

ON BÉZOUT DOMAINS

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Recall that a commutative domain R in which every finitely generated ideal is principal is called a *Bézout domain*. By definition, a noetherian Bézout domain is a principal ideal domain. Several examples of non-noetherian Bézout domains are listed in [1], 243-246.

Recall also that a commutative domain R is called an *Elementary Divisor domain* if, given any matrix A with coefficients in R , there exist invertible matrices P, Q with coefficients in R such that $PAQ = D$ with $D = \text{diag}(d_1, \dots)$ a diagonal matrix (such a matrix may be rectangular, but it has d_1, \dots on the main diagonal and zeroes elsewhere).

Kaplansky showed in [7], 5.2, that a Bézout domain is an Elementary Divisor domain if and only if it satisfies (*): For all $a, b, c \in R$ with $(a, b, c) = R$, there exist $p, q \in R$ such that $(pa, pb + qc) = R$ (see also [4], 6.3). It is well-known that a principal ideal domain is an Elementary Divisor domain. Consideration of the Elementary Divisor problem for a non-noetherian ring can be found as early as [11].

It is an open question dating back to Helmer [5] in 1942 to decide whether a Bézout domain¹ is always an Elementary Divisor domain. Leavitt and Mosbo in fact state in [8], Remark 8, that it has been conjectured that there exists a Bézout domain that is not an Elementary Divisor domain (see also Problem 5 in [4], p. 122). Our contribution to this question is the introduction of new chains of implications between *R is an Elementary Divisor domain* and *R is Bézout*, which may prove useful in an eventual solution to the above open question.

Let $M_n(R)$ denote the ring of $(n \times n)$ -matrices with coefficients in R . We make the following definitions.

Definition 1 Let $n \geq 1$. A ring R is called an $(SU)_n$ -ring if, given any $A \in M_n(R)$, there exist a symmetric matrix $S \in M_n(R)$ and an invertible matrix $U \in \text{GL}_n(R)$ such that $A = SU$. If R is an $(SU)_n$ -ring for all $n \geq 1$, we shall say that R is an *SU-ring*.

A ring R is called an $(SU')_n$ -ring if, given any $A \in M_n(R)$, there exist a symmetric matrix $S \in M_n(R)$ and an invertible matrix $U \in \text{SL}_n(R)$ such that $A = SU$. If R is an $(SU')_n$ -ring for all $n \geq 1$, we shall say that R is an *SU'-ring*.

Proposition 2. *Let R be any commutative domain. Consider the following properties:*

- a) *R is an Elementary Divisor domain.*
- b) *R is an SU' -domain.*
- c) *R is an SU -domain.*
- d) *R is a Bézout domain.*

Then $a) \implies b) \implies c) \implies d)$.

Proof. $a) \implies b)$. Let $A \in M_n(R)$. Choose $P, Q \in \text{GL}_n(R)$ such that $PAQ = D$ is a diagonal matrix. Let $\epsilon := \det(P) \det(Q)^{-1}$. Let E denote any invertible diagonal matrix with determinant ϵ . Then $PAQE = DE$ is still symmetric. We find that

$$AQE(P^{-1})^t = P^{-1}DE(P^{-1})^t$$

Date: May 9, 2008.

¹What we now call Bézout domain is called a Prüfer domain in [5], first paragraph.

is symmetric, with $\det(QE(P^{-1})^t) = 1$. It is obvious that $b) \implies c)$. The last implication $c) \implies d)$ follows from our next lemma, which shows that an $(SU)_2$ -domain is a Bézout domain.

Lemma 3. *Let R be a domain, with $a, b \in R$. Let $A := \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$. Then there exists $V := \begin{pmatrix} u & v \\ s & t \end{pmatrix} \in \text{GL}_2(R)$ such that AV is symmetric if and only if the ideal (a, b) is principal.*

Proof. The cases where $ab = 0$ are easy and left to the reader. Assume that $ab \neq 0$. For the product AV to be symmetric, we need $av = bu$. For U to be invertible, we need that $ut - sv = \epsilon \in R^*$. Then $aut - asv = a\epsilon = u(at - bs)$, and $at - bs$ divides a . Similarly, $at - bs$ divides b . Therefore, $(at - bs) \subseteq (a, b) \subseteq (at - bs)$, and we find that the ideal (a, b) is principal.

