SEMINAR TALK: EDGE-COLORINGS OF K_n WHICH FORBID RAINBOW CYCLES

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Abstract. An introduction to a paper soon-to-be-published in *Utilitas Mathematica* by Adam Gouge, Dean Hoffman and Peter Johnson, Laura Nunley, and Luke Paben, entitled **Edge-colorings of K_n Which Forbid Rainbow Cycles**. In this talk, I hope to share one of the major results of the paper: It is known that the greatest number of colors that can appear on the edges of K_n in a coloring that forbids rainbow K_3's, and thus all rainbow cycles, is $n-1$. We characterize all such colorings (with $n-1$ colors): for $n \geq 3$ the essentially different such colorings are in natural one-to-one correspondence with the full binary trees with n leafs.

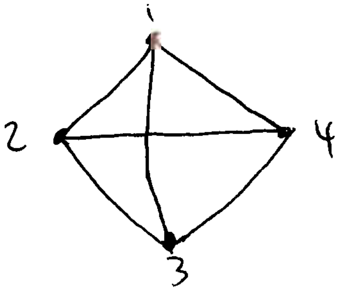
1. Basic Definitions

Definition 1. A **graph** is an ordered pair $G := (V, E)$ comprising a set V of vertices or nodes, together with a set E of edges or lines, which are two-element subsets of V.

Definition 2. A **simple graph** is a graph which is undirected, has no loops, and has no more than one edge between any two vertices.

Note. A **finite graph** is a graph G such that $V(G)$ and $E(G)$ are finite sets.

Example 1. Let $V := \{1, 2, 3, 4\}$, and $E := \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$.



Definition 3. A **complete graph** is a simple graph in which each pair of vertices is joined by an edge. In other words, the graph contains all possible edges.

Note. If a complete graph contains n vertices, denoted K_n, how many edges are there? *Answer:* $\binom{n}{2}$.

Definition 4. An **edge coloring** is an assigning of colors to the edges of a graph. They could be numbers, if one wishes, but for the purposes of colored chalk, we will use actual colors. :-)

Suppose that G and H are finite simple graphs. An edge-coloring of G **forbids rainbow H's** (copies of H) if each copy of H in G has two edges with the same color. (Thus, if H does not occur as a subgraph of G, the edge-coloring of G forbids rainbow H's.) If $\mathcal{H}$ is a family or collection of graphs, then an edge-coloring of G forbids rianbow members of $\mathcal{H}$ if the coloring forbids rainbow H's for each $H \in \mathcal{H}$.

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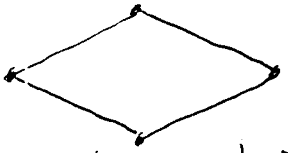
This paper is about edge-colorings of complete graphs which forbid rainbow cycles, as announced in the title. Our first result reveals that this announcement was perhaps unnecessarily dramatic.

Proposition 1 (1.1). *Suppose that G is chordal. Then an edge-coloring of G forbids rainbow cycles if and only if it forbids rainbow K_3's.*

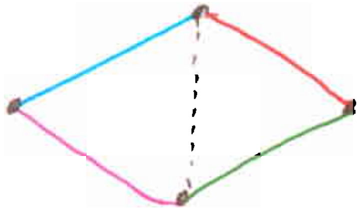
Definition 5. A **chordal** graph is a simple graph with no induced cycle of length greater than 3. According to Wikipedia, "*a graph is chordal if each of its cycles of four or more nodes has a chord, which is an edge joining two nodes that are not adjacent in the cycle. An equivalent definition is that any chordless cycles have at most three nodes. Chordal graphs are a subset of the perfect graphs. They are sometimes also called triangulated graphs.*"

Chordal



Not chordal

Proof. The "only if" assertion is clear. Now suppose an edge-coloring of G does not forbid rainbow cycles. We shall see that it must admit a rainbow K_3. Let C be the shortest rainbow cycle in G. If the length of C were 3, there would be a "chord", an edge of G whose end-vertices divide C into two paths of lengths ≥ 2. This chord would create two cycles in G both shorter than C, of which at least one must be rainbow. Therefore, C must be of length 3, meaning $C \simeq K_3$. ☐



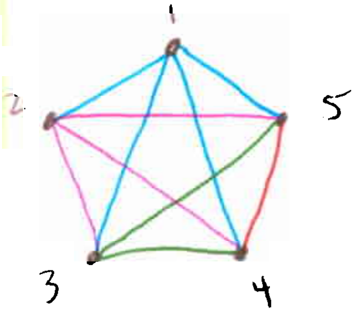
The inspiration for what follows is a cute Hungarian-schoolboy-prize-exam type result that we first heard from Jenö Lehel. We have since learned that it descends from [1].

