HYPERSURFACES OF PROJECTIVE SCHEMES AND A MOVING LEMMA

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Abstract. Let X/S be a quasi-projective morphism over an affine base. We develop in this article a technique for proving the existence of closed subschemes H/S of X/S with various favorable properties. We offer several applications of this technique, including the existence of finite quasi-sections in certain projective morphisms, and the existence of hypersurfaces in X/S containing a given closed subscheme C, and intersecting properly a closed set F.

Assume now that the base S is the spectrum of a Dedekind domain R such that for any finite surjective morphism $Z \to S$, $\mathrm{Pic}(Z)$ is a torsion group. This condition is satisfied if R is the ring of integers of a number field, or the ring of functions of a smooth affine curve over a finite field. We prove in this context a moving lemma pertaining to horizontal 1-cycles on a regular scheme X quasi-projective and flat over S. We also show the existence of a finite S-morphism to $\mathbb{P}^d_S$ for any scheme X projective over S when X/S has all its fibers of a fixed dimension d.

keywords. Hypersurface, Quasi-section, Multisection, Moving lemma, 1-cycles, Rationally equivalent.

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1. Introduction

Let $S = \operatorname{Spec} R$ be an affine scheme, and let X/S be a quasi-projective scheme. The core of this article is a method, summarized in 1.4 below, for proving the existence of closed subschemes Y of X with various favorable properties. As the technical details can be somewhat complicated, we start this introduction by discussing the applications of the method that the reader will find in this article.

Date: July 8, 2011.

Recall that a global section f of an invertible sheaf $\mathcal{L}$ on X defines a closed subset H_f of X, consisting of all points $x \in X$ where the stalk f_x does not generate $\mathcal{L}_x$. Since $f\mathcal{O}_X \subset \mathcal{L}$, the ideal sheaf $\mathcal{I} := f\mathcal{O}_X \otimes \mathcal{L}^{-1}$ endows H_f with the structure of closed subscheme of X.

It is classical that if X/k is a projective scheme over a field, $C \subset X$ a proper closed subset, and $\xi_1, \dots, \xi_r$ are points of X not contained in C, then there exists a hypersurface H in X such that $C \subset H$ and $\xi_1, \dots, \xi_r \notin H$. Such a statement is commonly referred to as an Avoidance Lemma (see, e.g., 4.3). Our next theorem establishes an Avoidance Lemma for Families. As usual, $\mathrm{Ass}(X)$ denotes the set of associated points of a scheme X, and this set is finite when X is noetherian.

Theorem 5.3 *Let S be an affine noetherian scheme and let $X \to S$ be a quasi-projective morphism. Let C be a closed subscheme of X, proper over S, and such that $C \cap \mathrm{Ass}(X) = \emptyset$. Let F be a closed subset of X. Assume that for all $s \in S$, C_s does not contain any irreducible component of positive dimension of F_s and of X_s.*

Fix an ample invertible sheaf $\mathcal{O}_X(1)$ on X. Then there exist $n > 0$ and a global section f of $\mathcal{O}_X(n)$ such that the ideal sheaf $\mathcal{I}$ of the associated closed subscheme H_f is invertible, and such that

(1) *C is a closed subscheme of H_f,*
(2) *H_f does not contain any irreducible component of positive dimension of F_s and of X_s, for all $s \in S$.*

We use Theorem 5.3 to establish in 6.2 the existence of finite quasi-sections in certain projective morphisms X/S, as we now discuss. Let $X \to S$ be a surjective morphism. Following EGA [11], IV.14, p. 200, we call a closed subscheme C of X a *finite quasi-section* when $C \to S$ is finite and surjective. Some authors call *multisection* a finite quasi-section $C \to S$ which is also flat, with C irreducible (see e.g., [15], p. 12 and 4.7).

When $\dim S = 1$ and $X \to S$ is proper, the existence of a finite quasi-section C is well-known and easy to establish. It suffices to take C to be the Zariski closure of a closed point of the generic fiber of $X \to S$. When $\dim S > 1$, the process of taking the closure of any closed point of the generic fiber does not always produce a closed subset *finite* over S (see 6.1).

