ELEMENTARY DIVISOR DOMAINS AND BÉZOUT DOMAINS

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Abstract. It is well-known that an Elementary Divisor domain R is a Bézout domain, and it is a classical open question to determine whether the converse statement is false. In this article, we provide new chains of implications between *R is an Elementary Divisor domain* and *R is Bézout* defined by hyperplane conditions in the general linear group. Motivated by these new chains of implications, we construct given any commutative ring R new Bézout domains associated with R which are expected in general not to be Elementary Divisor domains.

keywords Elementary Divisor domain, Bézout domain, Rings defined by matrix properties, Symmetric matrix, Trace zero matrix.

MSC: 13F10, 15A33, 15A21

1. Introduction

A commutative ring R in which every finitely generated ideal is principal is called a *Bézout ring*. By definition, a noetherian Bézout domain is a principal ideal domain. Examples of non-noetherian Bézout domains can be found for instance in [1], 243-246.

A commutative ring R is called an *Elementary Divisor domain* if, given any matrix A with coefficients in R, there exist invertible matrices P, Q with coefficients in R such that $PAQ = D$ with $D = \operatorname{diag}(d_1, \dots)$ a diagonal matrix (such a matrix may be rectangular, but it has $d_1, \dots$ on the main diagonal and zeroes elsewhere). Kaplansky showed in [9], 5.2, that a Bézout domain is an Elementary Divisor domain if and only if it satisfies:

(∗) For all $a, b, c \in R$ with $(a, b, c) = R$, there exist $p, q \in R$ such that $(pa, pb + qc) = R$.

(See also [5], 6.3.) It is well-known that a principal ideal domain is an Elementary Divisor domain. Consideration of the Elementary Divisor problem for a non-noetherian ring can be found as early as 1915 in Wedderburn [12].

It is an open question dating back at least to Helmer [8] in 1942 to decide whether a Bézout domain is always an Elementary Divisor domain. Gillman and Henriksen gave examples of Bézout rings that are not Elementary Divisor rings in [7]. In 1977, Leavitt and Mosbo in fact stated in [10], Remark 8, that it has been conjectured that there exists a Bézout domain that is not an Elementary Divisor domain (see also Problem 5 in [5], p. 122).

Our contribution to this question is the introduction, in 2.5 and 3.14, of new chains of implications between *R is an Elementary Divisor domain* and *R is Bézout*, which may prove useful in an eventual solution to the above open question. In particular, motivated by these new chains of implications, we construct, given any commutative

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ring R which is not Bézout, new Bézout domains associated with R (see 2.8 and 3.13). We have not been able to determine if these rings are Elementary Divisor domains.

2. Condition $(SU)_n$

Let $M_n(R)$ denote the ring of $(n \times n)$-matrices with coefficients in R. We make the following definitions.

Definition 2.1 Let $n \geq 1$. A ring R is called an *$(SU)_n$-ring* if, given any $A \in M_n(R)$, there exist a symmetric matrix $S \in M_n(R)$ and an invertible matrix $U \in \mathrm{GL}_n(R)$ such that $A = SU$. If R is an $(SU)_n$-ring for all $n \geq 1$, we shall say that R is an *SU-ring*.

A ring R is called an *$(SU')_n$-ring* if, given any $A \in M_n(R)$, there exist a symmetric matrix $S \in M_n(R)$ and an invertible matrix $U \in \mathrm{SL}_n(R)$ such that $A = SU$. If R is an $(SU')_n$-ring for all $n \geq 1$, we shall say that R is an *SU'-ring*.

Remark 2.2 Suppose that every matrix $A \in M_n(R)$ has a factorization $A = SU$ with S symmetric and U invertible. Then every matrix A has a factorization $A = VT$, with T symmetric and V invertible. Indeed, transpose the SU-factorization of the transpose matrix A^t.

Remark 2.3 We note that if R satisfies Condition $(SU)_n$ or $(SU')_n$, and I is any proper ideal of R, then R/I also satisfies Condition $(SU)_n$ or $(SU')_n$. It is also true that if $T \subset R$ is a multiplicative subset, then the localization ring $T^{-1}(R)$ satisfies Condition $(SU)_n$ or $(SU')_n$.

Remark 2.4 Suppose that a matrix $A \in M_n(R)$ has a factorization $A = SU$ with S symmetric and U invertible. In general, such a factorization is not unique. For instance, if $A \in \mathrm{GL}_n(R)$, and $A = SU$, then for all n, $A = S^n(S^{1-n}U)$. When S is not of finite order, $S^n(S^{1-n}U)$ is a factorization of A in infinitely many different ways. For an example where S is torsion, let $A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We clearly have the trivial factorization $A = A \cdot \mathrm{Id}_2$, and infinitely many SU'-factorizations

$$A = \begin{pmatrix} x+1 & x \\ x & x-1 \end{pmatrix} \begin{pmatrix} x & -(x-1) \\ -(x+1) & x \end{pmatrix}.$$

One may wonder whether the number of distinct SU'-factorizations of $A \in \mathrm{SL}_n(\mathbb{Z})$ is always infinite.

