 1^7)$
	2	$(8^4, 7^{12}, 2, 1^{15})$
$A_3+A_2+A_1$	3	$(7, 6^8, 5^3, 3^{21})$
	2	$(4^{30}, 3^2, 2, 1^5)$
A_4	3	$(9, 7^7, 5^9, 3^7, 1^9)$
	2	$(8^2, 7^6, 5^8, 4^2, 3^6, 1^9)$
$(A_3+A_2)^{(2)}$	2	$(4^{30}, 3^2, 2, 1^5)$
A_3+A_2	3	$(7, 6^8, 5, 4^4, 3^{18}, 1^3)$
	2	$(4^{30}, 3^2, 1^7)$
$D_4(a_1)+A_1$	3	$(7, 6^6, 5^4, 4^4, 3^{16}, 1^6)$
	2	$(4^{24}, 3^4, 2^{10}, 1^5)$
D_4	3	$(9^2, 7^{13}, 3, 1^{21})$
	2	$(4^{30}, 3^2, 1^7)$
A_3+2A_1	3	$(7, 6^4, 5^7, 4^4, 3^{11}, 2^6, 1^6)$
	2	$(4^{24}, 3^4, 2^{11}, 1^3)$
$D_4(a_1)$	3	$(7, 6^2, 5^{12}, 3^{15}, 1^9)$
	2	$(4^{20}, 3^{12}, 1^{17})$
$(A_3+A_1)'$	3	$(7, 6^2, 5^7, 4^{10}, 3^6, 2^6, 1^9)$
	2	$(4^{24}, 3^4, 2^{10}, 1^5)$
$2A_2+A_1$	3	$(3^{43}, 2^2)$
	2	$(4^{24}, 3^4, 2^{10}, 1^5)$
$(A_3+A_1)''$	3	$(7, 5^{15}, 3^{10}, 1^{21})$
	2	$(4^{20}, 3^{12}, 2, 1^{15})$
A_2+3A_1	3	$(3^{42}, 1^7)$
	2	$(4^{14}, 3^6, 2^{29}, 1)$
$2A_2$	3	$(3^{42}, 1^7)$
	2	$(4^{20}, 3^{12}, 1^{17})$
A_3	3	$(7, 5^7, 4^{16}, 3, 1^{24})$
	2	$(4^{20}, 3^{12}, 1^{17})$
A_2+2A_1	3	$(3^{37}, 2^8, 1^6)$
	2	$(4^{14}, 3^6, 2^{28}, 1^3)$
A_2+A_1	3	$(3^{34}, 2^8, 1^{15})$
	2	$(4^{10}, 3^{14}, 2^{18}, 1^{15})$
$4A_1$	3	$(3^{28}, 2^{14}, 1^{21})$
	2	$(2^{63}, 1^7)$
A_2	3	$(3^{33}, 1^{34})$
	2	$(4^2, 3^{30}, 1^{35})$
$(3A_1)'$	3	$(3^{19}, 2^{26}, 1^{24})$
	2	$(2^{62}, 1^9)$
$(3A_1)''$	3	$(3^{27}, 1^{52})$
	2	$(2^{53}, 1^{27})$
$2A_1$	3	$(3^{10}, 2^{32}, 1^{39})$
	2	$(2^{52}, 1^{29})$
A_1	3	$(3, 2^{32}, 1^{66})$
	2	$(2^{34}, 1^{65})$
$\{0\}$	3, 2	(1^{133})

Type E_8 (I)