Theorem 6.2 *Let S be an affine scheme and let $X \to S$ be a projective and finitely presented morphism. Suppose that all fibers of $X \to S$ are of the same dimension $d \geq 0$. Let C be a finitely presented closed subscheme of X, with $C \to S$ finite but not necessarily surjective. Then:*

(1) *There exists a finite quasi-section T which contains C.*
(2) *Assume that S is noetherian. If $X \to S$ is flat with Cohen-Macaulay fibers, then there exists a finite quasi-section which contains C and is flat over S.*

This theorem is used in the proof of Theorem 8.1 discussed below. As an application to Theorem 6.2, we obtain in 6.5 a strengthening in the affine case of the classical splitting lemma for vector bundles. Another application to the problem of extending a given family of stable curves $\mathcal{D} \to Z$ after a finite surjective base change is found in 6.7.

When $X \to S$ is quasi-projective and S is the spectrum of a Dedekind domain R, an irreducible finite quasi-section is also called an *integral point* in [26], 1.4. The existence of a finite quasi-section in the quasi-projective case over $S = \operatorname{Spec} \mathbb{Z}$ when the generic fiber is geometrically irreducible is Rumely's famous Local-Global Principle [32]. This existence result was extended in [26], 1.6, and holds when S is excellent and satisfies Condition (T) recalled below. That Condition (T) is necessary in the Local-Global Principle is showed by an example of Raynaud ([25], 3.2).

1.1 As in [26], Définition 1.5, we say that S satisfies *Condition* (T) if:

(a) For any finite extension L of the field of fractions K of R, the normalization S' of S in $\operatorname{Spec} L$ has torsion Picard group $\operatorname{Pic}(S')$, and
(b) The residue fields at all closed points of S are algebraic extensions of finite fields.

For example, S satisfies Condition (T) if S is an affine integral curve over a finite field, or if S is the spectrum of the ring of P-integers in a number field K, where P is a finite set of finite places of K.

The following weaker condition is needed for our next two theorems:

1.2 We say that S satisfies *Condition* (T*) if $\operatorname{Pic}(Z)$ is a torsion group for any finite surjective morphism $Z \to S$.

It is clear that a semi-local Dedekind domain R satisfies Condition (T*), but may not satisfy Condition (T). It is shown in [26], 2.3, that if an excellent[1] Dedekind domain satisfies Condition (T), then it satisfies Condition (T*).

Assume that $\dim(S) = 1$. An integral subscheme C of X/S such that $C \to S$ is a finite quasi-section is called an irreducible *horizontal* 1*-cycle* on X. A *horizontal* 1*-cycle* on X is an element of the free abelian group generated by the irreducible horizontal 1-cycles. Our next application of the method developed in this article is a Moving lemma for horizontal 1-cycles.

Theorem 7.2 *Let R be a Dedekind domain satisfying Condition* (T*)*, and let $S := \operatorname{Spec} R$. Let $X \to S$ be a flat and quasi-projective morphism, with X integral and regular. Let C be a horizontal* 1*-cycle on X. Let F be a closed subset of X such that for all $s \in S$, $F \cap X_s$ has codimension at least* 1 *in X_s.*

Then some positive multiple mC of C is rationally equivalent to a horizontal 1*-cycle C' on X whose support does not meet F. When R is semi-local, we can take $m = 1$.*

Example 7.4 shows that the Condition (T)(a) is necessary for Theorem 7.2 to hold. A different proof of Theorem 7.2 when S is semi-local is given in [9], 3.3, where it is then used to prove a formula for the index of an algebraic variety over a Henselian field ([9], 8.4).

Even for schemes of finite type over a finite field, Theorem 7.2 is not a consequence of the classical Chow's Moving Lemma. Indeed, let X be a quasi-projective variety over a field k. Let X^{sing} denote the singular locus of X. The classical Chow's Moving Lemma [31] and its generalization ([5], II.9, assuming k algebraically closed) immediately imply the following statement:

[1]The hypothesis that R be excellent is not explicitly stated in [26], 2.3. However, it is likely needed in its proof, particularly in 2.5, where the conductor of a domain into its integral closure needs to be non-zero.

1.3 *Let Z be a 1-cycle on X with* $\mathrm{Supp}(Z) \cap X^{\mathrm{sing}} = \emptyset$*. Assume that* $\dim(X^{\mathrm{sing}}) < \mathrm{codim}(Z, X)$*. Let F be a closed subset of X of codimension at least 2 in X. Then there exists a 1-cycle Z' on X, rationally equivalent to Z, and such that* $\mathrm{Supp}(Z') \cap (F \cup X^{\mathrm{sing}}) = \emptyset$*.*

Consider a morphism $X \to S$ as in Theorem 7.2, and assume in addition that S is a smooth affine curve over a finite field k. Let F be a closed subset as in 7.2. Such a subset may be of codimension 1 in X. Thus, Theorem 7.2 is not a consequence of Chow's Moving Lemma for 1-cycles just recalled, since 1.3 can only be applied to $X \to S$ when F is a closed subset of codimension at least 2 in X.