Recall that in a *Hermite ring*, every matrix can reduced to triangular form, and we have the implications *R is a Elementary Divisor ring* implies *R is a Hermite ring* implies *R is a Bézout ring*. Gillman and Henriksen have proved in [6], Theorem 3, that a commutative ring is an Hermite ring if and only if the following condition is satisfied: for every $a, b \in R$, there exists c, d and g in R such that $a = cg$, $b = dg$, and $(c, d) = R$. It follows immediately that a Bézout domain is a Hermite domain.

Proposition 2.5. *Let R be any commutative ring. Consider the following properties:*

a) *R is an Elementary Divisor ring.*
b) *R is an SU'-ring.*
c) *R is an SU-ring.*
d) *R is a Hermite ring.*

Then a) $\Longrightarrow$ b) $\Longrightarrow$ c) $\Longrightarrow$ d).

Proof. *a)* $\Longrightarrow$ *b)*. Let $A \in M_n(R)$. Choose $P, Q \in \mathrm{GL}_n(R)$ such that $PAQ = D$ is a diagonal matrix. Let $\epsilon := \det(P)\det(Q)^{-1}$. Let E denote any invertible diagonal matrix with determinant ϵ. Then $PAQE = DE$ is still symmetric since D is diagonal. We find that

$$AQE(P^{-1})^t = P^{-1}DE(P^{-1})^t$$

is symmetric, with $\det(QE(P^{-1})^t) = 1$. It is obvious that *b)* $\Longrightarrow$ *c)*. The last implication *c)* $\Longrightarrow$ *d)* follows from our next lemma.

Lemma 2.6. *Let R be any ring, and let $a, b \in R$. Let* $A := \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$.

(i) *If there exists* $V := \begin{pmatrix} u & v \\ s & t \end{pmatrix} \in \mathrm{GL}_2(R)$ *such that AV is symmetric, then there exists $g \in R$ such that $a = ug$, $b = vg$, and $(u, v) = R$. Moreover, if R is a $(SU)_n$-ring for some $n \geq 2$, then R is a Hermite ring.*

(ii) *If R is an Hermite ring, then there exists $V \in \mathrm{SL}_2(R)$ such that AV is symmetric.*

Proof. (i) The cases where $a = 0$ or $b = 0$ are easy and left to the reader. Assume that $a \neq 0$ and $b \neq 0$. The product AV is symmetric if and only if $av = bu$. The matrix V is invertible if and only if $ut - sv = \epsilon \in R^*$. Then $aut - asv = a\epsilon = u(at - bs)$, and $at - bs$ divides a. Similarly, $v(at - bs) = b$. Therefore, $(at - bs) \subseteq (a, b) \subseteq (at - bs)$, and we find that the ideal (a, b) is principal. We also have $(u, v) = R$, as desired.

Suppose now that is a $(SU)_n$-ring for some $n \geq 2$. Let $a, b \in R$. Consider the square $(n \times n)$-matrix $A = (a_{ij})$ with all null entries, except for $a_{11} := a$ and $a_{21} := b$. Assume that there exists $V = (v_{ij}) \in \mathrm{GL}_n(R)$ such that AV is symmetric. Then we find that $v_{13} = \cdots = v_{1n} = 0$, and $av_{12} = bv_{11}$. Expanding the determinant of V using the first row, we find that we can write $\det(V) = v_{11}s - v_{12}t \in R^*$ for some $s, t \in R$. We conclude as above with $g = at - bs$.

(ii) Let $a, b \in R$. Assume that there exists $c, d, g \in R$ such that $a = gc$ and $b = gd$, and there exist $s, t \in R$ such that $cs + dt = 1$. We can write

$$\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} c & d \\ -t & s \end{pmatrix} = \begin{pmatrix} ac & ab/g \\ ab/g & bd \end{pmatrix}.$$

□

It is completely obvious from the previous proposition that if R is an Elementary Divisor domain and an $(SU)_n$-domain, than it is also an $(SU)_{n-1}$-domain. We can strengthen this assertion as follows.

Proposition 2.7. *Let R be a commutative $(SU)_n$-domain for $n \geq 3$. Then R is a Bézout domain, and an $(SU)_{n-1}$-domain.*

Proof. Lemma 2.6 shows that the domain R is Bézout. Let $A \in M_{n-1}(R)$. Since R is Bézout, it is possible to find two invertible matrices $P, Q \in \mathrm{GL}_{n-1}(R)$ such that PAQ consists in its upper left corner of a nonsingular matrix A' of rank equal to $\mathrm{rank}(A)$, and such that all other coefficients of PAQ are zeros.