Orbit	Prime	Partition
E_8	5	$(25^8, 24^2)$
	3	$(27^7, 26^2, 3, 2^2)$
	2	$(16^8, 15^8)$
$E_8(a_1)$	5	$(25^8, 24^2)$
	3	$(27^7, 26^2, 3, 2^2)$
	2	$(16^8, 15^8)$
$E_8(a_2)$	5	$(25^6, 24^2, 20^2, 5^2)$
	3	$(27^5, 26^2, 20^2, 9^2, 3)$
	2	$(16^8, 15^8)$
$E_8(a_3)$	5	$(25^4, 24^2, 22^2, 15^2, 10^2, 3^2)$
	3	$(27^3, 20^2, 18^4, 9^4, 8^2, 3)$
	2	$(16^8, 15^8)$
$E_8(a_4)$	5	$(25^2, 24^2, 20^2, 15^4, 10^4, 5^2)$
	3	$(27, 26^2, 21^2, 18^2, 15^2, 12^2, 9^2, 8^2, 3)$
	2	$(16^8, 15^8)$
E_7	5	$(25^4, 23^4, 15^2, 10^2, 3, 1^3)$
	3	$(27^3, 19^4, 18^2, 9^4, 8^2, 1^3)$
	2	$(16^6, 15^2, 14^8, 2^2, 1^6)$
$E_8(b_4)$	5	$(25^2, 23, 21, 19, 17^2, 15^2, 13, 11^3, 7^2, 5, 3^2)$
	3	$(27, 20^2, 19, 18^2, 17^3, 9^8, 3)$
	2	$(16^6, 15^2, 14^8, 2^2, 1^6)$
$E_8(a_5)$	5	$(23^2, 21, 19, 17, 15^3, 13^2, 11^3, 10^2, 7, 5, 3^3)$
	3	$(23^2, 21, 19, 17, 15^3, 12^4, 11^2, 9, 7, 5, 3^3)$
	2	$(16^4, 15^4, 12^8, 4^4, 3^4)$
$E_7(a_1)$	5	$(25^2, 22^2, 19, 17, 16^2, 15, 12^2, 11^2, 7, 6^2, 3, 1^3)$
	3	$(27, 19^5, 17^3, 9^8, 1^3)$
	2	$(16^6, 15^2, 14^8, 2^2, 1^6)$
$E_8(b_5)$	5	$(23, 19^2, 17^3, 15^3, 11, 10^4, 9, 7^2, 5, 3^4)$
	3	$(19, 18^2, 17^3, 11^3, 9^{11}, 7, 3)$
	2	$(16^4, 15^4, 12^8, 4^4, 3^4)$
$(D_7)^{(2)}$	2	$(16^2, 15^6, 8^{14}, 7^2)$
D_7	5	$(23, 22^2, 19, 16^2, 15, 13^3, 12^2, 11, 10^2, 7, 4^2, 3, 1^3)$
	3	$(23, 22^2, 19, 16^2, 15, 13^3, 12^2, 11, 10^2, 7, 4^2, 3, 1^3)$
	2	$(8^{24}, 7^8)$
$E_8(a_6)$	5	$(19^2, 15^5, 11, 10^8, 9, 5^7)$
	3	$(19, 18^2, 15^3, 13^3, 9^9, 7, 6^2, 3^3)$
	2	$(16^2, 15^6, 8^{14}, 7^2)$
$E_7(a_2)$	5	$(23, 19, 18^2, 17, 16^2, 15^2, 11, 10^4, 8^2, 7, 4^2, 3^2, 1^3)$
	3	$(19, 18^2, 17^3, 10^6, 9^7, 8^2, 1^3)$
	2	$(16^4, 14^4, 13^4, 12^4, 4^2, 3^2, 2^6, 1^2)$
$E_6 + A_1$	5	$(23, 18^2, 17^3, 16^2, 15, 11, 10^2, 9^3, 8^2, 3^2, 2^4, 1^3)$
	3	$(9^{25}, 8^2, 3, 2^2)$
	2	$(16^4, 14^4, 13^4, 12^4, 4^2, 3^2, 2^6, 1^2)$
$(D_7(a_1))^{(2)}$	2	$(16^2, 15^2, 14^4, 9^4, 8^8, 6^4, 2^2, 1^2)$

Type E_8 (II)