The following theorem generalizes to schemes X/S of any dimension Theorem 2 in [10] and Theorem 1.2 in [4].

Theorem 8.1 *Let R be a Dedekind domain satisfying Condition* (T*)*, and let* $S := \mathrm{Spec}\, R$*. Let* $d \geq 0$*, and let $X \to S$ be a projective morphism such that* $\dim X_s = d$ *for all $s \in S$. Then there exists a finite S-morphism $X \to \mathbb{P}^d_S$.*

Now that the main applications of our method for proving the existence of hypersurfaces H_f in projective schemes X/S with certain desired properties have been discussed, let us summarize the method.

1.4 Let $X \to S$ be a projective morphism with $S = \mathrm{Spec}\, R$ affine. Let $\mathcal{O}(1)$ be a very ample sheaf on X relative to S. Let $C \subset X$ be a closed subscheme defined by an ideal $\mathcal{I}$, and set $\mathcal{I}(n) := \mathcal{I} \otimes \mathcal{O}(n)$. Our goal is to show the existence, for some n large enough, of a global section f of $\mathcal{I}(n)$ such that the associated subscheme H_f has the desired properties.

To do so, we fix a system of generators $f_1, \dots, f_N$ of $H^0(X, \mathcal{I}(n))$, and we consider for each $s \in S$ a subset $\Sigma(s) \subset \mathbb{A}^N(k(s))$ consisting of all the vectors $(\alpha_1, \dots, \alpha_N)$ such that $\sum_i \alpha_i f_{i|X_s}$ does not have the desired properties. We show then that all these subsets $\Sigma(s)$ are contained in a single constructible subset T of $\mathbb{A}^N/S$ (which depends on n). To find a desired global section $f := \sum_i a_i f_i$ with $a_i \in R$ which avoids the subset T of 'bad' sections, we show that for some n large enough the constructible subset T satisfies the hypotheses of the following proposition. The section σ whose existence follows from 2.3 provides the desired vector $(a_1, \dots, a_N) \in R^N$.

Proposition 2.3. *Let* $S = \mathrm{Spec}\, R$ *be a noetherian affine scheme. Let T be a constructible subset of $\mathbb{A}^N_S$. Suppose that:*

(1) $\dim(T) < N$.
(2) *For all $s \in S$, there exists a $k(s)$-rational point in $\mathbb{A}^N_{k(s)}$ which does not belong to T_s.*

Then there exists a section σ of $\pi : \mathbb{A}^N_S \to S$ such that $\sigma(S) \cap T = \emptyset$.

The proof of (a slightly stronger version of) Proposition 2.3 is given in section 2, along with the construction exhibiting a non-trivial constructible subset T containing all $\Sigma(s)$ (2.2).

We present our next theorem as a final illustration of the strength of the method. This theorem, stated in a slightly stronger form in section 3, is the key to the proof of Theorem 7.2, as it allows for a reduction to the case of relative dimension 1. Note that Condition (T*) is not needed as a hypothesis in this theorem.

Theorem 3.1 *Let S be an affine noetherian scheme of dimension 1, and let $X \to S$ be a quasi-projective morphism. Let C be a closed irreducible subscheme of X, of codimension $d > 0$ in X. Assume that $C \to S$ is finite and surjective, and that $C \to X$ is a regular immersion. Let F be a closed subset of X. Fix an ample invertible sheaf $\mathcal{O}_X(1)$ on X. Then there exist $n > 0$ and a section f of $\mathcal{O}_X(n)$ such that:*

(i) *C is a closed subscheme of codimension $d-1$ in H_f, and $C \to H_f$ is a regular immersion;*
(ii) *For all $s \in S$, any irreducible component Γ of $F \cap X_s$ is such that $\dim(\Gamma \cap (H_f)_s) \leq \max(\dim(\Gamma) - 1, 0)$.*

The proof of Theorem 3.1 is quite subtle and spans sections 3 and 4. Even though the scheme X is assumed to be regular in Theorem 7.2, it is not possible in general to expect that a hypersurface H_f containing C can be chosen to also be regular. Thus, when the total space is not regular, we impose regularity conditions by assuming that C is regularly immersed in X. Great care is then needed in the proof of 3.1 to insure that a hypersurface H_f can be found with the property that C is regularly immersed in H_f. Our proof uses the hypothesis that $\dim(S) = 1$ (3.9).