Let $B \in M_n(R)$ be the matrix with A' in the upper left corner, and with all other entries zeros. By hypothesis, there exists $U \in \mathrm{GL}_n(R)$ such that BU is symmetric. Clearly, the last $n - \mathrm{rank}(A)$ rows of BU consists only in zeros. Since the matrix BU is symmetric, its last $n - \mathrm{rank}(A)$ columns also consists only in zeros. Let W

denote any vector in $R^{\mathrm{rank}(A)}$ obtained from one of the $n-\mathrm{rank}(A)$ last columns of U by removing from the column its last $n-\mathrm{rank}(A)$ coefficients. Then $A'W=0$. Since $\det(A')\neq 0$, we find that $W=0$. Let V denote the square rank(A)-matrix in the upper left corner of U, and let V' denote the square $(n-\mathrm{rank}(A))$-matrix in the lower right corner of U. Then $\det(U)=\det(V)\det(V')$. Hence, V is invertible, and we have $A'V$ symmetric.

Consider now the square matrix T of size $(n-1)$ consisting of two blocks: V in the upper left corner, and an identity matrix of the appropriate size in the lower right corner. The matrix T is invertible. By construction, $PAQT$ is symmetric. Then $AQT(P^{-1})^t$ is also symmetric, with $QT(P^{-1})^t$ invertible. □

It is clear that if R is an $(SU)_n$-ring with the property that every unit $r\in R^*$ is an n-th power in R, then it is also an $(SU')_n$-ring. Indeed, if $A=SU$ with S symmetric and $\det(U)\in R^*$, write $\det(U)=\epsilon^n$, and $D:=\mathrm{diag}(\epsilon,\dots,\epsilon)$. Then $A=(SD)(D^{-1}U)$ with SD symmetric, and $D^{-1}U\in\mathrm{SL}_n(R)$.

It is natural to ask whether any of the implications in our last propositions can be reversed in general. We can also ask whether a commutative Bézout $(SU')_n$-domain is also an $(SU')_{n-1}$-domain.

Example 2.8 Proposition 2.5 suggests the following construction of new Bézout rings. Let R be any commutative ring and fix $n>1$. Let $X=(x_{ij})_{1\leq i,j\leq n}$ denote the square $n\times n$-matrix in the indeterminates $x_{ij}, 1\leq i,j\leq n$. For each matrix $A\in M_n(R)$, consider the subset $I(A)$ of $R[x_{11},\dots,x_{nn}]$ consisting of $\det(X)-1$ and of the $(n^2-2)/2$ polynomial equations obtained by imposing the condition that the matrix AX is symmetric. Let $<I(A)>$ denote the ideal of $R[x_{11},\dots,x_{nn}]$ generated by the elements of $I(A)$. We claim that $<I(A)>\neq R[x_{11},\dots,x_{nn}]$. Indeed, choose a maximal ideal M of R, and consider the field $K:=R/M$. If $1\in<I(A)>$, then 1 is also contained in the ideal of $K[x_{11},\dots,x_{nn}]$ generated by the images of the elements of $I(A)$ modulo M. This is not possible since K is a principal ideal domain, and Proposition 2.5 shows then that K satisfies Condition $(SU')_n$.

Consider the set $\mathcal{I}$ of all subsets $I(A)$, $A\in M_n(R)$, such that there exists no homomorphism of R-algebras between $R[x_{11},\dots,x_{nn}]/<I(A)>$ and R (i.e., such that there exists no matrix $Y\in\mathrm{SL}_n(R)$ with AY symmetric). For each subset $I=I(A)\in\mathcal{I}$, we let $\mathbf{x}^I$ denote the set of n^2 variables labeled $x^I_{11},\dots,x^I_{nn}$, and we denote by $(\mathbf{x}^I)$ the matrix (x^I_{ij}). We now let $I(A,\mathbf{x}^I)$ be the subset of $R[\mathbf{x}^I]$ consisting of $\det((\mathbf{x}^I))-1$ and of the $(n^2-n)/2$ polynomial equations obtained by imposing the condition that the matrix $A(\mathbf{x}^I)$ is symmetric. It is not difficult to check that the ideal $<I(A,\mathbf{x}^I), I\in\mathcal{I}>$ is a proper ideal of the polynomial ring $R[\mathbf{x}^I, I\in\mathcal{I}]$. We define the quotient ring

$$s_n(R):=R[\mathbf{x}^I, I\in\mathcal{I}]/<I(A,\mathbf{x}^I), I\in\mathcal{I}>.$$

Note that if R is a $(SU')_n$-ring, then $\mathcal{I}=\emptyset$ and, in particular, $s_n(R)=R$. It is clear that we have a natural morphism of R-algebras $R\to s_n(R)$. By construction, given any matrix $B\in M_n(R)$, there exists $U\in\mathrm{SL}_n(s_n(R))$ such that BU is symmetric. Indeed, it suffices to take $U:=(\text{class of}(x^{I(B)}_{ij})\text{ in } s_n(R))$.