Assume now that $(a, b) = (g)$. Then there exists $c, d \in R$ such that $a = gc$ and $b = gd$, and there exist $s, t \in R$ such that $as + bt = g$. Hence, $g(cs + dt) = g$, and since R is a domain, we find that $cs + dt = 1$. We can write

$$\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} c & d \\ -t & s \end{pmatrix} = \begin{pmatrix} ac & ab/g \\ ab/g & bd \end{pmatrix}.$$

□

We also have the following infinite sequence of implications.

Proposition 4. *Let R be a commutative Bézout domain and $n > 1$. If R is an $(SU)_n$ -domain, then it is an $(SU)_{n-1}$ -domain.*

Proof. Let $A \in M_{n-1}(R)$. Since R is Bézout, it is possible to find two invertible matrices $P, Q \in \text{GL}_{n-1}(R)$ such that PAQ consists in its upper left corner of a nonsingular matrix A' of rank equal to $\text{rank}(A)$, and such that all other coefficients of PAQ are zeros.

Let $B \in M_n(R)$ be the matrix with A' in the upper left corner, and with all other entries zeros. Let $U \in \text{GL}_n(R)$ be such that BU is symmetric. Clearly, the last $n - \text{rank}(A)$ rows of BU consists only in zeros. Since the matrix BU is symmetric, its last $n - \text{rank}(A)$ columns also consists only in zeros. Let W denote any vector in $R^{\text{rank}(A)}$ obtained from one of the $n - \text{rank}(A)$ last columns of U by removing from the column its last $n - \text{rank}(A)$ coefficients. Then $A'W = 0$. Since $\det(A') \neq 0$, we find that $W = 0$. Let V denote the square $\text{rank}(A)$ -matrix in the upper left corner of U , and let V' denote the square $(n - \text{rank}(A))$ -matrix in the lower right corner of U . Then $\det(U) = \det(V) \det(V')$. Hence, V is invertible, and we have $A'V$ symmetric.

Consider now the $(n - 1)$ -matrix T consisting of two blocks: V in the upper left corner, and an identity matrix of the appropriate size in the lower right corner. The matrix T is invertible. By construction, $PAQT$ is symmetric. Then $AQT(P^{-1})^t$ is also symmetric, with $QT(P^{-1})^t$ invertible. □

We do not know if any of the implications in our last two propositions can be reversed. We introduce now a natural strengthening of Property $(SU')_2$ of a more ‘arithmetic geometry’ flavor, which we call Property H_2 , with the letter H referring to a hyperplane condition.

Let R be any domain. Let $V_n := ((x_{ij}))_{1 \leq i, j \leq n}$ denote a square matrix in the indeterminates $x_{ij}, 1 \leq i, j \leq n$. Set $d_n := \det(V_n) \in R[x_{ij}, 1 \leq i, j \leq n]$. For $\mu \in R$,

denote by $Z_{d_n - \mu}(R)$ the set of solutions to the equation $d_n - \mu = 0$ in R^{n^2} . Clearly, $\text{SL}_n(R) = Z_{d_n - 1}(R)$.

Let $h(x_{11}, \dots, x_{nn}) := \sum_{1 \leq i, j \leq n} a_{ij} x_{ij} \in R[x_{ij}, 1 \leq i, j \leq n]$ be a non-zero homogeneous linear polynomial (i.e., without constant term). Let $\nu \in R$, and let $Z_{h - \nu}(R)$ denote the set of solutions to the equation $h - \nu = 0$ in R^{n^2} . When $(a_{ij}, 1 \leq i, j \leq n) = R$, $Z_{h - \nu}(R) \neq \emptyset$ for all ν .

Definition 5 We say that a commutative domain *satisfies condition H_n* if, for all linear homogeneous polynomials $h(x_{11}, \dots, x_{nn})$ and all $\nu \in R$ such that $Z_{h - \nu}(R) \neq \emptyset$, and for all $\mu \in R$, we have $Z_{d_n - \mu}(R) \cap Z_{h - \nu}(R) \neq \emptyset$.

Proposition 6. *Let R be any commutative domain. Consider the following properties:*

- a) R is an Elementary Divisor domain.
- b) R satisfies Condition H_2 .
- c) R satisfies Condition $(SU')_2$.
- d) R is a Bézout domain.

