ON A GENERALIZATION OF THE FROBENIUS NUMBER

ALEXANDER BROWN, ELEANOR DANNENBERG, JENNIFER FOX, JOSHUA HANNA,
KATHERINE KECK, ALEXANDER MOORE, ZACHARY ROBBINS, BRANDON SAMPLES,
AND JAMES STANKEWICZ

Abstract. We consider a generalization of the Frobenius Problem where the object of interest is the greatest integer which has exactly j representations by a collection of relatively prime integers. We prove an analogue of a theorem of Brauer and Shockley and show how it can be used for computation.

The *linear diophantine problem of Frobenius* has long been a celebrated problem in number theory. Most simply put, the problem is to find the *Frobenius Number* of k positive relatively prime integers $(a_1, \ldots, a_k)$, i.e. the greatest integer M for which there is no way to express M as the non-negative integral linear combination of the given integers.

A generalization, which has drawn interest both from classical study of the Frobenius Problem ([Al05, Problem A.2.6]) and from the perspective of partition functions and integer points on polytopes (as in [BR04]), is to ask for the greatest integer M which can be expressed in exactly j different ways. We make this precise with the following definitions:

A *representation* of M by a k-tuple $(a_1, \ldots, a_k)$ of positive, relatively prime integers is a solution $(x_1, \ldots, x_k) \in \mathbb{Z}^k_{\geq 0}$ to the equation $M = \sum_{i=1}^k a_i x_i$.

We define the *j-Frobenius Number* of a k-tuple $(a_1, \ldots, a_k)$ of relatively prime positive integers to be the greatest integer M with exactly j representations of M by $(a_1, \ldots, a_k)$ if such a positive integer exists and zero otherwise. We refer to this quantity as $g_j(a_1, \ldots, a_k)$.

Finally we define $f_j(a_1, \ldots, a_k)$ exactly as we defined $g_j(a_1, \ldots, a_k)$, except that we consider only *positive representations* $(x_1, \ldots, x_k) \in \mathbb{Z}^k_{>0}$.

Note that the 0-Frobenius of $(a_1, \ldots, a_k)$ is the classical Frobenius Number. The purpose of this paper is to show the following generalization of a result of Brauer and Shockley [BS62] on the classical Frobenius Number.

Theorem 1. *If $d = \gcd(a_2, \ldots, a_k)$ and $j \geq 0$, either*

$$g_j(a_1, a_2, \ldots, a_k) = d \cdot g_j(a_1, \frac{a_2}{d}, \ldots, \frac{a_k}{d}) + (d-1)a_1$$

or $g_j(a_1, a_2, \ldots, a_k) = g_j(a_1, \frac{a_2}{d}, \ldots, \frac{a_k}{d}) = 0$.

The research done in this note was completed as part of the undergraduate number theory VIGRE research seminar directed by Professor Dino Lorenzini and assisted by graduate students Brandon Samples and James Stankewicz at the University of Georgia in Fall 2008. Support for the seminar was provided by the Mathematics Department's NSF VIGRE grant.

Date: April 23, 2009.

Lemma 1. *If* $f_j(a_1, \ldots, a_k)$ *is nonzero, there exist integers* $x_2, \ldots, x_k > 0$ *such that*

$$f_j(a_1, \ldots, a_k) = \sum_{i=2}^{k} a_i x_i.$$

Proof. Let $f_j := f_j(a_1, \ldots, a_k)$. By the definition of f_j, we can write $f_j = \sum_{i=1}^{k} a_i x_{i,\ell}$ with $x_{i,\ell} > 0$ for $1 \leq \ell \leq j$. Since

$$f_j + a_1 = \sum_{i=1}^{k} a_i x_{i,\ell} + a_1 = a_1(x_{1,\ell} + 1) + \sum_{i=2}^{k} a_i x_{i,\ell},$$

we obtain at least j positive representations of $f_j + a_1$. As f_j is the largest number with exactly j positive representations, there must be at least $j + 1$ distinct ways to represent $f_j + a_1$. Specifically, we have $f_j + a_1 = \sum_{i=1}^{k} a_i x'_{i,\ell}$ with $x'_{i,\ell} > 0$ for all $1 \leq \ell \leq j + 1$. Subtract a_1 from both sides of these $j + 1$ equations to obtain $f_j = (x'_{1,\ell} - 1)a_1 + \sum_{i=2}^{k} a_i x'_{i,\ell}$. Evidently, there exists some $\ell_0 \in [1, j + 1]$ for which $x'_{1,\ell_0} - 1 = 0$ because f_j cannot have $j + 1$ positive representations. Therefore, $f_j(a_1, \ldots, a_k) = \sum_{i=2}^{k} a_i x'_{i,\ell_0}$. $\square$

