

# ON BÉZOUT DOMAINS

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Recall that a commutative domain  $R$  in which every finitely generated ideal is principal is called a *Bézout domain*. By definition, a noetherian Bézout domain is a principal ideal domain. Several examples of non-noetherian Bézout domains are listed in [1], 243-246.

Recall also that a commutative domain  $R$  is called an *Elementary Divisor domain* if, given any matrix  $A$  with coefficients in  $R$ , there exist invertible matrices  $P, Q$  with coefficients in  $R$  such that  $PAQ = D$  with  $D = \text{diag}(d_1, \dots)$  a diagonal matrix (such a matrix may be rectangular, but it has  $d_1, \dots$  on the main diagonal and zeroes elsewhere).

Kaplansky showed in [7], 5.2, that a Bézout domain is an Elementary Divisor domain if and only if it satisfies (\*): For all  $a, b, c \in R$  with  $(a, b, c) = R$ , there exist  $p, q \in R$  such that  $(pa, pb + qc) = R$  (see also [4], 6.3). It is well-known that a principal ideal domain is an Elementary Divisor domain. Consideration of the Elementary Divisor problem for a non-noetherian ring can be found as early as [11].

It is an open question dating back to Helmer [5] in 1942 to decide whether a Bézout domain<sup>1</sup> is always an Elementary Divisor domain. Leavitt and Mosbo in fact state in [8], Remark 8, that it has been conjectured that there exists a Bézout domain that is not an Elementary Divisor domain (see also Problem 5 in [4], p. 122). Our contribution to this question is the introduction of new chains of implications between *R is an Elementary Divisor domain* and *R is Bézout*, which may prove useful in an eventual solution to the above open question.

Let  $M_n(R)$  denote the ring of  $(n \times n)$ -matrices with coefficients in  $R$ . We make the following definitions.

**Definition 1** Let  $n \geq 1$ . A ring  $R$  is called an  $(SU)_n$ -ring if, given any  $A \in M_n(R)$ , there exist a symmetric matrix  $S \in M_n(R)$  and an invertible matrix  $U \in \text{GL}_n(R)$  such that  $A = SU$ . If  $R$  is an  $(SU)_n$ -ring for all  $n \geq 1$ , we shall say that  $R$  is an *SU-ring*.

A ring  $R$  is called an  $(SU')_n$ -ring if, given any  $A \in M_n(R)$ , there exist a symmetric matrix  $S \in M_n(R)$  and an invertible matrix  $U \in \text{SL}_n(R)$  such that  $A = SU$ . If  $R$  is an  $(SU')_n$ -ring for all  $n \geq 1$ , we shall say that  $R$  is an *SU'-ring*.

**Proposition 2.** *Let  $R$  be any commutative domain. Consider the following properties:*

- a)  *$R$  is an Elementary Divisor domain.*
- b)  *$R$  is an  $SU'$ -domain.*
- c)  *$R$  is an  $SU$ -domain.*
- d)  *$R$  is a Bézout domain.*

*Then  $a) \implies b) \implies c) \implies d)$ .*

*Proof.*  $a) \implies b)$ . Let  $A \in M_n(R)$ . Choose  $P, Q \in \text{GL}_n(R)$  such that  $PAQ = D$  is a diagonal matrix. Let  $\epsilon := \det(P) \det(Q)^{-1}$ . Let  $E$  denote any invertible diagonal matrix with determinant  $\epsilon$ . Then  $PAQE = DE$  is still symmetric. We find that

$$AQE(P^{-1})^t = P^{-1}DE(P^{-1})^t$$

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<sup>1</sup>What we now call Bézout domain is called a Prüfer domain in [5], first paragraph.

is symmetric, with  $\det(QE(P^{-1})^t) = 1$ . It is obvious that  $b) \implies c)$ . The last implication  $c) \implies d)$  follows from our next lemma, which shows that an  $(SU)_2$ -domain is a Bézout domain.