Then a) \implies b) \implies c) \implies d).

Proof of a) \implies b). Let $h = ax + by + cu + dv \in R[x, y, u, v]$, with $\gcd(a, b, c, d) = 1$. Consider the matrix

$$A := \begin{pmatrix} b & d \\ -a & -c \end{pmatrix}.$$

We need to show that for any $\mu, \nu \in R$, there exist $x, y, u, v \in R$ with $xv - yu = \mu$ such that

$$A \begin{pmatrix} x & y \\ u & v \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\beta - \gamma = \nu$. The key of the proof is the following easy fact. Let $B := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and let P be any two matrices in $M_2(R)$. Then

$$PB(P^t) = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix},$$

with $\beta' - \gamma' = \det(P)(\beta - \gamma)$. We leave the verification of this fact to the reader.

By hypothesis, there exist $P, Q \in \text{GL}_2(R)$ such that $PAQ = \text{diag}(e, f)$. Since $\gcd(a, b, c, d) = 1$, we may assume that $e \in R^*$. Multiplying both sides on the left by $\text{diag}(1, \det(P)^{-1})$ and both sides on the right by $\text{diag}(1, \det(Q)^{-1})$ if necessary, we may assume that the matrices P and Q are in $\text{SL}_2(R)$.

Let $V := \begin{pmatrix} 1 & e^{-1}\nu \\ 0 & \mu \end{pmatrix}$. Then

$$PAQV = \text{diag}(e, f)V = \begin{pmatrix} e & \nu \\ 0 & \mu f \end{pmatrix}.$$

Hence,

$$AQV(P^{-1})^t = P^{-1} \begin{pmatrix} e & \nu \\ 0 & \mu f \end{pmatrix} (P^{-1})^t = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix},$$

with $\beta'' - \gamma'' = \det(P^{-1})\nu = \nu$. Hence, R satisfies Condition H_2 , since $\det(QV(P^{-1})^t) = \mu$.

Proof of b) \implies c). Let $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$. Consider the polynomial $h := cx - ay + du - bv$. Clearly, $Z_h(R) \neq \emptyset$. Using Condition H_2 , we find that $Z_{d_2 - 1}(R) \cap Z_h(R) \neq \emptyset$.

Hence, we can find $x, y, u, v \in R$ such that $xv - yu = 1$ and

$$A \begin{pmatrix} x & y \\ u & v \end{pmatrix}$$

symmetric, since the condition $h = cx - ay + du - bv = 0$ implies that $ay + bv = cx + du$.

The implication $c) \implies d)$ follows from the previous lemma. \square

Proposition 7. *Let R be Bézout domain satisfying Condition H_2 , and let $n \geq 2$. Let $h(x_{11}, \dots, x_{nn}) := \sum_{1 \leq k, \ell \leq n} a_{k\ell} x_{k\ell} \in R[x_{k\ell}, 1 \leq k, \ell \leq n]$ with $\gcd(a_{k\ell}) = 1$. Consider indices $i \neq j$ such that $A_{ij} := \gcd(a_{ii}, a_{ij}, a_{ji}, a_{jj}) \neq 0$. Then*

$$Z_{d_n - \mu}(R) \cap Z_{h - \nu}(R) \neq \emptyset$$

for all $\mu \in R$ and all $\nu \in R$ with A_{ij} dividing $\nu - \sum_{k \neq i, j} a_{kk}$.

Proof. Set $x_{kk} = 1$ for all $k \neq i, j$. Set $x_{k\ell} = 0$ for all $(k, \ell) \neq (i, j), (j, i), (i, i), (j, j)$ and $(k, k), k \neq i, j$. Whenever A_{ij} divides $-\sum_{k \neq i, j} a_{kk} + \nu$, we can use Condition H_2 to obtain the existence of x_{ii}, x_{ij}, x_{ji} , and x_{jj} in R such that $x_{ii}x_{jj} - x_{ij}x_{ji} = \mu$ and $a_{ii}x_{ii} + a_{ij}x_{ij} + a_{ji}x_{ji} + a_{jj}x_{jj} = -\sum_{k \neq i, j} a_{kk} + \nu$. With our condition that $x_{kk} = 1$ for all $k \neq i, j$, we find that $h(x_{11}, \dots, x_{nn}) = \nu$ and $d_n(x_{11}, \dots, x_{nn}) = \mu$, as desired. \square

