Edge-colorings of K_n Which Forbid Rainbow Cycles

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Abstract

The greatest number of colors that can appear on the edges of K_n in a coloring that forbids rainbow K_3's, and thus all rainbow cycles, is $n-1$. We charactrize all such colorings (with $n-1$ colors): for $n \geq 3$ the essentially different such colorings are in natural one-to-one correspondence with the full binary trees with n leafs. Related results:

(i) if $K_n^{(\lambda)}$ is the multigraph obtained by multiplying each edge of K_n by λ, then the greatest number of colors appearing on the edges of $K_n^{(\lambda)}$ in a coloring that forbids rainbow K_3's is $n-1+(\lambda-1)\lfloor \frac{n}{2} \rfloor$;

(ii) there is an edge-coloring of K_n using t colors which forbids rainbow K_3's and uses the different colors as close to equally often as possible if and only if $t \in \{1, \ldots, \lceil \frac{n}{2} \rceil\}$.

1 Introduction

Suppose that G and H are finite simple graphs. An edge-coloring of G *forbids rainbow* H's (copies of H) if each copy of H in G has two edges with the same color. (Thus, if H does not occur as a subgraph of G, every edge-coloring of G forbids rainbow H's). If $\mathcal{H}$ is a family or collection of graphs, then an edge-coloring of G forbids rainbow members of $\mathcal{H}$ if the coloring forbids rainbow H's for each $H \in \mathcal{H}$.

This paper is about edge-colorings of complete graphs which forbid rainbow cycles, as announced in the title. Our first result reveals that this

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announcement was perhaps unnecessarily dramatic. A *chordal* graph is a simple graph with no induced cycle of length greater than 3.

Proposition 1.1 *Suppose that G is chordal. Then an edge-coloring of G forbids rainbow cycles if and only if it forbids rainbow K_3's.*

Proof The "only if" assertion is clear. Now suppose that an edge-coloring of G does not forbid rainbow cycles. We shall see that it must admit a rainbow K_3. Let C be a shortest rainbow cycle in G. If the length of C were > 3 there would be a "chord", an edge of G whose end-vertices divide C into two paths of lengths ≥ 2. This chord would create two cycles in G both shorter than C, of which at least one must be rainbow. Therefore, C must be of length 3, meaning $C \simeq K_3$. □

The inspiration for what follows is a cute Hungarian-schoolboy-prize-exam type result that we first heard from Jenö Lehel. We have since learned that it descends from [1].

Theorem JL *If the edges of K_n are colored with n or more colors actually appearing then there is a rainbow K_3 somewhere. On the other hand, K_n can be edge-colored with $n-1$ colors appearing and with rainbow K_3's forbidden.*

We will not spoil your enjoyment by giving a formal proof of Theorem JL. Our main result is about all possible colorings verifying the second assertion. The first can be proved by induction on n, or, more beautifully: assuming such a coloring, look at the graph induced by n edges of different colors.

A $JL(n)$ coloring is a coloring of the edges of K_n with exactly $n-1$ colors, forbidding rainbow K_3's (and thus rainbow cycles, by Proposition 1.1).

2 Characterization of $JL(n)$ Colorings

Lemma 2.1 *Suppose that $n \geq 2$ and that K_n is colored with a $JL(n)$ coloring. Then for each $v \in V(K_n)$ there is exactly one color that appears on edges incident to v which does not appear in $K_n - v$. It follows that the coloring restricted to $K_n - v$ is a $JL(n-1)$ coloring.*

Proof If all $n-1$ colors appear in $K_n - v$ then by Theorem JL there is a rainbow K_3 in $K_n - v$. Therefore, at least one color appears on the edges incident to v which does not appear in $K_n - v$. If as many as two colors appear on the edges incident to v which do not appear in $K_n - v$, then there is a rainbow K_3. □

Corollary 2.2 *Suppose that* $S \subset V(K_n)$, $1 \leq |S| \leq n$, *and* K_n *is colored with a* $JL(n)$ *coloring. Then the coloring restricted to* $\langle S\rangle$, *the clique induced in* K_n *by* S, *is a* $JL(|S|)$ *coloring.*

Proof If $|S| = n$, the conclusion is clear. Otherwise, let $v_1, \ldots, v_k$ be some ordering of $V(K_n)\backslash S$. Then $\langle S\rangle = \ldots(\ldots((K_n - v_1) - v_2)\ldots)\ldots$, and each intermediate graph is a clique. Apply Lemma 2.1 k times. $\square$

Corollary 2.3 *In a* $JL(n)$ *coloring no* K_3 *is monochromatic.*

Proof This follows from Corollary 2.2; take $|S| = 3$. $\square$

Theorem 2.4 *Suppose that* $n \geq 2$. *Every* $JL(n)$ *coloring is obtainable as follows: choose positive integers* r, s *satisfying* $r + s = n$; *partition* $V(K_n)$ *into sets* R, S *satisfying* $|R| = r$, $|S| = s$. *Color all* R*-to-*S *edges in* K_n *with one color–say green. Color* $\langle R\rangle$ *with a* $JL(r)$ *coloring and* $\langle S\rangle$ *with a* $JL(s)$ *coloring with disjoint sets of colors on the two cliques, and with green not appearing in* $\langle R\rangle$, *nor in* $\langle S\rangle$.