**Lemma 3.** *Let  $R$  be a domain, with  $a, b \in R$ . Let  $A := \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ . Then there exists  $V := \begin{pmatrix} u & v \\ s & t \end{pmatrix} \in \text{GL}_2(R)$  such that  $AV$  is symmetric if and only if the ideal  $(a, b)$  is principal.*

*Proof.* The cases where  $ab = 0$  are easy and left to the reader. Assume that  $ab \neq 0$ . For the product  $AV$  to be symmetric, we need  $av = bu$ . For  $U$  to be invertible, we need that  $ut - sv = \epsilon \in R^*$ . Then  $aut - asv = a\epsilon = u(at - bs)$ , and  $at - bs$  divides  $a$ . Similarly,  $at - bs$  divides  $b$ . Therefore,  $(at - bs) \subseteq (a, b) \subseteq (at - bs)$ , and we find that the ideal  $(a, b)$  is principal.

Assume now that  $(a, b) = (g)$ . Then there exists  $c, d \in R$  such that  $a = gc$  and  $b = gd$ , and there exist  $s, t \in R$  such that  $as + bt = g$ . Hence,  $g(cs + dt) = g$ , and since  $R$  is a domain, we find that  $cs + dt = 1$ . We can write

$$\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} c & d \\ -t & s \end{pmatrix} = \begin{pmatrix} ac & ab/g \\ ab/g & bd \end{pmatrix}.$$

□

We also have the following infinite sequence of implications.

**Proposition 4.** *Let  $R$  be a commutative Bézout domain and  $n > 1$ . If  $R$  is an  $(SU)_n$ -domain, then it is an  $(SU)_{n-1}$ -domain.*

*Proof.* Let  $A \in M_{n-1}(R)$ . Since  $R$  is Bézout, it is possible to find two invertible matrices  $P, Q \in \text{GL}_{n-1}(R)$  such that  $PAQ$  consists in its upper left corner of a nonsingular matrix  $A'$  of rank equal to  $\text{rank}(A)$ , and such that all other coefficients of  $PAQ$  are zeros.

Let  $B \in M_n(R)$  be the matrix with  $A'$  in the upper left corner, and with all other entries zeros. Let  $U \in \text{GL}_n(R)$  be such that  $BU$  is symmetric. Clearly, the last  $n - \text{rank}(A)$  rows of  $BU$  consists only in zeros. Since the matrix  $BU$  is symmetric, its last  $n - \text{rank}(A)$  columns also consists only in zeros. Let  $W$  denote any vector in  $R^{\text{rank}(A)}$  obtained from one of the  $n - \text{rank}(A)$  last columns of  $U$  by removing from the column its last  $n - \text{rank}(A)$  coefficients. Then  $A'W = 0$ . Since  $\det(A') \neq 0$ , we find that  $W = 0$ . Let  $V$  denote the square  $\text{rank}(A)$ -matrix in the upper left corner of  $U$ , and let  $V'$  denote the square  $(n - \text{rank}(A))$ -matrix in the lower right corner of  $U$ . Then  $\det(U) = \det(V) \det(V')$ . Hence,  $V$  is invertible, and we have  $A'V$  symmetric.

Consider now the  $(n - 1)$ -matrix  $T$  consisting of two blocks:  $V$  in the upper left corner, and an identity matrix of the appropriate size in the lower right corner. The matrix  $T$  is invertible. By construction,  $PAQT$  is symmetric. Then  $AQT(P^{-1})^t$  is also symmetric, with  $QT(P^{-1})^t$  invertible. □

We do not know if any of the implications in our last two propositions can be reversed. We introduce now a natural strengthening of Property  $(SU')_2$  of a more ‘arithmetic geometry’ flavor, which we call Property  $H_2$ , with the letter  $H$  referring to a hyperplane condition.

Let  $R$  be any domain. Let  $V_n := ((x_{ij}))_{1 \leq i, j \leq n}$  denote a square matrix in the indeterminates  $x_{ij}, 1 \leq i, j \leq n$ . Set  $d_n := \det(V_n) \in R[x_{ij}, 1 \leq i, j \leq n]$ . For  $\mu \in R$ ,

denote by  $Z_{d_n-\mu}(R)$  the set of solutions to the equation  $d_n - \mu = 0$  in  $R^{n^2}$ . Clearly,  $\text{SL}_n(R) = Z_{d_n-1}(R)$ .