Section 3 contains most of the proof of Theorem 3.1. We have discussed separately in section 4 all results using properties of m-regular sheaves needed in the proof of 3.1, such as results pertaining to the existence of a constructible set T satisfying Condition (2) in 2.3. Sections 5, 6, 7, and 8, contain the proofs of the applications of our method. It is our pleasure to thank Robert Varley for Example 8.5.

2. Constructible subsets

2.1 Let $S = \operatorname{Spec} R$ be an affine noetherian scheme. Let $\mathbb{A}^N_S \to S$ be an affine space over S. Let $T \subseteq \mathbb{A}^N_S$ be a subset. Endow T with the topology induced by that of $\mathbb{A}^N_S$, and define the *dimension of* T to be the Krull dimension of the topological space T. As usual, $\dim(T) < 0$ if and only if $T = \emptyset$. For all $s \in S$, we denote by T_s the subset $T \cap \mathbb{A}^N_{k(s)}$ of $\mathbb{A}^N_{k(s)} \subset \mathbb{A}^N_S$.

Let $Z \to S$ be a proper morphism and let $\mathcal{F}$ be a coherent sheaf on Z with support equal to Z. Fix $f_1, \dots, f_N \in H^0(Z, \mathcal{F})$. For $s \in S$, denote by $f_{i|Z_s}$, or simply by $f_{i,s}$ the pullback in $H^0(Z_s, \mathcal{F}_s)$ of the section f_i. Recall that a section of $H^0(Z_s, \mathcal{F}_s)$ *vanishes at* $\xi \in Z_s$ if its image in $\mathcal{F}_s(\xi) := \mathcal{F}_s \otimes k(\xi)$ is zero. Consider the set

$$\Sigma(s) := \left\{ (\alpha_1, \dots, \alpha_N) \in k(s)^N \;\middle|\; \sum_i \alpha_i f_{i,s} \text{ vanishes at some generic point of } Z_s \right\}.$$

When $Z_s = \emptyset$, we set $\Sigma(s) := \emptyset$.

Proposition 2.2. *Let $S = \operatorname{Spec} R$ be an affine noetherian scheme, let $\pi : Z \to S$ be a proper morphism, and let $\mathcal{F}$ be a coherent sheaf on Z with support equal to Z. Then there exists a constructible subset T of $\mathbb{A}^N_S$ such that for all $s \in S$, $\Sigma(s)$ is equal to the set of $k(s)$-rational points of $\mathbb{A}^N_S$ contained in T_s.*

The constructible subset T whose existence is proved in 2.2 is not unique.[2] The proof of 2.2 exhibits one construction of such a set T, and within this proof, we will make notes on how to bound the dimension of the sets T_s, for $s \in S$. We will then apply the construction of 2.2 to provide explicit bounds on the dimension of T_s in the special cases discussed in 3.6 and 3.7. These explicit bounds will allow us to apply Proposition 2.3, where the hypothesis that $\dim T \leq N-1$ is essential.

Proof of Proposition 2.2. We start with the following observations:

(i) Let $Z' \subseteq Z$ be a closed subscheme defined by a nilpotent ideal sheaf. Consider the sheaf $\mathcal{F}|_{Z'}$, with global sections $f'_1, \dots, f'_N$, images of $f_1, \dots, f_N$ in $H^0(Z', \mathcal{F}|_{Z'})$. Suppose that T' satisfies the statement of the proposition for the morphism $Z' \to S$, sheaf $\mathcal{F}|_{Z'}$, and sets $\Sigma'(s)$. Then T' also satisfies the statement of the proposition for $Z \to S$ and $\mathcal{F}$, since $\Sigma(s) = \Sigma'(s)$ for all $s \in S$. In particular, it suffices to prove the proposition when both Z and S are reduced.

(ii) Suppose that S_1 and S_2 are two affine, open or closed, subschemes of S, and that as a set, S is the disjoint union of S_1 and S_2. Let $Z_i := Z \times_S S_i$. Suppose that there exist constructible subsets $T_i \subset \mathbb{A}^N_{S_i}$, $i = 1, 2$, satisfying the statement of the proposition for $Z_i \to S$, $\mathcal{F}|_{Z_i}$, and with respect to the sections $f_1|_{Z_i}, \dots, f_N|_{Z_i}$. Then $T := T_1 \cup T_2$ satisfies the statement of the proposition for $Z \to S$.