Further, any edge-coloring of K_n *arrived at by following the directions above is a* $JL(n)$ *coloring.*

Proof First, if a coloring is arrived at by following the directions, then it is straightforward to verify that there is no rainbow K_3, and the number of colors used is $(r - 1) + (s - 1) + 1 = n - 1$, so the coloring is a $JL(n)$ coloring.

Now suppose that we have a $JL(n)$ coloring. We proceed by induction on n to show that it must be achievable as described above. When $n = 2$ the claim is clear. Suppose that $n > 2$. Let v be any vertex of K_n. By Lemma 2.1, the coloring restricted to $K_n - v \simeq K_{n-1}$ is a $JL(n-1)$ coloring. By the induction hypothesis there are positive integers r_0 and s_0 satisfying $r_0 + s_0 = n - 1$ and a partition of $V(K_n - v)$ into sets R_0, S_0 with $|R_0| = r_0$, $|S_0| = s_0$, such that all R_0-to-S_0 edges are one color, say green, which does not appear on the edges of $\langle R_0\rangle$ or of $\langle S_0\rangle$, and the sets of colors appearing on $\langle R_0\rangle$ and $\langle S_0\rangle$ are disjoint.

Let red be the color on an edge incident to v which does not appear in $K_n - v$ (Lemma 2.1, again). Without loss of generality, red appears on a v-to-R_0 edge. Then every v-to-S_0 edge must be either red or green – otherwise there is a rainbow K_3. If there is a red edge and a green edge among the v-to-S_0 edges, then, because neither red nor green appears in $\langle S_0\rangle$, there is a rainbow K_3. Therefore, the v-to-S_0 edges are either all green or all red. If all green then no v-to-R_0 edge is green by Corollary 2.3. Also, no v-to-R_0 edge bears a color appearing in $\langle S_0\rangle$, because such a color does not appear in $\langle R_0\rangle$, so such a v-to-R_0 edge together with a red v-to-R_0 edge and an edge in $\langle R_0\rangle$ would make a rainbow K_3.

Thus, if all the v-to-S_0 edges are green, taking $R = R_0 \cup \{v\}$ and $S = S_0$ verifies the conclusion of the theorem. If all the v-to-S_0 edges are red, then all the v-to-R_0 edges are red as well: none can be green, for then there would be a rainbow K_3 formed by a red v-to-R_0 edge, a green v-to-R_0 edge, and an edge in $\langle R_0\rangle$, and none can be a color other than red or green because, otherwise, there would be a rainbow K_3 formed by a v-to-R_0 edge, an R_0-to-S_0 edge, and a v-to-S_0 edge. Therefore, if all v- to-S_0 edges are red we can take

$$R = \{v\} \text{ and } S = R_0 \cup S_0.$$

□

Two edge-colorings of K_n are *essentially the same* if there is a permutation of the vertices of K_n and a renaming of the colors that takes one coloring into the other; otherwise, the two colorings are *essentially different*. Theorem 2.4 gives a simple construction process for producing $JL(n)$ colorings essentially the same as each and every $JL(n)$ coloring: if $n > 1$, partition n into two positive integers, and then iterate the procedure on each part which is > 1. If $n > 2$, to each instance of this process corresponds a *full binary tree*, which is a tree with one vertex of degree 2 and the others of degrees 3 and 1. Full binary trees are normally represented as rooted trees, rooted at the vertex of degree 2. For example, in Figure 1 is a full binary tree corresponding to an instance of the construction process for $n = 9$.

Observe that if the tree in Figure 1 were given without the integral weights shown, the weighting could be retrieved by weighting each leaf (vertex of degree one) with 1 and then completing the weighting by weighting each “parent” vertex with the sum of the weights of its “children”. Generalizing this example without putting it all into words, it is (hopefully) clear that for $n > 1$ the essentially different instances of the Theorem 2.4 construction process for $JL(n)$ colorings are in one-to-one correspondence with the full binary trees with n leafs. But do essentially different instances of that construction process necessarily produce essentially different $JL(n)$ colorings?