Corollary 8. *Let R be a valuation domain. Then R satisfies Condition H_n for all $n \geq 2$.*

Proof. By definition, in a valuation domain R , (a, b) is equal to either (a) or to (b) . In particular, R satisfies Kaplansky's condition $(*)$, and is an Elementary Divisor domain. So R satisfies condition H_2 . Given $h(x_{11}, \dots, x_{nn}) := \sum_{1 \leq k, \ell \leq n} a_{k\ell} x_{k\ell} \in R[x_{k\ell}, 1 \leq k, \ell \leq n]$ and $\nu \in R$ such that $Z_{h - \nu}(R) \neq \emptyset$, we find that $g := \gcd(a_{k\ell}, 1 \leq k, \ell \leq n)$ divides ν . Dividing by g , we are reduced to consider the case where $(a_{k\ell}, 1 \leq k, \ell \leq n) = R$. Then there exist indices $i \neq j$ such that $A_{ij} := \gcd(a_{ii}, a_{ij}, a_{ji}, a_{jj}) = 1$, and the previous proposition shows that R satisfies Condition H_n for all $n \geq 2$. \square

Remark 9 It is natural to wonder whether a domain R may satisfy Condition H_n for some $n > 2$ and not satisfy Condition H_2 . Proposition 6 implies that a principal ideal domain satisfies Condition H_2 . It is natural to wonder whether it then must also satisfies Condition H_n for all $n \geq 3$.

This latter question is open even when $R = \mathbb{Z}$. The above corollary implies that the answer to this question is positive for the local principal ideal domain $\mathbb{Z}_{(p)} := \{c/d \in \mathbb{Q}, \gcd(c, d) = 1, p \nmid d\}$, p any prime.

We note that knowing that an affine variety over \mathbb{Z} has a point in $\mathbb{Z}_{(p)}$ for all prime p does not in general imply the existence of an integer point. For instance, consider $(ax)^2 \pm (by)^2 = 1$, with $a, b > 1$, $a, b \in \mathbb{Z}$, coprime. Since $a, b > 1$, we find that this equation never has an integer solution. This equation always has the rational solution $(1/a, 0)$, and in the case of $(ax)^2 + (by)^2 = 1$, it also has the solution $(0, 1/b)$. Hence, in the latter case, since a and b are coprime, we find that this equation has a solution in $\mathbb{Z}_{(p)}$ for all prime p . In the case $(5x)^2 - (3y)^2 = 1$, we have the solutions $(1/5, 0)$ and $(1/4, 1/4)$ and, thus, a solution in $\mathbb{Z}_{(p)}$ for all prime p .

It is a classic result that the integral closure $\overline{\mathbb{Z}}$ of \mathbb{Z} in the algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} is a Bézout domain (see, e.g., [6], Theorem 102). In fact, given $(b, c) = \overline{\mathbb{Z}}$, there exists $q \in \overline{\mathbb{Z}}$ such that $(b, c) = (b + qc)$ ([2], 3.3, [10], 1.2). Using Kaplansky's criterion, it follows then that $\overline{\mathbb{Z}}$ is an Elementary Divisor domain.

Proposition 10. *Let $R = \overline{\mathbb{Z}}$ be the ring of all algebraic integers. Let $n \geq 2$. Then, for all linear homogeneous polynomials $h(x_{11}, \dots, x_{nn})$ and all $\nu \in R$ such that $Z_{h-\nu}(R) \neq \emptyset$, and for all $\mu \in R \setminus \{0\}$, we have $Z_{d_n-\mu}(R) \cap Z_{h-\nu}(R) \neq \emptyset$.*

Proof. Proposition 6 implies that Condition H_2 is satisfied. Let $n > 2$. Fix a linear homogeneous polynomial $h(x_{11}, \dots, x_{nn}) := \sum_{1 \leq k, \ell \leq n} a_{k\ell} x_{k\ell} \in R[x_{k\ell}, 1 \leq k, \ell \leq n]$ with $\gcd(a_{k\ell}) = 1$. Pick any (i, j) such that $a_{ij} \neq 0$ and use the relation $h(x_{11}, \dots, x_{nn}) = \nu$ to express the variable x_{ij} in terms of the others. Substitute this expression for x_{ij} in the polynomial $d_n - \mu$ to obtain a polynomial F in $n^2 - 1$ variables, of degree n . Moreover, expanding the determinant along a row containing the position (i, j) , we find that we can write

$$F = F_n + F_{n-1} + F_0,$$

where F_i is a homogeneous polynomial of degree i . Any monomial appearing in F_{n-1} is a product of $n - 1$ distinct variables. Any monomial appearing in F_n is divisible by a product of $n - 1$ distinct variables. When $\nu = 0$, $F_{n-1} = 0$.