(iii) We claim that given any $Z \to S$ as in the proposition, the proposition is proved if we can prove the statement of the proposition for $Z_U \to U$, $\mathcal{F}|_U$, and the sections $f_1|_U, \dots, f_N|_U$, where $U \subseteq S$ is some dense affine open subset. Indeed, let $Z_1 := S \setminus U$ and consider a dense open subset U_1 of Z_1 where the statement of the proposition holds. Construct in this way a sequence of closed subsets $Z \supset Z_1 \supset \dots$. The noetherian hypothesis implies that this sequence is finite, and the proposition follows using (ii) repeatedly.

(iv) Any S contains a dense open subset consisting in a finite disjoint union of irreducible affine open subsets. Using (i), (ii), and (iii), we conclude that it suffices to prove the proposition when S is integral. Moreover, given S integral and $Z \to S$ as in the proposition, it suffices as in (iii) to prove the proposition for $Z_U \to U$ for some non-empty affine open subset U.

(v) It suffices to prove the proposition after a finite surjective base change $g : S' \to S$, in the following sense. Let $Z' := Z \times_S S'$, with $h : Z' \to Z$. Let $\mathcal{F}' := h^*\mathcal{F}$. The support of $\mathcal{F}'$ is equal to Z'. Let $f'_1, \dots, f'_N$ denote the images of $f_1, \dots, f_N$ under the natural map $H^0(Z, \mathcal{F}) \to H^0(Z', \mathcal{F}')$. For any $s' \in S'$, we define $\Sigma'(s')$ as in 2.1 for the above data. Suppose that $T' \subseteq \mathbb{A}^N_{S'}$ is a constructible subset which satisfies the conclusion of the proposition for the morphism $Z' \to S'$ and associated data. Let T be the image of T' in $\mathbb{A}^N_S$ under the natural morphism $\mathbb{A}^N_{S'} \to \mathbb{A}^N_S$. By Chevalley's Theorem ([11], IV.1.8.4), T is constructible. We claim that T satisfies the conclusion of the proposition for $Z \to S$. It is clear that $\dim T_s = \dim T'_{s'}$.

[2]However, one can show that the set T we will construct satisfies $T_s(k(s')) = \Sigma'(s')$ for all affine morphisms $S' \to S$ and all $s' \in S'$ lying over s, where $\Sigma'(s')$ is defined in a natural way as in Observation (v). This fact is not used in the sequel.

Let us show first that when $s \in S$, then $\Sigma(s)$ is equal to the set of $k(s)$-rational points of $(\mathbb{A}^N_S)_s$ contained in T_s. Let $(\alpha_1, \ldots, \alpha_N) \in \Sigma(s)$. Let $f_s := \sum_i \alpha_i f_{i,s} \in H^0(Z_s, \mathcal{F}_s)$. Then f_s vanishes at a generic point ξ of Z_s by hypothesis. Let $s' \in S'$ be a preimage of s, and let ξ' be a point of $Z'_{s'} = (Z_s)_{k(s')}$ lying over ξ. Then ξ' is a generic point of $Z'_{s'}$ because $k(s')/k(s)$ is finite, and f_s, viewed as an element of $H^0(Z'_{s'}, \mathcal{F}'_{s'})$, vanishes at ξ'. Therefore, $(\alpha_1, \ldots, \alpha_N) \in k(s)^N \subset k(s')^N$ belongs to $\Sigma'(s')$ and thus corresponds to a $k(s')$-rational point of $\mathbb{A}^N_{S'}$ contained in $T'_{s'}$. Its image under $\mathbb{A}^N_{S'} \to \mathbb{A}^N_S$ is therefore a $k(s)$-rational point of T_s.

Conversely, let $(\alpha_1, \ldots, \alpha_N)$ be $k(s)$-rational point of T_s. As $T' \to T$ is surjective, there exists $s' \in S'$ lying over s such that $(\alpha_1, \ldots, \alpha_N)$ is a $k(s')$-rational point of $\mathbb{A}^N_{S'}$ contained in $T'_{s'}$. So f_s, viewed as an element of $H^0(Z'_{s'}, \mathcal{F}'_{s'})$, vanishes at some generic point ξ' of $Z'_{s'}$. The image ξ of ξ' in Z is a generic point of Z_s. We have $\mathcal{F}'_{s'}(\xi') = \mathcal{F}_s(\xi) \otimes_{k(\xi)} k(\xi')$, so f_s vanishes at ξ, and $(\alpha_1, \ldots, \alpha_N) \in \Sigma(s)$.