Theorem 2. *If* $\gcd(a_2, \ldots, a_k) = d$*, then*

$$f_j(a_1, a_2, \ldots, a_k) = d \cdot f_j(a_1, \frac{a_2}{d}, \ldots, \frac{a_k}{d}),$$

Proof. Let $a_i = da'_i$ for $i = 2, \ldots, k$ and $N = f_j(a_1, \ldots, a_k)$.
Assuming $N > 0$, we know by Lemma 1 that

$$N = \sum_{i=2}^{k} a_i x_i = d \sum_{i=2}^{k} a'_i x_i$$

with $x_i > 0$. Let $N' = \sum_{i=2}^{k} a'_i x_i$. We want to show that $N' = f_j(a_1, a'_2, \ldots, a'_k)$ and will do this in three steps.

Step 1: First, we know that N' does not have $j + 1$ or more positive representations by $a_1, a'_2, \ldots, a'_k$. If N' could be so represented, then for $1 \leq l \leq j + 1$ we would have

$$N' = a_1 y_{1,\ell} + \sum_{i=2}^{k} a'_i y_{i,\ell}.$$

Therefore,

$$dN' = d(a_1 y_{1,\ell}) + \sum_{i=2}^{k} da'_i y_{i,\ell},$$

which occurs if and only if

$$N = a_1(dy_{1,\ell}) + \sum_{i=2}^{k} a_i y_{i,\ell},$$

which is a contradiction since there cannot be $j+1$ distinct positive representations of N.

Step 2: Next, we know that

$$f_j(a_1, \dots, a_k) = N = a_1 x_{1,\ell} + \sum_{i=2}^{k} a_i x_{i,\ell}$$

for $1 \leq l \leq j$ and $x_i > 0$, so

$$\frac{N}{d} = \frac{a_1 x_{1,\ell}}{d} + \sum_{i=2}^{k} \frac{a_i x_{i,\ell}}{d}.$$

Since $d|N$ and $d|a_i$ for $i \geq 2$, we must have $d|a_1 x_{1,\ell}$ for $1 \leq \ell \leq j$. In addition, $\gcd(a_1, d) = 1$ so we must have $d|x_{1,\ell}$ for $1 \leq \ell \leq j$. So

$$N' = a_1 \frac{x_{1,\ell}}{d} + \sum_{i=2}^{k} a'_i x_{i,\ell},$$

hence N' has at least j distinct positive representations. But we have already shown that N' cannot have $j+1$ or more positive representations, thus N' has exactly j positive representations.

Step 3: Finally we will show that N' is the largest number with exactly j positive representations. Consider any $n > N'$. Since $dn > dN' = N$, we know that dn can be represented as a linear combination of $a_1, \dots, a_k$ in exactly X ways with $X \neq j$.
Thus, for $1 \leq l \leq X$ and $X \neq j$ we have

$$dn = a_1 x_{1,\ell} + \sum_{i=2}^{k} a_i x_{i,\ell}$$

and as in Step 2,

$$n = a_1 (\frac{x_{1,\ell}}{d}) + \sum_{i=2}^{k} a'_i x_{i,\ell}.$$

If $X > j$ then we certainly do not have exactly j representations, so assume $X < j$. Assume now that we can write $n = a_1 y_1 + \sum_{i=2}^{k} a'_i y_i$ where $y_i \neq x_{i,\ell}$ for any such ℓ. By multiplying by d we get a new representation for dn, which is a contradiction because dn is represented in exactly $X \neq j$ ways.

Therefore N' is the greatest number with exactly j positive representations and so

$$N' = f_j(a_1, a'_2, \dots, a'_k).$$

Thus

$$f_j(a_1, a_2, \dots, a_k) = d \cdot f_j(a_1, \frac{a_2}{d}, \dots, \frac{a_k}{d}).$$

□

Having established our results about $f_j(a_1, \dots, a_k)$, we show that we can translate these results to results about the j-Frobenius Numbers.

Lemma 2. *Either* $f_j(a_1, \dots, a_k) = g_j(a_1, \dots, a_k) = 0$ *or,*

$$f_j(a_1, \dots, a_k) = g_j(a_1, \dots, a_k) + \sum_{i=1}^{k} a_i$$

Proof. For ease, write f_j for $f_j(a_1, \dots, a_k)$, g_j for $g_j(a_1, \dots, a_k)$, and $K = \sum_{i=1}^{k} a_i$.