Theorem 1 (JL). *If the edges of K_n are colored with n or more colors actually appearing then there is a rainbow K_3 somewhere. On the other hand, K_n can be edge-colored with n-1 colors appearing and with rainbow K_3's forbidden.*

Proof. Left to the reader. ☐

A $JL(n)$ **coloring** is a coloring of the edges of K_n with exactly $n-1$ colors, forbidding rainbow K_3's (and thus rainbow cycles, by Proposition 1.1).

"My" construction of a $JL(5)$ coloring:

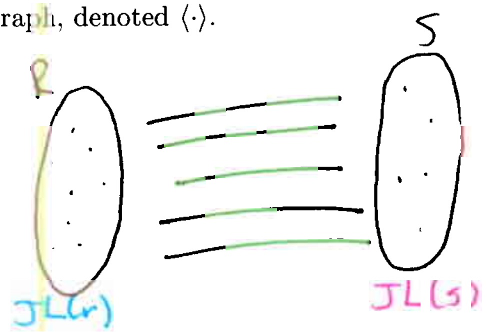


K_5

2. CHARACTERIZATION OF $JL(n)$ COLORINGS

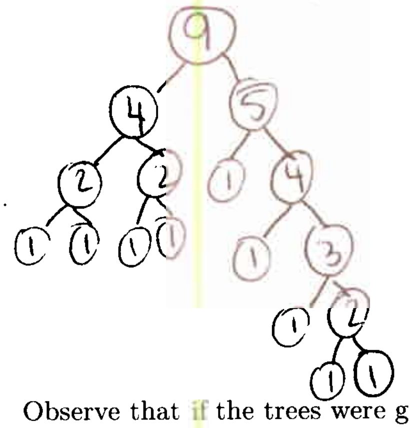
Theorem 2 (2.4). *Suppose that $n \geq 2$. Every $JL(n)$ coloring is obtainable as follows: choose positive integers r, s satisfying $r + s = n$; partition $V(K_n)$ into sets R, S satisfying $|R| = r$, $|S| = s$. Color all $R - to - S$ edges in K_n with one color - say green. Color $\langle R \rangle$ with a $JL(r)$ coloring and $\langle S \rangle$ with a $JL(s)$ coloring with disjoint sets of colors on the two cliques, and with green not appearing in $\langle R \rangle$, nor in $\langle S \rangle$.*
Further, any edge-coloring of K_n arrived at by following the directions above is a $JL(n)$ coloring.

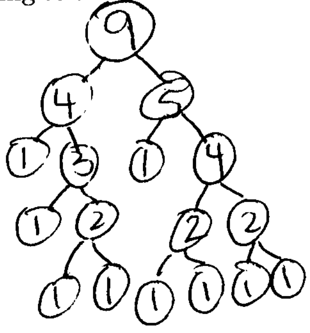
Definition 6. A **clique** in a simple undirected graph is a set of pairwise adjacent vertices, or a complete subgraph, denoted $\langle \cdot \rangle$.