Let  $h(x_{11}, \dots, x_{nn}) := \sum_{1 \leq i, j \leq n} a_{ij} x_{ij} \in R[x_{ij}, 1 \leq i, j \leq n]$  be a non-zero homogeneous linear polynomial (i.e., without constant term). Let  $\nu \in R$ , and let  $Z_{h-\nu}(R)$  denote the set of solutions to the equation  $h - \nu = 0$  in  $R^{n^2}$ . When  $(a_{ij}, 1 \leq i, j \leq n) = R$ ,  $Z_{h-\nu}(R) \neq \emptyset$  for all  $\nu$ .

**Definition 5** We say that a commutative domain *satisfies condition  $H_n$*  if, for all linear homogeneous polynomials  $h(x_{11}, \dots, x_{nn})$  and all  $\nu \in R$  such that  $Z_{h-\nu}(R) \neq \emptyset$ , and for all  $\mu \in R$ , we have  $Z_{d_n-\mu}(R) \cap Z_{h-\nu}(R) \neq \emptyset$ .

**Proposition 6.** *Let  $R$  be any commutative domain. Consider the following properties:*

- a)  $R$  is an Elementary Divisor domain.
- b)  $R$  satisfies Condition  $H_2$ .
- c)  $R$  satisfies Condition  $(SU')_2$ .
- d)  $R$  is a Bézout domain.

Then a)  $\implies$  b)  $\implies$  c)  $\implies$  d).

*Proof of a)  $\implies$  b).* Let  $h = ax + by + cu + dv \in R[x, y, u, v]$ , with  $\gcd(a, b, c, d) = 1$ . Consider the matrix

$$A := \begin{pmatrix} b & d \\ -a & -c \end{pmatrix}.$$

We need to show that for any  $\mu, \nu \in R$ , there exist  $x, y, u, v \in R$  with  $xv - yu = \mu$  such that

$$A \begin{pmatrix} x & y \\ u & v \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with  $\beta - \gamma = \nu$ . The key of the proof is the following easy fact. Let  $B := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  and let  $P$  be any two matrices in  $M_2(R)$ . Then

$$PB(P^t) = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix},$$

with  $\beta' - \gamma' = \det(P)(\beta - \gamma)$ . We leave the verification of this fact to the reader.

By hypothesis, there exist  $P, Q \in \text{GL}_2(R)$  such that  $PAQ = \text{diag}(e, f)$ . Since  $\gcd(a, b, c, d) = 1$ , we may assume that  $e \in R^*$ . Multiplying both sides on the left by  $\text{diag}(1, \det(P)^{-1})$  and both sides on the right by  $\text{diag}(1, \det(Q)^{-1})$  if necessary, we may assume that the matrices  $P$  and  $Q$  are in  $\text{SL}_2(R)$ .

Let  $V := \begin{pmatrix} 1 & e^{-1}\nu \\ 0 & \mu \end{pmatrix}$ . Then

$$PAQV = \text{diag}(e, f)V = \begin{pmatrix} e & \nu \\ 0 & \mu f \end{pmatrix}.$$

Hence,

$$AQV(P^{-1})^t = P^{-1} \begin{pmatrix} e & \nu \\ 0 & \mu f \end{pmatrix} (P^{-1})^t = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix},$$

with  $\beta'' - \gamma'' = \det(P^{-1})\nu = \nu$ . Hence,  $R$  satisfies Condition  $H_2$ , since  $\det(QV(P^{-1})^t) = \mu$ .

*Proof of b)  $\implies$  c).* Let  $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$ . Consider the polynomial  $h := cx - ay + du - bv$ . Clearly,  $Z_h(R) \neq \emptyset$ . Using Condition  $H_2$ , we find that  $Z_{d_2-1}(R) \cap Z_h(R) \neq \emptyset$ .

Hence, we can find  $x, y, u, v \in R$  such that  $xv - yu = 1$  and

$$A \begin{pmatrix} x & y \\ u & v \end{pmatrix}$$

symmetric, since the condition  $h = cx - ay + du - bv = 0$  implies that  $ay + bv = cx + du$ .