Let $s^{(1)}_n(R):=s_n(R)$, and for each $i\in\mathbb{N}$, we set $s^{(i)}_n(R):=s_n(s^{(i-1)}_n(R))$. Finally, we let

$$\mathcal{S}_n(R):=\lim_{i} s^{(i)}_n(R).$$

Let $C \in M_n(\mathcal{S}_n(R))$. Then the finitely many coefficients of C all lie in a single ring $s_n^{(i)}(R)$ for some $i > 0$. By construction, there exist $U := (u_{ij}) \in \mathrm{SL}_n(s_n^{(i)}(R))$ such that BU is symmetric. It follows that $\mathcal{S}_n(R)$ satisfies Condition $(SU')_n$.

Given any prime ideal P of $\mathcal{S}_n(R)$, the quotient $\mathcal{S}_n(R)/P$ is also $(SU')_n$-domain and, thus, a Bézout domain (2.7). It is natural to wonder whether one could show for a well-chosen ring R that one such domain is not an Elementary Divisor domain, for instance by showing that $\mathcal{S}_n(R)/P$ is not a $(SU')_{n+1}$-domain.

3. Condition H_n

Let R be any commutative ring. Let $X_n := ((x_{ij}))_{1\leq i,j\leq n}$ denote the square matrix in the indeterminates $x_{ij}, 1 \leq i,j \leq n$. Set

$$d_n := \det(X_n) \in R[x_{11}, \ldots, x_{nn}].$$

For $\mu \in R$, denote by $Z_{d_n-\mu}(R)$ the set of solutions to the equation $d_n - \mu = 0$ in R^{n^2}. Clearly, $\mathrm{SL}_n(R) = Z_{d_n-1}(R)$. (The notation $Z_f(R)$ stands for the *zeroes* of f in R^{n^2}.)

Let $h(x_{11}, \ldots, x_{nn}) := \sum_{1\leq i,j\leq n} a_{ij}x_{ij} \in R[x_{11}, \ldots, x_{nn}]$ be a non-zero homogeneous linear polynomial (i.e., without constant term). Let $\nu \in R$, and let $Z_{h-\nu}(R)$ denote the set of solutions to the equation $h - \nu = 0$ in R^{n^2}.

By definition, when $(a_{ij}, 1 \leq i,j \leq n) = R$, there exist $c_{ij} \in R$ such that $\sum a_{ij}c_{ij} = 1$. Thus, in this case, $Z_{h-\nu}(R) \neq \emptyset$ for all ν. In a Bézout ring R, $Z_{h-\nu}(R) \neq \emptyset$ if and only if a generator of $(a_{ij}, 1 \leq i,j \leq n)$ divides ν.

Definition 3.1 We say that a commutative ring R *satisfies Condition* H_n if, given any linear homogeneous polynomial $h(x_{11}, \ldots, x_{nn})$ and any $\nu \in R$ such that $Z_{h-\nu}(R) \neq \emptyset$, then for all $\mu \in R$, we have $Z_{d_n-\mu}(R) \cap Z_{h-\nu}(R) \neq \emptyset$.

In other words, stratify $M_n(R)$ using the determinant, so that

$$M_n(R) = \sqcup_{\mu \in R} Z_{d_n-\mu}(R).$$

When R *satisfies Condition* H_n, any hyperplane in $R^{n^2} = M_n(R)$ which is not empty meets every stratum of the stratification.

Remark 3.2 We note that if R satisfies Condition H_n and I is any proper ideal of R, then R/I also satisfies Condition H_n. It is also true that if $T \subset R$ is a multiplicative subset, then the localization ring $T^{-1}(R)$ also satisfies Condition H_n.

The key ideas in the proof of the following proposition are due to Robert Varley.

Proposition 3.3. *Let R be an Elementary Divisor ring. Then R satisfies Condition H_n for all $n > 1$.*

Proof. Fix $h(x_{11}, \ldots, x_{nn}) = \sum a_{ij}x_{ij} \in R[x_{11}, \ldots, x_{nn}]$, and $\nu, \mu \in R$. Let $A := (a_{ij}) \in M_n(R)$ denote the associated matrix. Assume that $Z_{h-\nu}(R) \neq \emptyset$. Then any generator of the ideal $(a_{11}, \ldots, a_{nn})$ divides ν. We need to show the existence of $U \in M_n(R)$ such that $\det(U) = \mu$, and AU has trace $\mathrm{Tr}(AU) = \nu$.