Orbit	Prime	Partition
$D_7(a_1)$	5	$(19, 17^2, 15^3, 13, 11^3, 10^6, 7^2, 5^4, 3^3, 1)$
	3	$(19, 17^2, 15^3, 12^2, 11^4, 9^5, 6^4, 3^4, 1)$
	2	$(8^{14}, 7^{18}, 1^{10})$
$E_8(b_6)$	5	$(17, 15^3, 13^2, 11^3, 10^6, 7^3, 5^8, 3^2)$
	3	$(9^{25}, 8^2, 1^7)$
	2	$(16^2, 15^2, 12^4, 11^4, 8^6, 7^2, 4^8)$
$E_7(a_3)$	5	$(19, 17, 16^2, 15^2, 12^2, 11^2, 10^6, 7, 6^2, 5^3, 3, 2^2, 1^3)$
	3	$(19, 17, 16^2, 15^2, 12^2, 11^2, 10^4, 9^3, 6^4, 3^2, 2^2, 1^3)$
	2	$(16^2, 15^2, 14^4, 9^4, 8^8, 6^4, 2^2, 1^2)$
$E_6(a_1) + A_1$	5	$(17, 15, 14^2, 13^2, 12^2, 11, 10^4, 9^2, 8^2, 7, 5^7, 3^2, 2^2, 1)$
	3	$(9^{25}, 8^2, 3, 2^2)$
	2	$(16^2, 14^2, 13^2, 12^2, 11^2, 10^2, 9^2, 8^4, 6^2, 5^2, 4^4, 2^4)$
$(A_7)^{(3)}$	3	$(9^{25}, 8^2, 3, 2^2)$
A_7	5	$(16^2, 15, 13^3, 12^2, 11, 10^2, 9^3, 8^4, 7, 5^7, 3, 1^3)$
	3	$(9^{25}, 8^2, 1^7)$
	2	$(8^{24}, 7^8)$
$D_7(a_2)$	5	$(15, 14^2, 11, 10^{10}, 9^2, 5^{15}, 1)$
	3	$(9^{19}, 7^8, 3^5, 1^6)$
	2	$(8^{24}, 7^8)$
E_6	5	$(23, 17^7, 15, 11, 9^7, 3, 1^{14})$
	3	$(9^{25}, 8^2, 1^7)$
	2	$(16^4, 13^{12}, 4^2, 3^2, 1^{14})$
D_6	5	$(19, 16^4, 15, 11^6, 10^4, 6^4, 5^2, 1^{10})$
	3	$(19, 16^4, 15, 11^5, 10^4, 9^2, 6^4, 3, 1^{10})$
	2	$(8^{22}, 7^2, 6^8, 2^2, 1^6)$
$(D_5 + A_2)^{(2)}$	2	$(8^{22}, 7^2, 6^8, 2^2, 1^6)$
$D_5 + A_2$	5	$(15, 13^2, 11^3, 10^8, 9, 7^2, 5^{11}, 3^5, 1)$
	3	$(9^{19}, 8^4, 6^4, 3^7)$
	2	$(8^{20}, 7^4, 4^{12}, 3^4)$
$E_6(a_1)$	5	$(17, 15, 13^6, 11, 10^2, 9^6, 7, 5^7, 3, 1^8)$
	3	$(9^{25}, 8^2, 1^7)$
	2	$(16^2, 13^6, 11^2, 9^6, 8^2, 5^6, 4^2, 1^8)$
$E_7(a_4)$	5	$(15, 13, 12^2, 11^2, 10^8, 8^2, 7, 5^{10}, 4^2, 3^2, 2^2, 1^3)$
	3	$(9^{19}, 8^2, 7^4, 6^2, 3^6, 1^3)$
	2	$(8^{22}, 7^2, 6^8, 2^2, 1^6)$
$A_6 + A_1$	5	$(13^3, 12^2, 11, 10^2, 9^3, 8^4, 7^5, 6^4, 5^3, 4^2, 3^2, 2^2, 1^3)$
	3	$(9^{19}, 8^2, 7^4, 6^2, 3^6, 1^3)$
	2	$(8^{22}, 7^2, 6^8, 2^2, 1^6)$
$D_6(a_1)$	5	$(15, 12^4, 11, 10^6, 9^4, 7, 5^{10}, 3^5, 1^6)$
	3	$(9^{19}, 7^8, 3^5, 1^6)$
	2	$(8^8, 7^{12}, 6^{12}, 2^6, 1^{16})$
$(A_6)^{(2)}$	2	$(8^{22}, 7^2, 6^8, 2^2, 1^6)$
A_6	5	$(13^3, 11^5, 9^3, 7^{13}, 5^3, 3^5, 1^6)$
	3	$(9^{19}, 7^8, 3^5, 1^6)$
	2	$(8^{14}, 7^{18}, 1^{10})$

Type E_8 (III)