Rumely's Local-Global Principle [9] implies that if a system of equations with coefficients in $R = \overline{\mathbb{Z}}$ defines a irreducible affine variety over $\overline{\mathbb{Q}}$, and if this system of equations has a solution over each localization $\overline{\mathbb{Z}}_P$, where P is a maximal ideal of $\overline{\mathbb{Z}}$, then the system of equations has a solution in $\overline{\mathbb{Z}}$. We show below that the polynomial F is irreducible in $\overline{\mathbb{Q}}[x_{k\ell}, (k, \ell) \neq (i, j)]$ when $n \geq 3$. Given any prime ideal P in a Bézout domain R , the local ring R_P is a valuation domain, so Corollary 8 implies that F has a solution in R_P . Rumely's Local-Global Principle implies that F has a solution in $R = \overline{\mathbb{Z}}$. \square

Proposition 11. *Let K be any field. Let $n > 2$. Fix a non-zero linear homogeneous polynomial $h(x_{11}, \dots, x_{nn}) := \sum_{1 \leq k, \ell \leq n} a_{k\ell} x_{k\ell} \in K[x_{k\ell}, 1 \leq k, \ell \leq n]$. Let $\mu \in K^*$ and $\nu \in K$. Then the subvariety of the affine space \mathbb{A}^{n^2}/K defined by the equations $d_n - \mu$ and $h - \nu$ is irreducible.*

Proof. We keep the notation introduced in the proof of the previous proposition. To prove our statement, it suffices to prove that the polynomial F is irreducible. We start by proving its irreducibility² when $n = 3$. Suppose that it is reducible, and write a factorization

$$F = F_3 + F_2 + F_0 = (g_1 + g_0)(h_2 + h_1 + h_0),$$

where g_i, h_i are homogeneous polynomials of degree i over K .

Recall that we assume that $\mu \neq 0$. Then $F_0 = g_0 h_0 = -\mu$, and after dividing by F_0 , we can assume that $F_0 = g_0 = h_0 = 1$. It follows that $f_1 + g_1 = 0$, $f_2 + f_1 g_1 = F_2$, and $f_2 g_1 = F_3$. Hence, if x_{ij} appears as a monomial in g_1 , then x_{ij}^2 appears as monomial in $f_2 = F_2 + g_1^2$, so that x_{ij}^3 appears as a monomial of $F_3 = f_2 g_1$, which is a contradiction.

Assume now that $n > 3$. We proceed by induction on n . Pick k, ℓ such that $k \neq i$ and $\ell \neq j$. Substitute in F the values $x_{k\ell} = 1$, $x_{ks} = 0$ for $s \neq \ell$, and $x_{t\ell} = 0$ for $t \neq k$ to get a new polynomial \overline{F} . A renumbering of the variables shows that \overline{F} is nothing but the polynomial $d_{n-1} - \mu$ in which a linear relation has been substituted for one variable: we can apply the induction hypothesis and find that \overline{F} is irreducible. Assume that F is reducible and write $F = (g_a + \dots + g_0)(h_b + \dots + h_0)$, where g_i and h_i are homogeneous of degree i . Note the relation $g_1 h_0 + g_0 h_1 = 0$. After the substitutions as above, we find that $\overline{F} = (\overline{g}_a + \dots + g_0)(\overline{h}_b + \dots + h_0)$. Since \overline{F} is irreducible of positive degree, either $\overline{g}_a + \dots + g_0$ or $\overline{h}_b + \dots + h_0$ is constant. Thus, either g_1 or h_1 can only be sums of linear

²When $n = 2$, write $d_2 = xv - uy$. When $\mu = \nu = 0$, and $h = x - u$, we find that $F = x(v - y)$ is reducible.