Let P and Q in $\mathrm{GL}_n(R)$ be such that $PAQ = \mathrm{diag}(d_1, \ldots, d_n)$ and d_i divides d_{i+1} for all $i = 1, \ldots, n-1$. Then $(a_{11}, \ldots, a_{nn}) = (d_1)$. Multiply both sides of $PAQ = \mathrm{diag}(d_1, \ldots, d_n)$ on the right by $D := \mathrm{diag}(1, \ldots, 1, \mu \det(P)^{-1}\det(Q)^{-1})$. Write $\nu = d_1 s$ with $s \in R$, and add s times the first column of $PAQD$ to its last column. Permute the first row with the last row. If n is odd, permute the first and second column, then the third and forth column, etc, to obtain a matrix with $(0,\ldots,0,\nu)$ on the diagonal. If n is even, permute the second and third column, then the forth and fifth column, etc, to again obtain a matrix with $(0,\ldots,0,\nu)$ on the diagonal. We have thus proved the existence of P' and Q' in $\mathrm{GL}_n(R)$ such that $P'PAQDQ'$ is a matrix with $(0,\ldots,0,\nu)$ on the diagonal, and $\det(P'PDQQ') = \pm\mu$. Multiplying both sides by $\mathrm{diag}(-1,1,\ldots,1)$ if necessary, we may assume that $\det(P'PDQQ') = \mu$. By construction, $\mathrm{Tr}((P'P)(AQDQ')) = \nu$, so that $\mathrm{Tr}(AQDQ'P'P) = \nu$. We can therefore choose $U := QDQ'P'P$ to satisfy the conditions of the proposition. □

Definition 3.4 Let n and s be positive integers. We say that a commutative ring R *satisfies Condition* $H_{n,s}$ if, given any system of s linear homogeneous polynomials $h_i \in R[x_{11},\ldots,x_{nn}]$, $i = 1,\ldots,s$, we have

$$\mathrm{GL}_n(R) \cap (\bigcap_{i=1}^{s} Z_{h_i}(R)) \neq \emptyset.$$

It is clear that if a ring satisfies Condition $H_{n,s}$, then $s < n$ (indeed, take $h_i(x_{11},\ldots,x_{nn}) := x_{1,i}$ for $i = 1,\ldots,n$ to show that Condition $H_{n,n}$ is never satisfied).

Remark 3.5 We note that if R satisfies Condition $H_{n,s}$, and I is any proper ideal of R, then R/I also satisfies Condition $H_{n,s}$. It is also true that if $T \subset R$ is a multiplicative subset, then the localization ring $T^{-1}(R)$ satisfies Condition $H_{n,s}$.

Proposition 3.3 shows that any Elementary Divisor ring satisfies Condition $H_{n,1}$. Our motivation for introducing the above definition is the following lemma.

Lemma 3.6. *Let R be a commutative ring satisfying Condition $H_{n,n-1}$ for some $n \geq 2$. Then R is a Hermite ring.*

Proof. Let $a, b \in R$. Condition $H_{n,n-1}$ implies the existence of $V = (v_{ij}) \in \mathrm{GL}_n(R)$ satisfying the following $n-1$ hyperplane conditions: $v_{13} = \cdots = v_{1n} = 0$, and $av_{12} = bv_{11}$. Expanding the determinant of V using the first row, we find that we can write $\det(V) = v_{11}s - v_{12}t \in R^*$ for some $s, t \in R$. We conclude as in the proof of 2.6 (i) that $g := (as - bt)\det(V)^{-1}$ is such that $gv_{11} = a$ and $gv_{12} = b$, with $(v_{11}, v_{12}) = R$. □

Proposition 3.7. *Any field K satisfies Condition $H_{n,n-1}$ for all $n \geq 2$.*

Proof. We thank J. Fresnel for making us aware of [4], Exer. 2.3.16, p. 112, which details a proof of the proposition under the assumption that K is infinite. The suggested proof in fact shows that the proposition holds if $|K| \geq r+1$. The key to 3.7 is the following statement, proved under the assumption that $|K| \geq r+1$ in [3], and in general in [11]: *If W is a subspace of the K-vector space $M_n(K)$ and $\dim(W) > rn$, then W contains an element of rank bigger than r.*