Orbit	Prime	Partition
$E_8(a_7)$	5	$(11, 10^6, 9^3, 7^5, 5^{20}, 3^5)$
	3	$(9^{14}, 7^2, 6^8, 5^6, 3^{10})$
	2	$(8^{20}, 7^4, 4^{12}, 3^4)$
$D_5 + A_1$	5	$(15, 12^2, 11^5, 10^4, 9^3, 8^2, 5^{10}, 3, 2^6, 1^6)$
	3	$(9^{15}, 8^8, 7^4, 3, 2^6, 1^6)$
	2	$(8^{20}, 6^4, 5^4, 4^6, 3^2, 2^6, 1^2)$
$E_7(a_5)$	5	$(11, 10^6, 9, 8^4, 7^2, 6^2, 5^{18}, 4^2, 3^3, 1^3)$
	3	$(9^{13}, 8^2, 7, 6^8, 5^3, 4^6, 3^6, 1^3)$
	2	$(8^{20}, 6^4, 5^4, 4^6, 3^2, 2^6, 1^2)$
$E_6(a_3) + A_1$	5	$(11, 10^4, 9^3, 8^4, 7^3, 6^2, 5^{17}, 3^3, 2^4, 1^3)$
	3	$(9^9, 8^2, 6^{16}, 3^{17}, 2^2)$
	2	$(8^{20}, 6^4, 5^4, 4^6, 3^2, 2^6, 1^2)$
$D_6(a_2)$	5	$(11, 10^6, 8^4, 7^5, 5^{17}, 4^4, 3, 1^6)$
	3	$(9^9, 7^4, 6^{14}, 3^{15}, 1^{10})$
	2	$(8^{12}, 7^4, 6^{16}, 2^{12}, 1^4)$
$D_5(a_1) + A_2$	5	$(11, 10^2, 9^3, 8^6, 7^4, 6^4, 5^{10}, 4^2, 3^7, 2^4, 1^3)$
	3	$(9^9, 8^2, 6^{16}, 3^{17}, 2^2)$
	2	$(8^6, 7^{10}, 5^4, 4^{20}, 3^8, 1^6)$
$A_5 + A_1$	5	$(11, 10^4, 9^3, 8^2, 7^4, 6^6, 5^{13}, 4^2, 3, 2^4, 1^6)$
	3	$(9^9, 7^4, 6^{14}, 3^{16}, 2^2, 1^3)$
	2	$(8^{12}, 7^4, 6^{16}, 2^{12}, 1^4)$
$A_4 + A_3$	5	$(5^{48}, 4^2)$
	3	$(9^7, 8^2, 7^6, 6^6, 5^6, 4^6, 3^{10}, 2^2, 1^3)$
	2	$(8^{10}, 7^6, 4^{30}, 3^2)$
D_5	5	$(15, 11^9, 9^7, 5^{10}, 1^{21})$
	3	$(9^{11}, 8^{16}, 1^{21})$
	2	$(8^{20}, 5^{12}, 4^2, 3^2, 1^{14})$
$E_6(a_3)$	5	$(11, 10^2, 9^7, 7^7, 5^{17}, 3^2, 1^{14})$
	3	$(9^9, 8^2, 6^{16}, 3^{16}, 1^7)$
	2	$(8^{20}, 5^{12}, 4^2, 3^2, 1^{14})$
$(D_4 + A_2)^{(2)}$	2	$(8^{10}, 7^2, 6^4, 5^4, 4^{24}, 2^6, 1^2)$
$D_4 + A_2$	5	$(11, 9^6, 7^{14}, 5^7, 3^{14}, 1^8)$
	3	$(9^8, 7^7, 6^{12}, 3^{16}, 1^7)$
	2	$(4^{56}, 3^8)$
$A_4 + A_2 + A_1$	5	$(5^{46}, 4^2, 3^2, 2^2)$
	3	$(9^3, 8^2, 7, 6^{18}, 5^3, 3^{25})$
	2	$(8^{10}, 7^2, 6^4, 5^4, 4^{24}, 2^6, 1^2)$
$D_5(a_1) + A_1$	5	$(11, 9^3, 8^6, 7^8, 6^6, 5^3, 4^2, 3^7, 2^{10}, 1^6)$
	3	$(9^5, 8^6, 7^4, 6^{12}, 3^{11}, 2^8, 1^6)$
	2	$(8^4, 7^6, 6^8, 5^8, 4^{10}, 3^6, 2^{10}, 1^8)$
A_5	5	$(11, 10^2, 9^7, 6^{14}, 5^9, 4^2, 1^{17})$
	3	$(9^9, 7^4, 6^{14}, 3^{15}, 1^{10})$
	2	$(8^8, 7^{12}, 6^{12}, 2^6, 1^{16})$

Type E_8 (IV)