The implication  $c) \implies d)$  follows from the previous lemma.  $\square$

**Proposition 7.** *Let  $R$  be Bézout domain satisfying Condition  $H_2$ , and let  $n \geq 2$ . Let  $h(x_{11}, \dots, x_{nn}) := \sum_{1 \leq k, \ell \leq n} a_{k\ell} x_{k\ell} \in R[x_{k\ell}, 1 \leq k, \ell \leq n]$  with  $\gcd(a_{k\ell}) = 1$ . Consider indices  $i \neq j$  such that  $A_{ij} := \gcd(a_{ii}, a_{ij}, a_{ji}, a_{jj}) \neq 0$ . Then*

$$Z_{d_n - \mu}(R) \cap Z_{h - \nu}(R) \neq \emptyset$$

for all  $\mu \in R$  and all  $\nu \in R$  with  $A_{ij}$  dividing  $\nu - \sum_{k \neq i, j} a_{kk}$ .

*Proof.* Set  $x_{kk} = 1$  for all  $k \neq i, j$ . Set  $x_{k\ell} = 0$  for all  $(k, \ell) \neq (i, j), (j, i), (i, i), (j, j)$  and  $(k, k), k \neq i, j$ . Whenever  $A_{ij}$  divides  $-\sum_{k \neq i, j} a_{kk} + \nu$ , we can use Condition  $H_2$  to obtain the existence of  $x_{ii}, x_{ij}, x_{ji}$ , and  $x_{jj}$  in  $R$  such that  $x_{ii}x_{jj} - x_{ij}x_{ji} = \mu$  and  $a_{ii}x_{ii} + a_{ij}x_{ij} + a_{ji}x_{ji} + a_{jj}x_{jj} = -\sum_{k \neq i, j} a_{kk} + \nu$ . With our condition that  $x_{kk} = 1$  for all  $k \neq i, j$ , we find that  $h(x_{11}, \dots, x_{nn}) = \nu$  and  $d_n(x_{11}, \dots, x_{nn}) = \mu$ , as desired.  $\square$

**Corollary 8.** *Let  $R$  be a valuation domain. Then  $R$  satisfies Condition  $H_n$  for all  $n \geq 2$ .*

*Proof.* By definition, in a valuation domain  $R$ ,  $(a, b)$  is equal to either  $(a)$  or to  $(b)$ . In particular,  $R$  satisfies Kaplansky's condition  $(*)$ , and is an Elementary Divisor domain. So  $R$  satisfies condition  $H_2$ . Given  $h(x_{11}, \dots, x_{nn}) := \sum_{1 \leq k, \ell \leq n} a_{k\ell} x_{k\ell} \in R[x_{k\ell}, 1 \leq k, \ell \leq n]$  and  $\nu \in R$  such that  $Z_{h - \nu}(R) \neq \emptyset$ , we find that  $g := \gcd(a_{k\ell}, 1 \leq k, \ell \leq n)$  divides  $\nu$ . Dividing by  $g$ , we are reduced to consider the case where  $(a_{k\ell}, 1 \leq k, \ell \leq n) = R$ . Then there exist indices  $i \neq j$  such that  $A_{ij} := \gcd(a_{ii}, a_{ij}, a_{ji}, a_{jj}) = 1$ , and the previous proposition shows that  $R$  satisfies Condition  $H_n$  for all  $n \geq 2$ .  $\square$

**Remark 9** It is natural to wonder whether a domain  $R$  may satisfy Condition  $H_n$  for some  $n > 2$  and not satisfy Condition  $H_2$ . Proposition 6 implies that a principal ideal domain satisfies Condition  $H_2$ . It is natural to wonder whether it then must also satisfies Condition  $H_n$  for all  $n \geq 3$ .

This latter question is open even when  $R = \mathbb{Z}$ . The above corollary implies that the answer to this question is positive for the local principal ideal domain  $\mathbb{Z}_{(p)} := \{c/d \in \mathbb{Q}, \gcd(c, d) = 1, p \nmid d\}$ ,  $p$  any prime.