Corollary 2.5 *For $n > 1$, for any color appearing in a $JL(n)$ coloring, the subgraph of K_n induced by the edges of that color is a complete bipartite graph. The subgraph is spanning in K_n for exactly one color.*

If two $JL(n)$ colorings are essentially the same then clearly the spanning complete bipartite color graphs in them are isomorphic, and the $JL(r)$, $JL(s)$ colorings on the cliques induced by the vertices in either part of the spanning monochromatic $K_{r,s}$'s are essentially the same. (When $r = s$ this means the two $JL(r)$ colorings in one $JL(n)$ coloring are paired with the

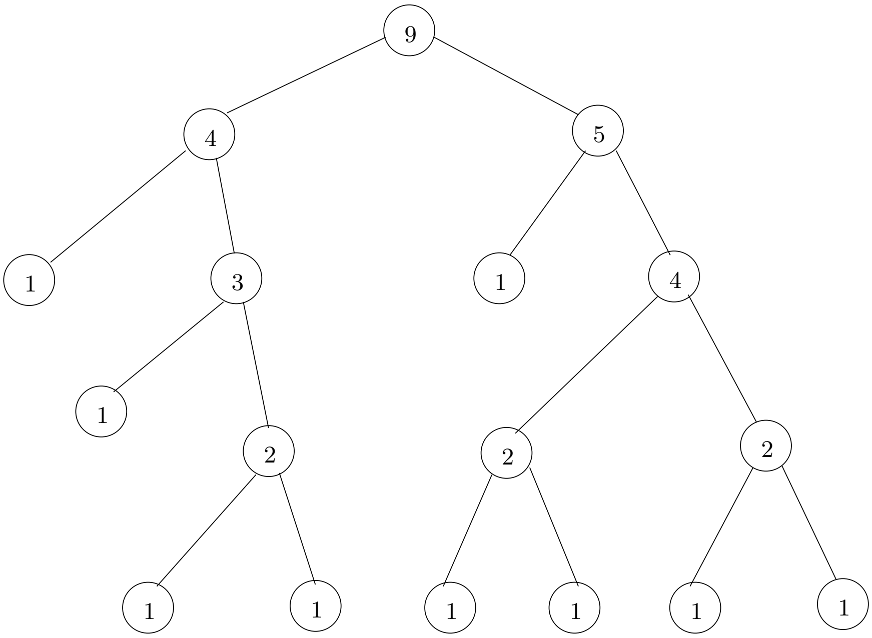

Figure 1: a $JL(9)$ coloring

two $JL(r)$ colorings in the other $JL(n)$ coloring, and within each pair, the two colorings are essentially the same.) Therefore different instances of the $JL(n)$ coloring construction procedure established by Theorem 2.4 produce essentially different colorings, which proves the following.

Corollary 2.6 *For $n > 2$ the essentially different $JL(n)$ colorings are in natural one-to-one correspondence with different full binary trees with n leafs.*

Corollary 2.7 *Let t_n be the number of essentially different $JL(n)$ colorings. (So t_n is also the number of full binary trees with n leafs, if $n > 2$.) Then for $n > 1$, if n is odd then*

$$t_n = \sum_{1 \le r < \frac{n}{2}} t_r t_{n-r} \tag{1}$$

and if n is even then

$$t_n = \sum_{1 \le r < \frac{n}{2}} t_r t_{n-r} + \binom{t_n/2}{2} + t_{n/2} \tag{2}$$

The proof of Corollary 2.7 is straightforward from Theorem 2.4 and remarks preceding Corollary 2.6. For $1 \leq r < \frac{n}{2}$, $t_r t_{n-r}$ is the number of different $JL(n)$ colorings resulting from a choice of r and $s = n - r$ in the partition of n at the start of the Theorem 2.4 construction process. When n is even, $r = n - r = n/2$ is a possibility; in that case, $t_{n/2}$ is the number of different colorings resulting from putting the same $JL(n/2)$ coloring on $\langle R \rangle$ and $\langle S \rangle$, and $\binom{t_{n/2}}{2}$ is the number of different colorings achieved by putting different $JL(n/2)$ colorings on $\langle R \rangle$ and $\langle S \rangle$.

The sequence (t_n) is sequence $A001190$ of [3].

In any coloring of edges, the frequency of a color is the number of edges the color appears on. Since the number of edges in $K_{r,s}$ is rs the frequencies of the colors in a $JL(n)$ coloring can be obtained by multiplying the weights of siblings in the node-weighted full binary tree corresponding to the coloring. For example, the frequencies of the 8 colors in the $JL(9)$ coloring described by the tree in Figure 1 are, in non-increasing order, $20, 4, 4, 3, 2, 1, 1, 1$.

Corollary 2.8 *Suppose that* $n \geq 3$ *and that* T *is a full binary tree with* n *leafs. Let the vertices of* T *be weighted by weighting each leaf with* 1 *and then weighting each parent mode with the sum of the weights of its children. Then the sum of the products of the weights of the* $n - 1$ *sibling pairs in* T *is* $\binom{n}{2}$.

Proof That sum of products is the sum of the frequencies of the colors in a $JL(n)$ coloring corresponding to T, and is therefore the number of edges in K_n. $\square$

There are easy elementary proofs of Corollary 2.8. The main virtue of our approach lies in the result being noticed at all.