Orbit	Prime	Partition
A_4+A_2	5	$(5^{46},3^5,1^3)$
	3	$(9^3,7^5,6^{16},5^3,3^{24},1^3)$
	2	$(8^6,7^{10},5^4,4^{20},3^8,1^6)$
A_4+2A_1	5	$(5^{45},3^4,2^4,1^3)$
	3	$(9,8^4,7^3,6^{12},5^7,4^4,3^{17},2^4,1^4)$
	2	$(8^6,7^2,6^{12},5^4,4^{14},3^2,2^{16})$
$D_5(a_1)$	5	$(11,9,8^8,7^8,6^8,5,3^8,2^8,1^{15})$
	3	$(9^3,8^8,7^6,6^{10},3^8,2^8,1^{15})$
	2	$(8^2,7^{10},5^{20},4^2,3^{10},1^{24})$
$2A_3$	5	$(5^{38},4^{12},1^{10})$
	3	$(8^4,7^6,6^4,5^{10},4^{16},3^6,2^4,1^{10})$
	2	$(4^{56},3^8)$
A_4+A_1	5	$(5^{45},3,2^6,1^8)$
	3	$(9,8^2,7^7,6^8,5^9,4^6,3^{12},2^6,1^9)$
	2	$(8^4,7^6,6^8,5^8,4^{10},3^6,2^{10},1^8)$
$D_4(a_1)+A_2$	5	$(5^{36},3^{20},1^8)$
	3	$(7,6^{14},5^6,3^{42},1)$
	2	$(4^{46},3^{18},1^{10})$
D_4+A_1	5	$(11,8^6,7^{14},6^6,3^2,2^{14},1^{21})$
	3	$(9^2,8^6,7^{13},6^6,3^2,2^{14},1^{21})$
	2	$(4^{54},3^2,2^{10},1^6)$
$A_3+A_2+A_1$	5	$(5^{32},4^8,3^{10},2^{10},1^6)$
	3	$(7,6^{14},5^3,4^6,3^{37},2^2,1^3)$
	2	$(4^{54},3^2,2^{10},1^6)$
A_4	5	$(5^{45},1^{23})$
	3	$(9,7^{11},5^{21},3^{11},1^{24})$
	2	$(8^2,7^{10},5^{20},4^2,3^{10},1^{24})$
$(A_3+A_2)^{(2)}$	2	$(4^{54},3^2,2^{10},1^6)$
A_3+A_2	5	$(5^{30},4^8,3^{13},2^8,1^{11})$
	3	$(7,6^{12},5^5,4^8,3^{34},1^{10})$
	2	$(4^{46},3^{18},1^{10})$
$D_4(a_1)+A_1$	5	$(5^{29},4^6,3^{14},2^{14},1^9)$
	3	$(7,6^8,5^{12},4^6,3^{28},2^8,1^9)$
	2	$(4^{40},3^{12},2^{18},1^{16})$
A_3+2A_1	5	$(5^{25},4^{10},3^{14},2^{14},1^{13})$
	3	$(7,6^6,5^{11},4^{14},3^{19},2^{12},1^{13})$
	2	$(4^{44},3^4,2^{28},1^4)$
$2A_2+2A_1$	5	$(5^{18},4^{12},3^{20},2^{20},1^{10})$
	3	$(3^{80},2^4)$
	2	$(4^{44},3^4,2^{28},1^4)$
D_4	5	$(11,7^{26},3,1^{52})$
	3	$(9^2,7^{25},3,1^{52})$
	2	$(4^{54},3^2,1^{26})$
$D_4(a_1)$	5	$(5^{29},3^{25},1^{28})$
	3	$(7,6^2,5^{24},3^{27},1^{28})$
	2	$(4^{36},3^{20},1^{44})$
A_3+A_1	5	$(5^{21},4^{16},3^9,2^{14},1^{24})$
	3	$(7,6^2,5^{15},4^{18},3^{10},2^{14},1^{24})$
	2	$(4^{40},3^{12},2^{18},1^{16})$
$2A_2+A_1$	5	$(5^{14},4^{14},3^{23},2^{18},1^{17})$
	3	$(3^{79},2^2,1^7)$
	2	$(4^{40},3^{12},2^{18},1^{16})$
$2A_2$	5	$(5^{14},3^{50},1^{28})$
	3	$(3^{78},1^{14})$
	2	$(4^{28},3^{36},1^{28})$
A_2+3A_1	5	$(5^7,4^{14},3^{28},2^{28},1^{17})$
	3	$(3^{70},2^{14},1^{10})$
	2	$(4^{26},3^6,2^{62},1^2)$
A_3	5	$(5^{13},4^{32},1^{55})$
	3	$(7,5^{11},4^{32},3,1^{55})$
	2	$(4^{36},3^{20},1^{44})$
A_2+2A_1	5	$(5^3,4^{16},3^{27},2^{32},1^{24})$
	3	$(3^{65},2^{16},1^{21})$
	2	$(4^{22},3^{14},2^{52},1^{14})$
A_2+A_1	5	$(5,4^{12},3^{32},2^{32},1^{35})$
	3	$(3^{58},2^{20},1^{34})$
	2	$(4^{14},3^{30},2^{34},1^{34})$
$4A_1$	5	$(4^8,3^{28},2^{48},1^{36})$
	3	$(3^{44},2^{40},1^{36})$
	2	$(2^{120},1^8)$
A_2	5	$(5,3^{55},1^{78})$
	3	$(3^{57},1^{77})$
	2	$(4^2,3^{54},1^{78})$
$3A_1$	5	$(4^2,3^{27},2^{52},1^{55})$
	3	$(3^{31},2^{50},1^{55})$
	2	$(2^{110},1^{28})$
$2A_1$	5	$(3^{14},2^{64},1^{78})$
	3	$(3^{14},2^{64},1^{78})$
	2	$(2^{92},1^{64})$
A_1	5	$(3,2^{56},1^{133})$
	3	$(3,2^{56},1^{133})$
	2	$(2^{58},1^{132})$
$\{0\}$	5, 3, 2	(1^{248})