We note that knowing that an affine variety over  $\mathbb{Z}$  has a point in  $\mathbb{Z}_{(p)}$  for all prime  $p$  does not in general imply the existence of an integer point. For instance, consider  $(ax)^2 \pm (by)^2 = 1$ , with  $a, b > 1$ ,  $a, b \in \mathbb{Z}$ , coprime. Since  $a, b > 1$ , we find that this equation never has an integer solution. This equation always has the rational solution  $(1/a, 0)$ , and in the case of  $(ax)^2 + (by)^2 = 1$ , it also has the solution  $(0, 1/b)$ . Hence, in the latter case, since  $a$  and  $b$  are coprime, we find that this equation has a solution in  $\mathbb{Z}_{(p)}$  for all prime  $p$ . In the case  $(5x)^2 - (3y)^2 = 1$ , we have the solutions  $(1/5, 0)$  and  $(1/4, 1/4)$  and, thus, a solution in  $\mathbb{Z}_{(p)}$  for all prime  $p$ .

It is a classic result that the integral closure  $\overline{\mathbb{Z}}$  of  $\mathbb{Z}$  in the algebraic closure  $\overline{\mathbb{Q}}$  of  $\mathbb{Q}$  is a Bézout domain (see, e.g., [6], Theorem 102). In fact, given  $(b, c) = \overline{\mathbb{Z}}$ , there exists  $q \in \overline{\mathbb{Z}}$  such that  $(b, c) = (b + qc)$  ([2], 3.3, [10], 1.2). Using Kaplansky's criterion, it follows then that  $\overline{\mathbb{Z}}$  is an Elementary Divisor domain.

**Proposition 10.** *Let  $R = \overline{\mathbb{Z}}$  be the ring of all algebraic integers. Let  $n \geq 2$ . Then, for all linear homogeneous polynomials  $h(x_{11}, \dots, x_{nn})$  and all  $\nu \in R$  such that  $Z_{h-\nu}(R) \neq \emptyset$ , and for all  $\mu \in R \setminus \{0\}$ , we have  $Z_{d_n-\mu}(R) \cap Z_{h-\nu}(R) \neq \emptyset$ .*

*Proof.* Proposition 6 implies that Condition  $H_2$  is satisfied. Let  $n > 2$ . Fix a linear homogeneous polynomial  $h(x_{11}, \dots, x_{nn}) := \sum_{1 \leq k, \ell \leq n} a_{k\ell} x_{k\ell} \in R[x_{k\ell}, 1 \leq k, \ell \leq n]$  with  $\gcd(a_{k\ell}) = 1$ . Pick any  $(i, j)$  such that  $a_{ij} \neq 0$  and use the relation  $h(x_{11}, \dots, x_{nn}) = \nu$  to express the variable  $x_{ij}$  in terms of the others. Substitute this expression for  $x_{ij}$  in the polynomial  $d_n - \mu$  to obtain a polynomial  $F$  in  $n^2 - 1$  variables, of degree  $n$ . Moreover, expanding the determinant along a row containing the position  $(i, j)$ , we find that we can write

$$F = F_n + F_{n-1} + F_0,$$

where  $F_i$  is a homogeneous polynomial of degree  $i$ . Any monomial appearing in  $F_{n-1}$  is a product of  $n - 1$  distinct variables. Any monomial appearing in  $F_n$  is divisible by a product of  $n - 1$  distinct variables. When  $\nu = 0$ ,  $F_{n-1} = 0$ .

Rumely's Local-Global Principle [9] implies that if a system of equations with coefficients in  $R = \overline{\mathbb{Z}}$  defines a irreducible affine variety over  $\overline{\mathbb{Q}}$ , and if this system of equations has a solution over each localization  $\overline{\mathbb{Z}}_P$ , where  $P$  is a maximal ideal of  $\overline{\mathbb{Z}}$ , then the system of equations has a solution in  $\overline{\mathbb{Z}}$ . We show below that the polynomial  $F$  is irreducible in  $\overline{\mathbb{Q}}[x_{k\ell}, (k, \ell) \neq (i, j)]$  when  $n \geq 3$ . Given any prime ideal  $P$  in a Bézout domain  $R$ , the local ring  $R_P$  is a valuation domain, so Corollary 8 implies that  $F$  has a solution in  $R_P$ . Rumely's Local-Global Principle implies that  $F$  has a solution in  $R = \overline{\mathbb{Z}}$ .  $\square$