Indeed, let $h_i \in K[x_{11},\ldots,x_{nn}]$, $i = 1,\ldots,s$, be any system of s linear homogeneous polynomials. Then the set $(\bigcap_{i=1}^{s} Z_{h_i}(K))$ is in fact a subspace of $M_n(K)$ of dimension at least $n^2 - s$. If this vector space does not contain any element of $\mathrm{GL}_n(K)$, then all its elements have rank at most $n-1$, and its dimension would be at most $n(n-1)$. This is a contradiction since $n(n-1) < n^2 - s$ when $s = n-1$. □

Proposition 3.8. *Let $n > s > 0$ be integers. Let R be any commutative ring. Let P be a prime ideal of R, with localization R_P. Suppose that there exists $k > 0$ such* *that the R_P/PR_P-vector space $(PR_P)^k/(PR_P)^{k+1}$ has dimension greater than $n-s$. Then R does not satisfy Condition $H_{n,s}$.*

Assume now that R is noetherian and that it satisfies Condition $H_{n,s}$. Then $\dim(R) \leq 1$, and every maximal ideal M of R is such that MR_M can be generated by at most $n-s$ elements.

Proof. Let us assume that R satisfies Condition $H_{n,s}$. Then R_P also satisfies Condition $H_{n,s}$. By hypothesis, there exist $r > n-s$ and elements $a_1, \dots, a_r$ of $(PR_P)^k \subset R_P$ whose images in $(PR_P)^k/(PR_P)^{k+1}$ are linearly independent. Consider the following s linear homogeneous polynomials in $R_P[x_{11}, \dots, x_{nn}]$:

$$a_1x_{11} + a_2x_{12} + \cdots + a_{n-s+1}x_{1,n-s+1},\ x_{1,n-s+2},\ \dots,\ x_{1,n}.$$

Using Condition $H_{n,s}$, there exists a matrix $U = (u_{ij}) \in \mathrm{GL}_n(R_P)$ such that $a_1u_{11} + a_2u_{12} + \cdots + a_{n-s+1}u_{1,n-s+1} = 0$, and $u_{1,n-s+2} = \cdots = u_{1,n} = 0$. Expanding the determinant of U along the first row, we find that there exist $b_i \in R_P$, $i = 1, \dots, n-s+1$, such that $b_1u_{11} + b_2u_{12} + \cdots + b_{n-s+1}u_{1,n-s+1}$ is a unit in R_P. In particular, there exists at least one u_{1j} with $j \leq n-s+1$ which does not belong to PR_P. It follows that $a_1u_{11} + a_2u_{12} + \cdots + a_{n-s+1}u_{1,n-s+1} = 0$ produces a non-trivial linear relation between the images of $a_1, \dots, a_r$ in the R_P/PR_P-vector space $(PR_P)^k/(PR_P)^{k+1}$, and this is a contradiction.

Assume now that R is noetherian. To prove that $\dim(R) \leq 1$, it suffices to show that for any maximal ideal M of R, $\dim(R_M) \leq 1$. Since R_M is a noetherian local ring, the function $f(k) := \dim_{R_M/MR_M}((MR_M)^k/(MR_M)^{k+1})$ is given for k large enough by the values of a polynomial $g(k)$ of degree equal to $(\dim(R_M) - 1)$. In particular, if $\dim(R_M) > 1$, there always exists a value k such that $f(k) > n-s$. This implies by our earlier considerations that Condition $H_{n,s}$ cannot be satisfied, and this is a contradiction. Assume now that $\dim(R_M) \leq 1$, and that MR_M can be minimally generated by r elements $a_1, \dots, a_r$. Then the images of $a_1, \dots, a_r$ in $MR_M/(MR_M)^2$ are linearly independent. It follows that $r \leq n-s$. □

Remark 3.9 By definition, a Dedekind domain R is noetherian, and every maximal ideal M of R is such that MR_M is generated by one element. One may wonder, in view of Proposition 3.8, whether R satisfies Condition $H_{n,n-1}$. As Lemma 3.6 shows, the answer is negative when R is not a principal ideal domain.

Remark 3.10 Theorems in the literature pertaining to the completion of a partial integral matrix to a unimodular matrix may sometimes be interpreted in light of variations on Condition $H_{n,s}$. For instance, pick $n-1$ distinct variables x_{ij}, and denote by I the set of chosen indices (i,j). Pick $c_{ij} \in \mathbb{Z}$, for each $(i,j) \in I$. Then the set $(\bigcap_{(ij)\in I} Z_{x_{ij}-c_{ij}}(\mathbb{Z}))$ contains an element of $\mathrm{GL}_n(\mathbb{Z})$ ([13], cor. 3). See also [2], Thm. 1, for related results.

Remark 3.11 Let us note here an analogy between the Conditions $(SU)_n$ and $H_{n,1}$: Both express the fact that a natural submodule of $M_n(R)$ has 'maximal orbit' under the action of $\mathrm{GL}_n(R)$.