Are the frequency vectors (the frequencies of the colors in non-increasing order) of different $JL(n)$ colorings necessarily different? The answer is no. The $JL(9)$ coloring described in Figure 1 is different from that in Figure 2, with the same frequency vector.

Problem: Enumerate the frequency vectors of the $JL(n)$ colorings, as a function of n

3 Equalized edge-colorings of K_n forbidding rainbow cycles

In this section we answer the following question: given $n > 1$ for which t is it possible to color the edges of K_n with exactly t colors appearing so that

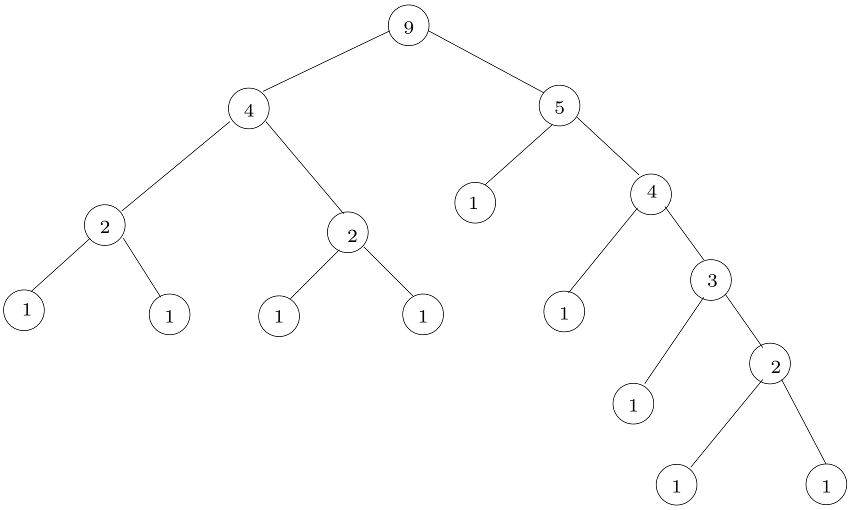

Figure 2: Another $JL(9)$ coloring

there are no rainbow K_3's and so that the different colors appear as nearly equally often as possible; that is, the greatest color frequency is no more than the least plus one. Let us call such a coloring an equalized rainbow-K_3-forbidding edge coloring of K_n with t colors. Our first result, about unrestricted K_3-forbidding edge colorings of K_n, is useful for our purpose, but is also of independent interest.

Theorem 3.1 *Suppose* $n > 1$ *and* K_n *is edge colored so that there are no rainbow* K_3*'s. Then for at least one color the subgraph of* K_n *induced by the edges of that color is connected and spanning in* K_n*.*

Proof The proof will be by induction on n. When $n = 2$ the claim is obviously true. Suppose that $n > 2$ and that K_n is edge-colored with no rainbow K_3's. Let $v \in V(K_n)$. By the induction hypothesis, the edges of some color, say green, induce a connected spanning subgraph H of $K_n - v \simeq K_{n-1}$. If any edge incident to v is green, then the green edges induce a connected spanning subgraph of K_n. So suppose that green does not appear on the edges incident to v.

Let $u \in V(K_n) \backslash \{v\}$ and suppose that the edge uv is red. We claim that every edge incident to v is red, which implies that the red edges induce a connected spanning subgraph of K_n. To see that the claim is true, suppose $w \in V(K_n) \backslash \{u, v\}$; we will see that vw is red. Because u and w are vertices in $K_n - v$ and H, the green edge subgraph, is connected and spanning in $K_n - v$, there is a path in H from u to w, say $u = x_0, x_1, \ldots, x_m = w$. Since there are no rainbow K_3's, $uv = x_0v$ is red, x_1x_0 is green, and vx_1

is not green, vx_1 must be red. By the same argument with x_1 replacing x_0 and x_2 replacing x_1, vx_2 is red. And so on; finally, $vx_m = vw$ is red. Since w was arbitrary, all edges incident to v are red. $\square$

Corollary 3.2 *If $n > 1$ and $t > \lceil \frac{n}{2} \rceil$ then there is no equalized rainbow-K_3-forbidding edge coloring of K_n with t colors.*

Proof If $n = 2$ the claim is obviously true. For $n > 2$ it can be verified straightforwardly that $\binom{n}{2}/t \leq \binom{n}{2}/(\lceil \frac{n}{2} \rceil + 1) \leq n - 2$. Therefore, in every equalized coloring of the edges of K_n with t colors, no color can appear more than $n-2$ times. By Theorem 3.1, no such coloring can be rainbow-K_3-forbidding. $\square$

Theorem 3.3 *If $n > 1$, there is an equalized rainbow-K_3-forbidding edge coloring of K_n with t colors if and only if $t \in \{1, \ldots, \lceil \frac{n}{2} \rceil\}$.*

Proof The "only if" assertion is Corollary 3.2. Obviously an equalized rainbow-K_3-forbidding edge coloring of K_n with t colors is possible for $t = 1$ and $t = 2$ ($n > 2$), so assume that $2 < t \leq \lceil \frac{n}{2} \rceil$.