Type F_4

Orbit	Prime	Partition
F_4	3	$(9^4, 8^2)$
	2	$(16^2, 4^2, 3^4)$
$F_4(a_1)$	3	$(9^4, 8^2)$
	2	$(8^4, 4^2, 3^4)$
$F_4(a_2)$	3	$(9^2, 8^2, 6^2, 3^2)$
	2	$(8^4, 4^2, 3^4)$
$(C_3)^{(2)}$	2	$(8^4, 4, 3^3, 2^3, 1)$
C_3	3	$(9^2, 7^4, 3, 1^3)$
	2	$(8^4, 2^8, 1^4)$
B_3	3	$(9^2, 7^4, 3, 1^3)$
	2	$(4^{10}, 3^4)$
$F_4(a_3)$	3	$(7, 6^2, 5^3, 3^6)$
	2	$(4^{10}, 3^4)$
$C_3(a_1)^{(2)}$	2	$(4^9, 3^3, 2^3, 1)$
$C_3(a_1)$	3	$(7, 6^2, 5, 4^4, 3^3, 1^3)$
	2	$(4^8, 2^8, 1^4)$
$(A_1 + \widetilde{A}_2)^{(2)}$	2	$(4^9, 3^3, 2^3, 1)$
$A_1 + \widetilde{A}_2$	3	$(3^{16}, 2^2)$
	2	$(4^8, 2^8, 1^4)$

Orbit	Prime	Partition
$(B_2)^{(2)}$	2	$(4^7, 3^7, 1^3)$
B_2	3	$(7, 5^4, 4^4, 3, 1^6)$
	2	$(4^6, 3^4, 2^5, 1^6)$
$A_2 + \widetilde{A}_1$	3	$(3^{16}, 2^2)$
	2	$(4^6, 3^4, 2^8)$
$\widetilde{A}_2$	3	$(3^{15}, 1^7)$
	2	$(4^8, 1^{20})$
$(A_2)^{(2)}$	2	$(4^2, 3^{12}, 1^8)$
A_2	3	$(3^{15}, 1^7)$
	2	$(4^2, 3^{12}, 1^8)$
$A_1 + \widetilde{A}_1$	3	$(3^{10}, 2^8, 1^6)$
	2	$(2^{24}, 1^4)$
$(\widetilde{A}_1)^{(2)}$	2	$(2^{21}, 1^{10})$
$\widetilde{A}_1$	3	$(3^7, 2^8, 1^{15})$
	2	$(2^{16}, 1^{20})$
A_1	3	$(3, 2^{14}, 1^{21})$
	2	$(2^{16}, 1^{20})$
$\{0\}$	3,2	(1^{52})

Type G_2

Orbit	Prime	Partition
G_2	3	$(9, 3, 2)$
	2	$(4^2, 3^2)$
$G_2(a_1)$	3	$(3^4, 2)$
	2	$(4^2, 3^2)$
$(\widetilde{A}_1)^{(3)}$	3	$(3^3, 2^2, 1)$
$\widetilde{A}_1$	3	$(3^3, 1^5)$
	2	$(2^6, 1^2)$
A_1	3	$(3, 2^4, 1^3)$
	2	$(2^6, 1^2)$
$\{0\}$	3,2	(1^{14})

7. Hasse diagrams

In this section, we provide the Hasse diagram for $\mathcal{U}_1(G)$ and $\mathcal{N}_1(\mathfrak{g})$ when Φ is of exceptional type. The inclusion of closures of unipotent classes over bad characteristics is given in [21,22,25]. For nilpotent orbits the inclusions of orbit closures are known for bad characteristics for G_2 [29], F_4 [13,27], E_6 [26]. The situation for E_7 and E_8 is still undetermined. However, by using our computations in Section 6, the new orbits which arise in these cases are not contained in $\mathcal{N}_1(\mathfrak{g})$. Therefore, the inclusion of orbit closures in $\mathcal{N}_1(\mathfrak{g})$ will be the same as in characteristic zero.