Any representation $(y_1, \dots, y_k)$ of M gives a representation $(y_1 + 1, \dots, y_k + 1)$ of $M + K$. Moreover, adding or subtracting K preserves the distinctness of representations because it adjusts every coefficient y_i by 1. Therefore if M has j representations, $M + K$ has at least j positive representations. Likewise, every positive representation of $M + K$ gives a representation of M. Thus $f_j = 0$ if and only if $g_j = 0$. Assume now that f_j and g_j are both nonzero.

Suppose that $f_j < g_j + K$. By definition, we can find exactly j representations $(y_1, \dots, y_k)$ for g_j and g_j has exactly j representations if and only if $g_j + K$ has exactly j *positive* representations $(x_1, \dots, x_k)$. However, by assumption $g_j + K > f_j$ and $g_j + K$ has exactly j positive representations. This contradicts the definition of f_j, hence $f_j \geq g_j + K$.

Suppose that $f_j > g_j + K$. By definition, we can find exactly j positive representations $(x_1, \dots, x_k)$ for f_j. The same argument as above shows that $f_j - K$ has exactly j representations in contradiction to the definition of g_j. Thus $f_j \leq g_j + K$. ☐

Proof of Theorem 1: Combine Theorem 2 with Proposition 2.

Corollary 1. *Let* a_1, a_2 *be coprime positive integers and let* m *be a positive integer. Suppose* $g_j = g_j(a_1, a_2, ma_1a_2) \neq 0$. *Then*

- $g_j = (j + 1)a_1a_2 - a_1 - a_2$ *for* $j < m + 1$ *and*
- $g_{m+2} = (m + 2)a_1a_2 - a_1 - a_2$.

Proof. Theorem 1 tells us that

$$\begin{aligned} g_j(a_1, a_2, ma_1a_2) &= a_2(g_j(a_1, 1, ma_1)) + (a_2 - 1)a_1 \\ &= a_2\left[a_1 g_j(1, 1, m) + (a_1 - 1)1\right] + (a_2 - 1)a_1 \\ &= a_1a_2(g_j(1, 1, m) + 2) - a_1 - a_2. \end{aligned}$$

We can now determine $g_j(1, 1, m)$ with the Taylor series $\frac{1}{(1-t)^2(1-t^m)} = \sum_{k=0}^{\infty} p_{1,1,m}(k)t^k$. Now recall that for $k < m$, $p_{1,1,m}(k) = p_{1,1}(k) = k + 1$ but $p_{1,1,m}(m) = m + 2$ and for all $k > m$, $p_{1,1,m}(k) > m + 2$. Thus $g_j(1, 1, m) = j - 1$ for $j < m$ and $g_{m+2}(1, 1, m) = m$. ☐

Remark. It is a consequence of the asymptotic in [Na00] that for a given tuple, there may be many j for which $g_j = 0$, so the ordering $g_0 < g_1 < \dots$ may not hold. In the process of discovering these equalities, we noted the somewhat stranger occurrence of tuples where $0 < g_{j+1} < g_j$.

Take for instance, the 3-tuple (3, 5, 8). The order $g_0 < g_1 < \dots$ holds until $g_{14} = 52$ and $g_{15} = 51$. As should also be the case, the 3-tuple increased by a factor of $d = 2$ creates the new "dependent" 3-tuple (3, 10, 16), which fails to hold order in the same position with $g_{14} = 107$ and $g_{15} = 105$. A few independent examples are:

$g_{17}(2,5,7) = 43$ and $g_{18}(2,5,7) = 42$,
$g_{38}(2,5,17) = 103$ and $g_{39}(2,5,17) = 102$,
$g_{35}(4,7,19) = 181$ and $g_{36}(4,7,19) = 180$, and
$g_{38}(9,11,20) = 376$ and $g_{39}(9,11,20) = 369$.

We do not as of yet know a lower bound on j for the above to occur. Indeed, in every case we have computed, if $g_0, g_1 > 0$ then $g_1 > g_0$, but to date neither a proof or a counterexample has presented itself.

References

[Al05] J. L. R. Alfonsín, *The Diophantine Frobenius Problem,* 2005 Oxford University Press.

[BR04] M. Beck; S. Robins, *A formula related to the Frobenius Problem in two dimensions,* Number Theory. New York Seminar 2003 (D. Chudnovsky, G. Chudnovsky, M. Nathanson, eds.), 2004, Springer, 17-23.

[BS62] A. Brauer; J.E. Shockley, *On a problem of Frobenius,* J. Reine Angew. Math. **211** (1962) 215-220.

[Na00] M. Nathanson, *Partitions with parts in a finite set,* Proc. AMS, 128 No. 5 (2000) 1269-1273.

E-mail address: `stankewicz@gmail.com`

Department of Mathematics, University of Georgia, Athens, GA 30602