There is $JL(n)$ coloring with frequency vector $(n-1, n-2, \ldots, 1)$: in the Theorem 2.4 construction, take $r = 1$, $s = n-1$, then partition $n-1$ into 1 and $n-2$, and so on. The idea of the proof of Theorem 3.3 is to partition $\{1, \ldots, n-1\}$ into sets $C_1, \ldots, C_t$ such that for each $j = 1, \ldots, t$, $\sum_{i \in C_j} i \in \{m, m+1\}$, where $m = \lfloor \binom{n}{2}/t \rfloor$. Then the coloring obtained from a $JL(n)$ coloring with frequency vector $(n-1, \ldots, 1)$ by merging the colors whose frequency is in C_j into a single color, $j = 1, \ldots, t$, is an equalized rainbow-K_3-forbidding edge coloring of K_n with t colors. The existence of such a partition, when $2 < t \leq \lceil \frac{n}{2} \rceil$, is guaranteed by Propositions 1.1 and 1.2 of [2]. $\square$

We noticed a curious phenomenon while struggling to prove Theorem 3.3 that is not particularly about rainbow-K_3-forbidding edge colorings, but that may be of interest. Given $n > 2$ and $t \in \{2, \ldots, \lceil \frac{n}{2} \rceil\}$, in looking for sets $C_1, \ldots, C_t$ partitioning $\{1, \ldots, n-1\}$ with $\sum_{i \in C_j} i = m_j \in \{m, m+1\}$, $m_1 \leq \cdots \leq m_t$, in every case we tried the C_j were obtainable by the following greedy method: pick the elements of C_1, then C_2, etc., and in each case start the picking for C_j with the largest unpicked integer not greater than m_j and continue by always picking the largest eligible integer, where an integer is eligible if it is unpicked and picking it doesn't cause the sum over C_j to exceed m_j. This greedy picking will fail if for some j we pick elements for C_j adding up to less than m_j before getting stuck. We can see no good reason why the method shouldn't fail occasionally, but it never has.

Example 1 Let $n = 15$ and $t = 6$. Then $m_1 = m_2 = m_3 = 17$ and $m_4 = m_5 = m_6 = 18$. By greedy picking from $\{1, \ldots, 14\}$ we get $C_1 = \{3, 14\}$,

$C_2 = \{4, 13\}$, $C_3 = \{5, 12\}$, $C_4 = \{7, 11\}$, $C_5 = \{8, 10\}$, $C_6 = \{1, 2, 6, 9\}$.

Problem Does greedy picking always work to obtain the sets $C_1, \ldots, C_t$ in the proof of Theorem 3.3, with their sums in non-decreasing order?

4 Edge-colorings of $K_n^{(\lambda)}$

As indicated in the abstract, $K_n^{(\lambda)}$ is the multigraph on n vertices with each pair of vertices connected by exactly λ edges. We will see that the edge colorings of $K_n^{(\lambda)}$ with no rainbow K_3's which use the maximum number of colors all arise from certain $JL(n)$ colorings.

Lemma 4.1 *Suppose $n \geq 3$. In any edge-coloring of K_n that forbids rainbow K_3's, any two edges bearing colors of frequency 1 in the coloring must be independent (non-incident).*

Proof If two such edges were incident then they would be two sides of a rainbow triangle. □

Corollary 4.2 *In any edge coloring of K_n that forbids rainbow K_3's, no more than $\lfloor \frac{n}{2} \rfloor$ of the colors have frequency 1.*

Lemma 4.3 *For each $n = 1, 2, \ldots$ there is a $JL(n)$ coloring in which $\lfloor \frac{n}{2} \rfloor$ of the colors have frequency 1.*

Proof The claim clearly holds for $n \leq 2$. If $n \geq 3$, construct a $JL(n)$ coloring by the process of Theorem 2.4 by partitioning n into 2, $n-2$, then partitioning $n-2$ into 2, $n-4$ (if $n \geq 5$), and so on. In the full binary tree corresponding to the coloring, there will be $\lfloor \frac{n}{2} \rfloor$ nodes with weight 2; each of these gives birth to two children of weight 1, and thus one edge bearing a color appearing on no other edge. □

A $JL(n)$ coloring in which $\lfloor \frac{n}{2} \rfloor$ colors have frequency 1 will be called a $\widehat{JL}(n)$ coloring. If u and v are distinct vertices in K_n, we will refer to the λ edges in $K_n^{(\lambda)}$ incident to u and v as the uv edge bundle. Here is a way to edge color $K_n^{(\lambda)}$ with $n - 1 + (\lambda - 1)\lfloor \frac{n}{2} \rfloor$ colors so that there is no rainbow K_3: start with a $\widehat{JL}(n)$ coloring; if uv bears a color with frequency > 1 in this coloring, color all of the edges in the uv edge bundle with that color; if uv bears a color with frequency 1 in this coloring, color the edges of the uv edge bundle with λ different colors, and arrange that each color appearing on an edge in such a bundle appears nowhere else in the coloring of $K_n^{(\lambda)}$. Such a coloring of $K_n^{(\lambda)}$, derived as described from a $\widehat{JL}(n)$ coloring, will be called a $JL(n, \lambda)$ coloring.