We should remark that the Hasse diagram for E_8 in [5, p. 444] is not correct. Eric Sommers has pointed out that there should be a line between the orbits $E_6 + A_1$ and $E_8(b_6)$.

We have also found that there should not be a line between $D_4(a_1)$ and A_3, but rather a line between $A_3 + A_1$ and A_3. These corrections agree with the Hasse diagram produced by Spaltenstein [25].

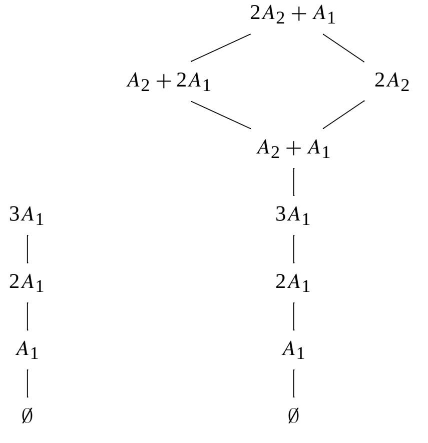

Fig. 1. E_6 for $p = 2$ (left) and $p = 3$ (right).

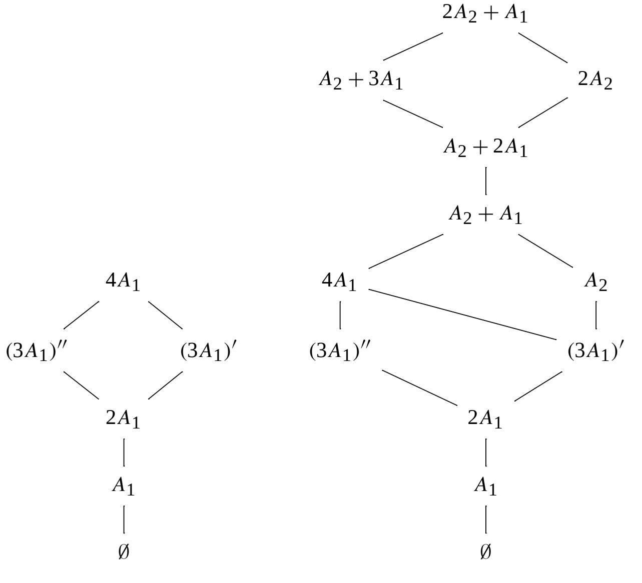

Fig. 2. E_7 for $p = 2$ (left) and $p = 3$ (right).