First, consider the R-submodule $T_n \subset M_n(R)$ consisting of all matrices having trace zero. Fix $s \geq 1$, and consider the product $(T_n)^s$ of s copies of T_n inside $(M_n(R))^s$. Let $\mathrm{GL}_n(R)$ act diagonally by left multiplication on the set $(M_n(R))^s$. Let $\mathcal{O}_{n,s}$ denote the orbit of $(T_n)^s$ in $(M_n(R))^s$ under the action of $\mathrm{GL}_n(R)$. Then *R satisfies Condition $H_{n,s}$ if and only if $\mathcal{O}_{n,s} = (M_n(R))^s$*. Indeed, let $h(x_{11}, \dots, x_{nn}) = \sum a_{ij}x_{ij}$ be any linear homogeneous polynomial. Let A denote the associated matrix $(a_{ij}) \in M_n(R)$. Then there exists $U = (x_{ij})$ in $\mathrm{GL}_n(R) \cap Z_h(R)$ if and only if the matrix UA^t has trace zero.

Second, let S_n denote the R-submodule of $M_n(R)$ consisting of all symmetric matrices. A ring satisfies Condition $(SU)_n$ if and only if $M_n(R)$ is equal to the orbit of S_n in $M_n(R)$ under the action of $\mathrm{GL}_n(R)$.

It is never the case that the orbit of $S_n \cap T_n$ is equal to $M_n(R)$, since for instance the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ cannot be written as UB with B symmetric and trace 0, and U invertible. Note also that the orbit of $S_n \times S_n$ in $M_n(R) \times M_n(R)$ under the diagonal action of $\mathrm{GL}_n(R)$ is never equal to $M_n(R) \times M_n(R)$. For instance, the element (B, C) is not in the orbit, where $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$.

For use in our next example, let us make the following definition.

Definition 3.12 Given a commutative ring R, we define *Condition* $H'_{n,s}$ analogously to Condition $H_{n,s}$, by replacing everywhere in 3.4 the group $\mathrm{GL}_n(R)$ by its subgroup $\mathrm{SL}_n(R)$.

Example 3.13 We can now use Lemma 3.6 to construct examples of new Bézout domains which satisfy Condition $H'_{n,n-1}$ for some $n > 1$. Let R be any commutative ring, and fix $n > 1$. For ease of notation, let us note here that the coefficients of a set of $n-1$ homogeneous linear polynomials in $R[x_{11}, \ldots, x_{nn}]$ determine a $n^2 \times (n-1)$ matrix A with entries in R. Conversely, such a matrix A determines $n-1$ linear homogeneous polynomials, namely the $n-1$ entries of the matrix $(x_{11}, \ldots, x_{nn})A$. Let $X = (x_{ij})$ denote the square $n \times n$-matrix in the indeterminates x_{ij}, $1 \leq i, j \leq n$.

For each matrix $A \in M_{n^2,n-1}(R)$, consider the subset $I(A)$ of $R[x_{11}, \ldots, x_{nn}]$ consisting of $\det(X) - 1$ and of the $n-1$ homogeneous linear polynomials obtained from A. Let $< I(A) >$ denote the ideal of $R[x_{11}, \ldots, x_{nn}]$ generated by $I(A)$. We claim that $< I(A) > \neq R[x_{11}, \ldots, x_{nn}]$. Indeed, choose a maximal ideal M of R, and let $K := R/M$. Let $I_M = \{\det(X) - 1, h_1, \ldots, h_{n-1}\}$ denote the subset of $K[x_{11}, \ldots, x_{nn}]$ consisting of the images modulo M of the elements of $I(A)$. Proposition 3.7 shows that the intersection $\mathrm{GL}_n(K) \cap (\cap_{i=1}^{n-1} Z_{h_i}(K))$ is not empty. Let C be a matrix in this intersection, and set $\det(C) = c$. It follows that over the field $L := K(\sqrt[n]{c})$, the matrix $\frac{1}{\sqrt[n]{c}} C$ belongs to $\mathrm{SL}_n(L) \cap (\cap_{i=1}^{n-1} Z_{h_i}(L))$. Therefore, the ideal $< I_M >$ is a proper ideal of $K[x_{11}, \ldots, x_{nn}]$, and $< I(A) > \neq R[x_{11}, \ldots, x_{nn}]$.