Theorem 4.4 *For positive integers n, λ the greatest number of colors appearing in an edge coloring of $K_n^{(\lambda)}$ which forbids rainbow K_3's is $n-1+(\lambda-1)\lfloor\frac{n}{2}\rfloor$, and if n, $\lambda > 1$ the only such colorings with that number of colors are $JL(n,\lambda)$ colorings.*

Proof We may as well assume that n, $\lambda > 1$. Consider λ fixed; the proof will be by induction on n. The claims of the theorem obviously hold if $n = 2$. Suppose that $n > 2$, and suppose that we are considering a rainbow-K_3-forbidding edge coloring of $K_n^{(\lambda)}$ using a maximum number of colors. Keep in mind that:

($*$) If u, v, w are distinct vertices of $K_n^{(\lambda)}$, and there is a color appearing in the vw bundle which appears in neither the uw nor the uv bundle, then the edges of the uw and uv bundles are all the same color.

Let v be any vertex of $K_n^{(\lambda)}$ and let $K^{(\lambda)} = K_n^{(\lambda)} - v \simeq K_{n-1}^{(\lambda)}$. By the induction hypothesis, there are no more than $n-2+(\lambda-1)\lfloor\frac{n-1}{2}\rfloor$ colors on $K^{(\lambda)}$, and that many only if that coloring of $K^{(\lambda)}$ is a $JL(n-1,\lambda)$ coloring. Since $n-2+(\lambda-1)\lfloor\frac{n-1}{2}\rfloor < n-1+(\lambda-1)\lfloor\frac{n}{2}\rfloor$, and since, by the remarks following Lemma 4.3, $K_n^{(\lambda)}$ can be edge-colored with $n-1+(\lambda-1)\lfloor\frac{n}{2}\rfloor$ colors appearing so that there is no rainbow K_3, and since we are assuming that $K_n^{(\lambda)}$ is colored with a maximum number of colors so that there is no rainbow K_3, there must be at least one edge incident to v bearing a color which does not appear in $K^{(\lambda)}$. We will call such colors *new* colors. Note that if n is even there are at least λ of them.

Because there are no rainbow K_3's, there cannot be two different new colors appearing in different edge bundles incident to v. Therefore, if some new color appears in two different edge bundles, then it is the only new color. Therefore, either

(a) there is only one new color, and it appears in at least two different edge bundles, or

(b) there is only one edge bundle incident to v in which new colors occur.

In case (a), by previous remarks it must be that n is odd and $K^{(\lambda)}$ is colored with a $JL(n-1,\lambda)$ coloring. Therefore every vertex of $K^{(\lambda)}$ is incident to an edge bundle on which λ different colors appear, unique to that edge bundle. Consequently, if the new color (say, red) appears in a uv edge bundle, there is a vertex $w \in V(K^{(\lambda)} - u)$ such that every edge in the vw edge bundle is colored red. But then every edge of the uv bundle must be red. Therefore, any edge bundle with a red edge is all red. Therefore any edge bundle incident to v is monochrome—if two different "old" colors occurred in some edge bundle, it would be possible to find a rainbow K_3 with one edge red. But now, returning from $K_n^{(\lambda)}$ to K_n, coloring each edge resulting from the collapse of a monochromatic edge bundle with the color of that bundle, and the color of each λ-polychromatic edge bundle to be a singular color, appearing nowhere else, and keeping in mind that $K^{(\lambda)} \simeq K_{n-1}^{\lambda}$ was colored with a $JL(n-1,\lambda)$ coloring, we find K_n edge-colored with no rainbow K_3's with $(n-1)-1+1 = n-1$ colors, $\frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor$ of them of frequency 1. This is a $\widehat{JL}(n)$ coloring. Therefore the coloring of $K_n^{(\lambda)}$ is a $JL(n,\lambda)$ coloring.