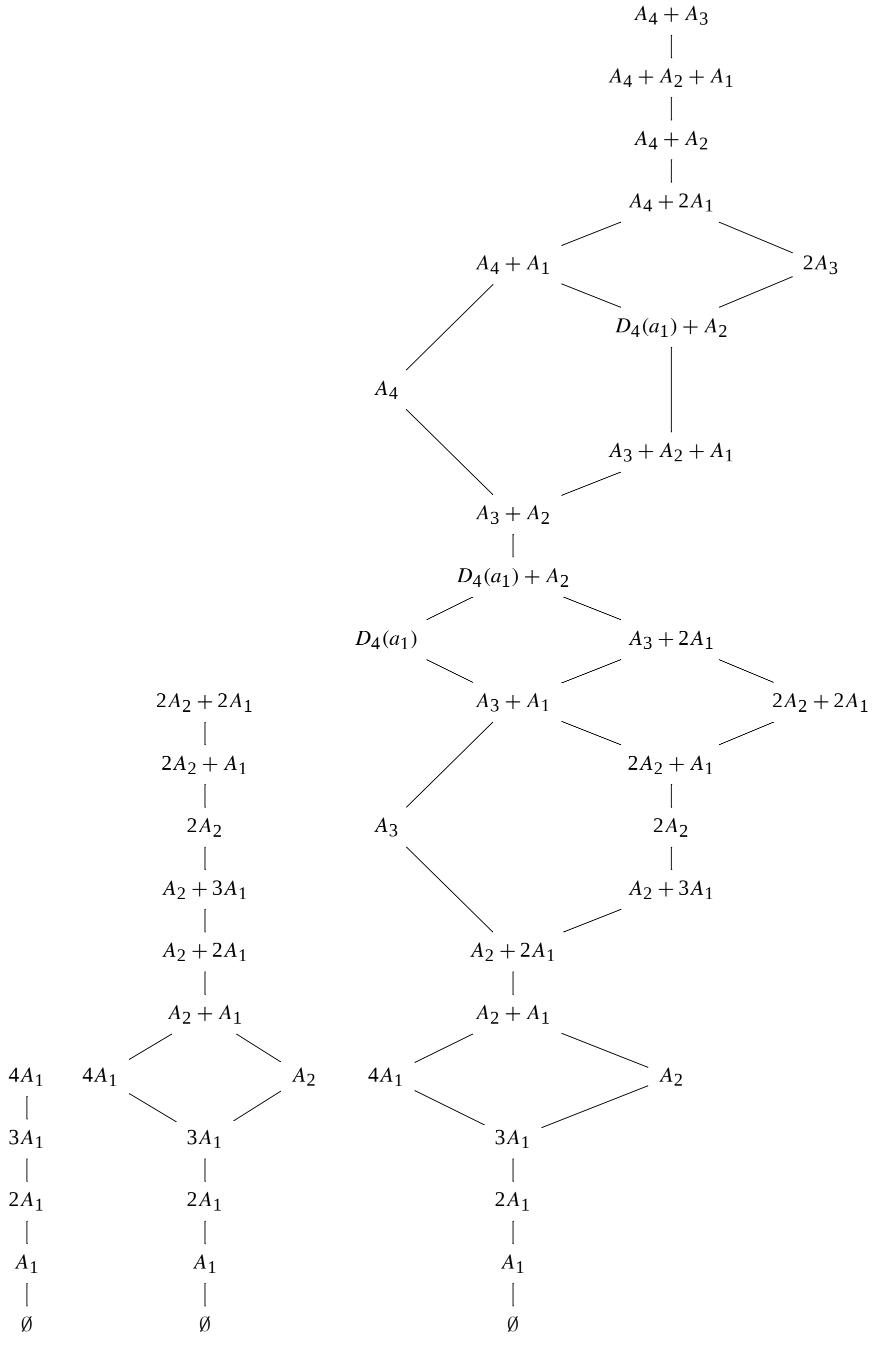

Fig. 3. E_8 for $p=2$ (left), $p=3$, (center), and $p=5$ (right).

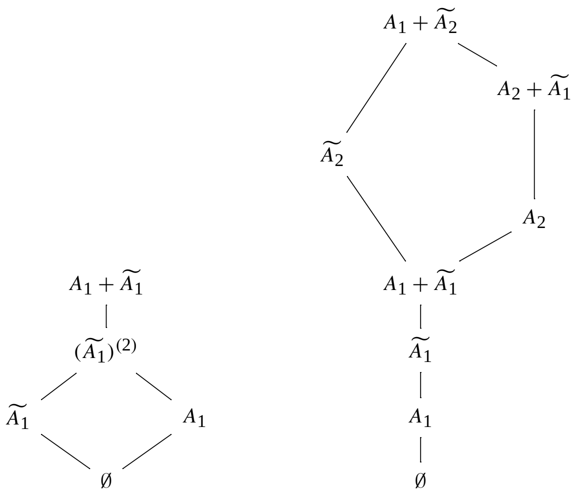

Fig. 4. F_4 with $p = 2$ (left) and $p = 3$ (right).

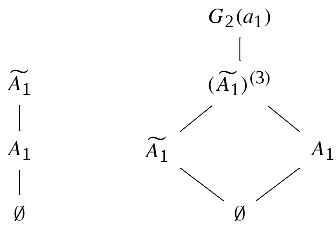

Fig. 5. G_2 for $p = 2$ (left) and $p = 3$ (right).

8. VIGRE Algebra Group at the University of Georgia

This project was initiated during 2003–2004 under the Vertical Integration of Research and Education (VIGRE) Program sponsored by the National Science Foundation (NSF) at the Department of Mathematics at the University of Georgia (UGA). The VIGRE Algebra Group consists of 4 faculty members, 4 postdoctoral fellows and 7 graduate students. The group was led by David J. Benson, Brian D. Boe and Daniel K. Nakano. The e-mail addresses of the members of the group are given below.

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