Consider the set $\mathcal{I}$ of all subsets $I(A)$, $A \in M_{n^2,n-1}(R)$, such that there exists no homomorphism of R-algebras between $R[x_{11}, \ldots, x_{nn}]/ < I(A) >$ and R. For each subset $I = I(A) \in \mathcal{I}$, let $\mathbf{x}^I$ denote the set of n^2 variables labeled $x_{11}^I, \ldots, x_{nn}^I$. Let $I(A, \mathbf{x}^I)$ be the subset of $R[\mathbf{x}^I]$ consisting of $\det((\mathbf{x}^I)) - 1$ and of the $n-1$ homogeneous linear polynomials obtained from A. It is not difficult to check that the ideal $< I(A, \mathbf{x}^I), I \in \mathcal{I} >$ is a proper ideal of $R[\mathbf{x}^I, I \in \mathcal{I}]$, so we can define the quotient ring

$$h_n(R) := R[\mathbf{x}^I, I \in \mathcal{I}]/ < I(A, \mathbf{x}^I), I \in \mathcal{I} > .$$

Note that if R is a $H'_{n,n-1}$-ring, then $\mathcal{I} = \emptyset$, and $h_n(R) = R$. It is clear that we have a natural morphism of R-algebras $R \to h_n(R)$. By construction, given any matrix $B \in M_{n^2,n-1}(R)$, there exists $U \in \mathrm{SL}_n(h_n(R))$ which also belongs to the zero-sets with coefficients in $h_n(R)$ of the $n-1$ homogeneous polynomials defined by B. Indeed, simply take $U :=$ (class of $x_{ij}^{I(B)}$ in $h_n(R))_{1\leq i,j\leq n}$.

Let $h_n^{(1)}(R) := h_n(R)$, and for each $i \in \mathbb{N}$, we set $h_n^{(i)}(R) := h_n(h_n^{(i-1)}(R))$. Finally, we let

$$\mathcal{H}_n(R) := \lim_i h_n^{(i)}(R).$$

Let $C \in M_{n^2,n-1}(\mathcal{H}_n(R))$. Then the finitely many coefficients of C all lie in a single ring $h_n^{(i)}(R)$ for some $i > 0$. By construction, there exist $U := (u_{ij}) \in \mathrm{SL}_n(h_n^{(i)}(R))$ which also belongs to the zero-sets with coefficients in $\mathcal{H}_n(R)$ of the $n-1$ homogeneous polynomials defined by C. It follows that $\mathcal{H}_n(R)$ satisfies Condition $H'_{n,n-1}$.

Given any prime ideal P of $\mathcal{H}_n(R)$, the quotient $\mathcal{H}_n(R)/P$ is also a $H'_{n,n-1}$-domain and, thus, a Bézout domain (3.6). It is natural to wonder whether one could show for a well-chosen ring R that one such domain is not an Elementary Divisor domain, for instance by showing that $\mathcal{H}_n(R)/P$ does not satisfy Condition $H_{n+1,1}$ and use 3.3.

In the simplest case where $n = 2$, the relationships between the two types of conditions introduced in this paper, $(SU)_n$ and H_n, can be summarized as follows.

Proposition 3.14. *Let R be any commutative ring. Consider the following properties:*

a) *R is an Elementary Divisor ring.*
b) *R satisfies Condition H_2.*
c') *R satisfies Condition $H'_{2,1}$.*
d') *R satisfies Condition $(SU')_2$.*
c) *R satisfies Condition $H_{2,1}$.*
d) *R satisfies Condition $(SU)_2$.*
e) *R is a Hermite ring.*

Then a) $\Longrightarrow$ b) $\Longrightarrow$ c') $\Longleftrightarrow$ d') $\Longrightarrow$ c) $\Longleftrightarrow$ d) $\Longrightarrow$ e).

Proof. The implication *a)* $\Longrightarrow$ *b)* is proved in Proposition 3.3. The implications *b)* $\Longrightarrow$ *c')*, *c')* $\Longrightarrow$ *c)*, and *d')* $\Longrightarrow$ *d)*, are obvious. The implication *d)* $\Longrightarrow$ *e)* is proved in Lemma 2.6.

Proof of c') $\Longleftrightarrow$ *d')* and *c)* $\Longleftrightarrow$ *d)*. Let $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$. Consider the polynomial $h := cX - aY + dU - bV$. Condition $H'_{2,1}$ implies that $\mathrm{SL}_2(R) \cap Z_h(R) \neq \emptyset$. Hence, we can find $x, y, u, v \in R$ such that $xv - yu = 1$ and such that

$$A \begin{pmatrix} x & y \\ u & v \end{pmatrix} =: S$$

with S symmetric, since the condition $h(x, y, u, v) = cx - ay + du - bv = 0$ implies that $ay + bv = cx + du$. This shows that *c')* $\Longrightarrow$ *d')*. The proof of *c)* $\Longrightarrow$ *d)* is similar. The proofs of the converses are left to the reader. □

Corollary 3.15. *Let R be a commutative noetherian domain. Then R is a Bézout domain if and only if R satisfies Condition H_2.*

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