In case (b), let $w \in V(K^{(\lambda)})$ be the vertex such that all new colors are in the vw edge bundles. Since the total number of colors is maximum it must be that λ different new colors appear on the edges of the vw bundle, and nowhere else. (Not in $K^{(\lambda)}$ because these are *new* colors, and not in other edge bundles incident to v because we are in case (b). By $(*)$ it follows that for every $u \in V(K^{(\lambda)})$, $u \neq w$, all edges in the uv and uw bundles have the same color. If $n = 3$ it is now apparent that the coloring of $K_3^{(\lambda)}$ is a $JL(3,\lambda)$ coloring, so suppose that $n > 3$. Let m be the number of colors appearing in the full coloring of $K_n^{(\lambda)}$. By previous remarks, if n is even, then

$$\begin{aligned} n-1+(\lambda-1)\tfrac{n}{2} \leq m \quad &\leq (\text{number of colors on } K^{(\lambda)}) + \lambda \\ &\leq (n-2) + (\lambda-1)(\tfrac{n}{2}-1) + \lambda \\ &= n-1+(\lambda-1)\tfrac{n}{2}. \end{aligned}$$

By the induction hypothesis, since $K^{(\lambda)}$ is colored with $n-2+(\lambda-1)\lfloor\frac{n-1}{2}\rfloor$ colors, the coloring of it is a $JL(n-1,\lambda)$ coloring; now it is easy to see that the coloring of $K_n^{(\lambda)}$ is a $JL(n,\lambda)$ coloring.

If n is odd then $n \geq 5$. Consider $G = K^{(\lambda)} - w = K_n^{(\lambda)} - v - w \simeq K_{n-2}^{(\lambda)}$. For any $u \in V(G)$, all uv and uw edges are the same color. If that color—let's call it green—does not appear on the edges of G then for every $x \in V(G) \backslash \{u\}$, either the vx, wx edges are green or they are the color of the necessarily monochromatic ux edge bundle; otherwise, there would be a rainbow K_3 with vertices v, x, u (and another with vertices w, x, u).

Therefore there is at most one color on the $\{v,w\}$-to-$V(G)$ edges that does not appear on the edges of G. By previous remarks and the application of the induction hypothesis to $G \simeq K_{n-2}^{(\lambda)}$, and keeping in mind that n is odd, we have

$$\begin{aligned} n-1+(\lambda-1)\lfloor\tfrac{n}{2}\rfloor \quad &= n-1+(\lambda-1)\tfrac{n-1}{2} \\ &\leq m \\ &\leq n-3+(\lambda-1)\lfloor\tfrac{n-2}{2}\rfloor + \lambda + 1 \\ &= n-1+(\lambda-1)\tfrac{n-1}{2}; \end{aligned}$$

Thus $m = n - 1 + (\lambda - 1)\lfloor \frac{n}{2} \rfloor$; further, the coloring restricted to G must be a $JL(n-2, \lambda)$ coloring, and from that it is easy to see that the coloring of $K_n^{(\lambda)}$ is a $JL(n, \lambda)$ coloring. $\square$

5 Forbidding rainbow directed triples in D_n

D_n will denote the complete directed graph on n vertices, the digraph in which for any two distinct vertices u, v there are two arcs, (u, v) and (v, u), joining them, one in each direction. There are two natural analogues concerning D_n of the question answered by Theorem JL for K_n, corresponding to the two different K_3-like subgraphs of D_n, cyclic triples and transitive triples. See Figure 3.

Cyclic triple

Transitive triple

Figure 3: directed triangles

The question of how many colors can be used to color the arcs of D_n so that there are no rainbow triples of either type is answered by Theorem 4.4 with $\lambda = 2$. The answer is $n - 1 + \lfloor \frac{n}{2} \rfloor$, and Theorem 4.4 with $\lambda = 2$. The answer is $n - 1 + \lfloor \frac{n}{2} \rfloor$, and Theorem 4.4 supplies the additional information that the only such colorings are really $JL(n, 2)$ colorings of $K_n^{(2)}$.

Theorem 5.1 *The greatest number of colors in a coloring of the arcs of* D_n *such that there are no rainbow transitive triples is* $n - 1 + \lfloor \frac{n}{2} \rfloor$.

Proof The proof will be by induction on n and will be very much like the proof of Theorem 4.4 in the case $\lambda = 2$, only easier. Most of the difficulty in the proof of Theorem 4.4 resides in carrying the claim of the $JL(n, \lambda)$ form over the induction step. We are burdened by no such claim here.

The statement of the theorem is obviously true for $n = 1, 2$, and we will start the induction by assuming $n \geq 3$, but it might be salutary for the reader to verify the claim for $n = 3$ separately and to note that there are essentially 3 different arc colorings of D_3 with 3 colors, with no rainbow transitive triples. One of the 3 arises from a $JL(3,2)$ coloring of $K_3^{(2)}$, and in the other two both cyclic triples in D_3 are rainbow. They differ in that in one, at each vertex both in-arcs are the same color, and in the other both out-arcs are of the same color.