**Proposition 11.** *Let  $K$  be any field. Let  $n > 2$ . Fix a non-zero linear homogeneous polynomial  $h(x_{11}, \dots, x_{nn}) := \sum_{1 \leq k, \ell \leq n} a_{k\ell} x_{k\ell} \in K[x_{k\ell}, 1 \leq k, \ell \leq n]$ . Let  $\mu \in K^*$  and  $\nu \in K$ . Then the subvariety of the affine space  $\mathbb{A}^{n^2}/K$  defined by the equations  $d_n - \mu$  and  $h - \nu$  is irreducible.*

*Proof.* We keep the notation introduced in the proof of the previous proposition. To prove our statement, it suffices to prove that the polynomial  $F$  is irreducible. We start by proving its irreducibility<sup>2</sup> when  $n = 3$ . Suppose that it is reducible, and write a factorization

$$F = F_3 + F_2 + F_0 = (g_1 + g_0)(h_2 + h_1 + h_0),$$

where  $g_i, h_i$  are homogeneous polynomials of degree  $i$  over  $K$ .

Recall that we assume that  $\mu \neq 0$ . Then  $F_0 = g_0 h_0 = -\mu$ , and after dividing by  $F_0$ , we can assume that  $F_0 = g_0 = h_0 = 1$ . It follows that  $f_1 + g_1 = 0$ ,  $f_2 + f_1 g_1 = F_2$ , and  $f_2 g_1 = F_3$ . Hence, if  $x_{ij}$  appears as a monomial in  $g_1$ , then  $x_{ij}^2$  appears as monomial in  $f_2 = F_2 + g_1^2$ , so that  $x_{ij}^3$  appears as a monomial of  $F_3 = f_2 g_1$ , which is a contradiction.

Assume now that  $n > 3$ . We proceed by induction on  $n$ . Pick  $k, \ell$  such that  $k \neq i$  and  $\ell \neq j$ . Substitute in  $F$  the values  $x_{k\ell} = 1$ ,  $x_{ks} = 0$  for  $s \neq \ell$ , and  $x_{t\ell} = 0$  for  $t \neq k$  to get a new polynomial  $\overline{F}$ . A renumbering of the variables shows that  $\overline{F}$  is nothing but the polynomial  $d_{n-1} - \mu$  in which a linear relation has been substituted for one variable: we can apply the induction hypothesis and find that  $\overline{F}$  is irreducible. Assume that  $F$  is reducible and write  $F = (g_a + \dots + g_0)(h_b + \dots + h_0)$ , where  $g_i$  and  $h_i$  are homogeneous of degree  $i$ . Note the relation  $g_1 h_0 + g_0 h_1 = 0$ . After the substitutions as above, we find that  $\overline{F} = (\overline{g}_a + \dots + g_0)(\overline{h}_b + \dots + h_0)$ . Since  $\overline{F}$  is irreducible of positive degree, either  $\overline{g}_a + \dots + g_0$  or  $\overline{h}_b + \dots + h_0$  is constant. Thus, either  $g_1$  or  $h_1$  can only be sums of linear

<sup>2</sup>When  $n = 2$ , write  $d_2 = xv - uy$ . When  $\mu = \nu = 0$ , and  $h = x - u$ , we find that  $F = x(v - y)$  is reducible.

monomials in the variables where the substitutions occurred. Since  $g_1h_0 + g_0h_1 = 0$  and  $g_0h_0 \neq 0$ , both  $g_1$  and  $h_1$  have that property. Choose now  $k', \ell'$  such that  $k' \neq i, k$  and  $\ell' \neq j, \ell$ . Repeat the same argument with  $k', \ell'$ , and find a contradiction on  $g_1$  and  $h_1$ .  $\square$

**Remark 12** Fix  $n \geq 2$  and a commutative domain  $R$ . It is natural to wonder what is the maximal integer  $s = s(n)$  such that, whenever the intersection of any  $s$  hyperplanes  $\bigcap_{i=1}^s Z_{h_i}(R)$  is not empty, then  $SL_n(R) \cap (\bigcap_{i=1}^s Z_{h_i}(R))$  is also not empty. When  $R$  is a valuation domain, the above corollary shows that  $s \geq 1$ . This question seems open even in the case where  $R$  is an algebraically closed field.