monomials in the variables where the substitutions occurred. Since $g_1h_0 + g_0h_1 = 0$ and $g_0h_0 \neq 0$, both g_1 and h_1 have that property. Choose now k', ℓ' such that $k' \neq i, k$ and $\ell' \neq j, \ell$. Repeat the same argument with k', ℓ' , and find a contradiction on g_1 and h_1 . \square

Remark 12 Fix $n \geq 2$ and a commutative domain R . It is natural to wonder what is the maximal integer $s = s(n)$ such that, whenever the intersection of any s hyperplanes $\bigcap_{i=1}^s Z_{h_i}(R)$ is not empty, then $SL_n(R) \cap (\bigcap_{i=1}^s Z_{h_i}(R))$ is also not empty. When R is a valuation domain, the above corollary shows that $s \geq 1$. This question seems open even in the case where R is an algebraically closed field.

Clearly, it is always possible to find n homogeneous linear polynomials h_i such that $SL_n(R) \cap (\bigcap_{i=1}^n Z_{h_i}(R)) = \emptyset$, so that $s(n) < n$. Indeed, simply take $h_i := X_{1,i}$ for $i = 1, \dots, n$. Let K denote the field of fractions of R . In this example, $SL_n(K) \cap (\bigcap Z_{h_i}(K))$ is also empty. A different example with $n = 2$ and $R = \mathbb{Z}$ is as follows. Consider $d_2 = xv - uy$, and $h_1 := v - 9x$, $h_2 := u + 4y$. Then $Z_{d_2-1}(\mathbb{Z}) \cap (\bigcap Z_{h_i}(\mathbb{Z})) = \emptyset$, but $Z_{d_2-1}(\mathbb{Q}) \cap (\bigcap Z_{h_i}(\mathbb{Q}))$ is not empty.

Theorems in the literature pertaining to the completion of a partial integral matrix to a unimodular matrix can be interpreted in light of the above question. For instance, pick $n - 1$ distinct variables x_{ij} , and denote by I the set of chosen indices (i, j) . Pick $c_{ij} \in R$, for each $(i, j) \in I$. Then the set $GL_n(R) \cap (\bigcap_{(ij) \in I} Z_{x_{ij}-c_{ij}}(R))$ is not empty ([12], cor. 3).

Now choose any number of hyperplanes in the $n^2 - n$ variables x_{ij} , $i \neq j$, such that in R^{n^2-n} , the intersection \mathcal{H} of these hyperplanes is not empty. Let $(u_{ij}, i \neq j)$ denote a vector in R^{n^2-n} that belongs to \mathcal{H} . Then the array U whose entries are u_{ij} for $i \neq j$ can be completed into a matrix in $GL_n(R)$ ([3], Thm. 1). Hence, $GL_n(R) \cap \mathcal{H}' \neq \emptyset$, where \mathcal{H}' denotes the common zeros in R^{n^2} of the equations of the chosen hyperplanes.

Remark 13 We note here the following easy fact. *Suppose that R is a Bézout domain with field of fractions K . Let J be any domain with $R \subseteq J \subseteq K$. If R satisfies $(SU)_n$ (resp. $(SU')_n$), then so does J .*

Indeed, it is well-known that J is also a Bézout domain. It is noted in [1], page 243, that every element of J can be written as α/β with $\alpha, \beta \in R$ and β a unit in J . It follows that any matrix $A \in M_n(J)$ can be written as $A = \text{diag}(\beta, \dots, \beta)A'$ with β invertible in J , and $A' \in M_n(R)$. Using property $(SU)_n$ for A' , we find $U \in GL_n(R)$ such that $A'U$ is symmetric (if $(SU')_n$ holds for R , we choose U with $\det(U) = 1$). Then $AU = \text{diag}(\beta, \dots, \beta)A'U$ is symmetric.

Remark 14 Suppose that a matrix $A \in M_n(R)$ has a factorization $A = SU$ with S symmetric and U invertible. In general, such a factorization is not unique. For instance, if $A \in GL_n(R)$, and $A = SU$, then for all n , $A = S^n(S^{1-n}U)$. Is the number of distinct factorizations of $A \in SL_n(\mathbb{Z})$ into a product SU always infinite?

Acknowledgement. Thanks to Lenny Chastkovsky and Jerry Hower for helpful comments and suggestions.

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