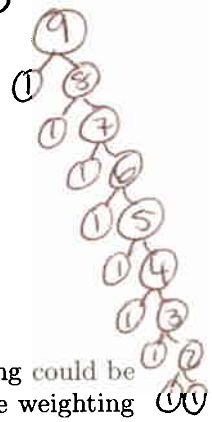
Proof. It's easy to verify that in any coloring arrived at by using the directions above, there is no rainbow K_3, and the number of colors used is $(r-1)+(s-1)+1 = n-1$, so the coloring is a $JL(n)$ coloring.
For the other direction, assuming you have a $JL(n)$ coloring and showing it can be arrived at using the directions above, the proof is very interesting, logical, and uses a Lemma and a couple of Corollaries we were not able to cover in the span of this talk (i.e. no monochromatic K_3's). ☐

Two edge-colorings of K_n are *essentially the same* if there is a permutation of the vertices of K_n and a renaming of the colors that takes one coloring into the other; otherwise, the two colorings are *essentially different.* Theorem 2.4 gives a simple construction process for producing $JL(n)$ colorings essentially the same as each and every $JL(n)$ coloring: if $n > 1$, partition n into two positive integers, and then iterate the procedure on each part which is > 1. If $n > 2$, to each instance of this process corresponds a **full binary tree**, which is a tree with one vertex of degree 2 and the others of degree 3 and 1. Full binary trees are normally represented as rooted trees, rooted at the vertex of degree 2. For example, consider "my" $JL(n)$ construction and the following are two full binary trees corresponding to an instance of the construction process for $n = 9$.





"my" JL(9)



Observe that if the trees were given without the integral weights shown, the weighting could be retrieved by weighting each leaf (vertex of degree one) with 1 and then completing the weighting by weighting each "parent" vertex with the sum of the weights of its "children". Generalizing this example without putting it all into words, it is (hopefully) clear that for $n > 1$ the essentially different instances of the Theorem 2.4 construction process for $JL(n)$ colorings are in one-to-one correspondence with the full binary trees with n leafs. But do essentially different instances of that construction process necessarily produce essentially different $JL(n)$ colorings?

If two $JL(n)$ colorings are essentially the same then clearly the spanning complete bipartite color graphs in them are isomorphic, and the $JL(r)$, $JL(s)$ colorings on the cliques induced by the vertices in either part of the spanning monochromatic $K_{r,s}$'s are essentially the same. (When $r = s$ this means the two $JL(r)$ colorings in one $JL(n)$ coloring are paired with the two $JL(r)$ colorings in the other $JL(n)$ coloring, and within each pair, the two colorings are essentially the same.) Therefore different instances of the $JL(n)$ coloring construction procedure established by Theorem 2.4 produce essentially different colorings, which proves the following.

Corollary 1 (2.6). *For $n \geq 3$ the essentially different $JL(n)$ colorings are in natural one-to-one correspondence with different full binary trees with n leafs.*

In any coloring of edges, the frequency of a color is the number of edges the color appears on. Since the number of edges in $K_{r,s}$ is rs the frequencies of the colors in a $JL(n)$ coloring can be obtained by multiplying the weights of the siblings in the node-weighted full binary tree corresponding to the coloring. For example, the frequencies of the 8 colors in the $JL(9)$ coloring described by the tree in above are, in non-increasing order, 20,4,4,3,2,1,1,1.

Corollary 2 (2.8). *Suppose that $n \geq 3$ and that T is a full binary tree with n leafs. Let the vertices of T be weighted by weighting each leaf with 1 and then weighting each parent node with the sum of the weights of its children. Then the sum of the products of the weights of the $n-1$ sibling pairs in T is $\binom{n}{2}$.*

Proof. That sum of products is the sum of the frequencies of the colors in a $JL(n)$ coloring corresponding to T, and is therefore the number of edges in K_n. ☐

There are easy elementary proofs of Corollary 2.8. The main virtue of our approach lies in the result being noticed at all.

Are the frequency vectors (the frequencies of the colors in non-increasing order) of different $JL(n)$ colorings necessarily different? The answer is no. The $JL(9)$ colorings described above are different, but they have the same frequency vector.

REFERENCES

[1] P. Erdös, M. Simonovits, and V. T. Sós, Anti-Ramsey theorems, *Infinite and finite sets* (Colloq. Kezthely, 1973, dedicated to P. Erdös on his 60^{th} birthday), Vol. II, pp. 633-643; *Colloq. Math. Soc. Janos Bolyai*, Vol. 10, North-Holland, Amsterdam, 1975. (Can be found at http://www.zblmath.fiz-karlsruhe.de/MATH/general/erdos/cit/31605111.htm).