Clearly, it is always possible to find  $n$  homogeneous linear polynomials  $h_i$  such that  $SL_n(R) \cap (\bigcap_{i=1}^n Z_{h_i}(R)) = \emptyset$ , so that  $s(n) < n$ . Indeed, simply take  $h_i := X_{1,i}$  for  $i = 1, \dots, n$ . Let  $K$  denote the field of fractions of  $R$ . In this example,  $SL_n(K) \cap (\bigcap Z_{h_i}(K))$  is also empty. A different example with  $n = 2$  and  $R = \mathbb{Z}$  is as follows. Consider  $d_2 = xv - uy$ , and  $h_1 := v - 9x$ ,  $h_2 := u + 4y$ . Then  $Z_{d_2-1}(\mathbb{Z}) \cap (\bigcap Z_{h_i}(\mathbb{Z})) = \emptyset$ , but  $Z_{d_2-1}(\mathbb{Q}) \cap (\bigcap Z_{h_i}(\mathbb{Q}))$  is not empty.

Theorems in the literature pertaining to the completion of a partial integral matrix to a unimodular matrix can be interpreted in light of the above question. For instance, pick  $n - 1$  distinct variables  $x_{ij}$ , and denote by  $I$  the set of chosen indices  $(i, j)$ . Pick  $c_{ij} \in R$ , for each  $(i, j) \in I$ . Then the set  $GL_n(R) \cap (\bigcap_{(ij) \in I} Z_{x_{ij}-c_{ij}}(R))$  is not empty ([12], cor. 3).

Now choose any number of hyperplanes in the  $n^2 - n$  variables  $x_{ij}$ ,  $i \neq j$ , such that in  $R^{n^2-n}$ , the intersection  $\mathcal{H}$  of these hyperplanes is not empty. Let  $(u_{ij}, i \neq j)$  denote a vector in  $R^{n^2-n}$  that belongs to  $\mathcal{H}$ . Then the array  $U$  whose entries are  $u_{ij}$  for  $i \neq j$  can be completed into a matrix in  $GL_n(R)$  ([3], Thm. 1). Hence,  $GL_n(R) \cap \mathcal{H}' \neq \emptyset$ , where  $\mathcal{H}'$  denotes the common zeros in  $R^{n^2}$  of the equations of the chosen hyperplanes.

**Remark 13** We note here the following easy fact. *Suppose that  $R$  is a Bézout domain with field of fractions  $K$ . Let  $J$  be any domain with  $R \subseteq J \subseteq K$ . If  $R$  satisfies  $(SU)_n$  (resp.  $(SU')_n$ ), then so does  $J$ .*

Indeed, it is well-known that  $J$  is also a Bézout domain. It is noted in [1], page 243, that every element of  $J$  can be written as  $\alpha/\beta$  with  $\alpha, \beta \in R$  and  $\beta$  a unit in  $J$ . It follows that any matrix  $A \in M_n(J)$  can be written as  $A = \text{diag}(\beta, \dots, \beta)A'$  with  $\beta$  invertible in  $J$ , and  $A' \in M_n(R)$ . Using property  $(SU)_n$  for  $A'$ , we find  $U \in GL_n(R)$  such that  $A'U$  is symmetric (if  $(SU')_n$  holds for  $R$ , we choose  $U$  with  $\det(U) = 1$ ). Then  $AU = \text{diag}(\beta, \dots, \beta)A'U$  is symmetric.

**Remark 14** Suppose that a matrix  $A \in M_n(R)$  has a factorization  $A = SU$  with  $S$  symmetric and  $U$  invertible. In general, such a factorization is not unique. For instance, if  $A \in GL_n(R)$ , and  $A = SU$ , then for all  $n$ ,  $A = S^n(S^{1-n}U)$ . Is the number of distinct factorizations of  $A \in SL_n(\mathbb{Z})$  into a product  $SU$  always infinite?

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