Suppose $n \geq 3$ and let $v \in V(D_n)$. Suppose that D_n is arc-colored with a maximum number m of colors so that no transitive triples are rainbow. Let $D = D_n - v \simeq D_{n-1}$. Let arcs (u,v), $u \in V(D)$, be called up-arcs, and arcs (v,u) be called down-arcs. A color on an arc with either end at v which does not occur in D will be called a new color. If a new color occurs on either (u,v) or (v,u) then no other new color occurs on any arc (w,v) of (v,w), $w \neq u$. Consequently, either there is no new color, or only one new color, in which cases the claim of the theorem follows easily by induction, or there are two new colors, say red on (u,v) and blue on (v,u), which occur on no other arcs.

If n is even then $n-1+\lfloor\frac{n}{2}\rfloor = n-2+\lfloor\frac{n-1}{2}\rfloor+2$, and the claim of the theorem follows easily by induction (and the fact that $m \geq n-1+\lfloor\frac{n}{2}\rfloor$, by Theoerm 4.4). So suppose that n is odd. Let $D' = D - u = D_n - \{u,v\} \simeq D_{n-2}$. Because there are no rainbow transitive triples and red and blue appear only on the arcs between u and v, for every $w \in V(D')$ all 4 arcs $(u,w),(w,u),(v,w),(w,v)$ must be the same color, say $c(w)$. If w, $x \in V(D')$ and $c(w) \neq c(x)$ then the colors on the arcs between w and x must be in the set $\{c(w),c(x)\}$. Consequently, at most one of the colors $c(w)$, $w \in V(D')$, does not appear in D'. Therefore, by the induction hypothesis, which says that the number of colors on D' is no greater than $n-3+\lfloor\frac{n-2}{2}\rfloor = n-3+\frac{n-3}{2}$, we have that

$$\begin{aligned} m \leq 3+n-3+\tfrac{n-3}{2} &= n-1+\tfrac{n-1}{2} \\ &= n-1+\lfloor\tfrac{n}{2}\rfloor. \end{aligned}$$

Therefore $m = n-1+\lfloor\frac{n}{2}\rfloor$. □

In fact, Theorem 5.1 is a special case of a more general theorem with the same proof, essentially. We thought it a good policy to give the prettier, less general theorem first and leave it to the reader to check the proof of the more general theorem.

Theorem 5.2 *Suppose that n, $\lambda \geq 2$ and D is a directed graph obtained by orienting the edges of $K_n^{(\lambda)}$ so that for any pair of distinct vertices there is at least one arc between them in each direction. Then the greatest number of colors with which the arcs of D can be colored so that there are no rainbow transitive triples is $n-1+(\lambda-1)\lfloor\frac{n}{2}\rfloor$.*

Since there are only 1/3 as many cyclic triples in D_n as there are transitive triples, it should be possible to forbid rainbow cyclic triples using more than $n-1+\lfloor \frac{n}{2} \rfloor \simeq \frac{3}{2}n$ colors, and, indeed, it is. The best we can do is a coloring using around $\frac{7}{4}n$ colors, and the question of the maximum number of colors to be used is still open.

Theorem 5.3 *Suppose that* n *is a positive integer and let integers* q *and* r *be defined by* $n = 4q + r$, $0 \le r < 4$. *Let*

$$g(n) = \begin{cases} 7q + r - 1 & \textit{if } r \in \{0,1\} \\ 7q + 2 & \textit{if } r = 2 \\ 7q + 4, & \textit{if } r = 3 \end{cases}$$

Then the arcs of D_n *can be colored with* $g(n)$ *colors appearing so that there are no rainbow cyclic triples.*

Proof The arcs of D_3 can be colored with 4 colors such that there are no rainbow cyclic triples: use two colors on one of the two cyclic triples and two other colors on the other cyclic triple. Also, it is possible to color the arcs of D_4 with 6 colors so that there are no rainbow cyclic triples: if the vertices are u, v, w, x, use colors 1 and 2 on the arcs joining u and v, 3 and 4 on the arcs joining w and x, and 5 and 6 on the two directed 4-cycles into which the remaining arcs can be decomposed, one color to a cycle. Partition $V(D_n)$ into q 4-sets $F_1, \dots, F_q$ and one r-set R, if $r > 0$. Use 6 different colors on the arcs of each D_4 induced by an F_i, with the q sets of 6 colors pairwise disjoint, and, if $r > 0$, $0, 2$, or 4 new colors on the D_1, D_2, or D_3 induced by R, so that there is no rainbow cyclic triple, so far. Finally, take any $JL(q)$ or $JL(q+1)$ (if $r > 0$) coloring of K_q or K_{q+1} considered to have vertices $F_1, \dots, F_q$ or $F_1, \dots, F_q$, R, respectively, with $q-1$ or q colors not yet appearing in D_n, and transport that coloring into D_n by coloring all arcs between F_i and F_j, $i \neq j$, or between F_i and R, if $r > 0$, with the color on the corresponding edge in the JL coloring. It is straightforward to verify that there are no rainbow cyclic triples in D_n with this coloring, and that the number of colors used is $g(n)$. $\square$

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