Using (i) and (iv), we can suppose that S is integral, with $\pi(Z) = S$. Let η be the generic point of S. Then there exists a finite extension $L/k(\eta)$ such that each irreducible component of $(Z_L)_{\mathrm{red}}$ is geometrically integral (see [11], IV.4.5.11 and IV.4.6.6). Restricting S to an open subscheme if necessary, we can find a finite surjective morphism $S' \to S$ with S' integral that extends $\operatorname{Spec} L \to \operatorname{Spec} k(\eta)$. Using (v), we are reduced to proving the proposition for $Z \times_S S' \to S'$. Using (i), it suffices to prove it for $Z' := (Z \times_S S')_{\mathrm{red}} \to S'$. Each irreducible component $Z'_1/S', \ldots, Z'_r/S'$ of Z'/S' has a geometrically integral generic fiber by construction. It follows from [11], IV.9.7.7, that there exists a dense affine subscheme U of S' such that $Z'_i \times_{S'} U \to U$ has geometrically integral fibers for all $i = 1, \ldots, r$. Restricting U further if necessary, we can suppose that the number of geometric irreducible components in the fibers of $Z' \times_{S'} U \to U$ is constant ([11], IV.9.7.8). Note now that for each $s \in U$, the irreducible components of Z'_s are exactly the fibers $(Z'_i)_s$, $i = 1, \ldots, r$. Hence, it suffices to prove that the proposition holds for each $Z'_i \times_{S'} U \to U$. We may thus assume in the statement of the proposition that Z is integral, and that $\pi : Z \to S$ has (geometrically) integral fibers.

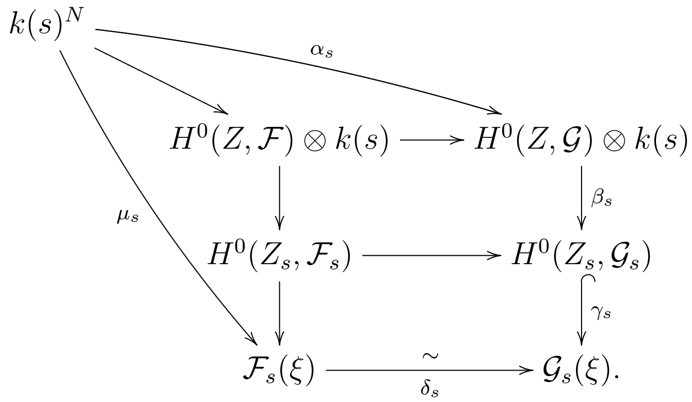
The reduction steps discussed above depend only of the morphism $Z \to S$. Our next reduction depends on the coherent sheaf $\mathcal{F}$. Consider the torsion subsheaf $\mathcal{F}_{\mathrm{tors}}$ of $\mathcal{F}$ (defined as the kernel of the morphism $\mathcal{F} \to \mathcal{F} \otimes k(Z)$, as Z is integral), and the associated exact sequence

$$0 \longrightarrow \mathcal{F}_{\mathrm{tors}} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

defining the coherent sheaf $\mathcal{G}$. The sheaf $\mathcal{G}$ is torsion-free and, thus, is torsion-free when restricted to the generic fiber of $Z \to S$. We may restrict S to an open subset W such that $\mathcal{G}|_{Z \times_S W}$ has torsion-free fibers $\mathcal{G}_s$ for all $s \in W$ ([11], IV.9.7.6). Since π is proper, $\pi_*(\mathcal{G})$ is coherent. Since the function $\varphi(s) := \dim_{k(s)}(\pi_*(\mathcal{G})_s \otimes k(s))$ is upper semi-continuous, we find that there exists a dense open subset U of W such that $\varphi(s)$ is constant on U, $\pi_*(\mathcal{G})|_U$ is free, and the natural map $H^0(Z \times_S U, \mathcal{G}|_{Z \times_S U}) \otimes k(s) \to H^0(Z_s, \mathcal{G}_s)$ is an isomorphism.

We may now assume that $S = \operatorname{Spec} R$, and that the morphism $Z \to S$ and the sheaf $\mathcal{F}$ have all the favorable properties discussed above. In particular, the R-module $H^0(Z, \mathcal{G})$ is free. Consider the linear composition $\alpha : R^N \to H^0(Z, \mathcal{F}) \to H^0(Z, \mathcal{G})$ defined by sending the standard basis vector e_i of R^N to the image in $H^0(Z, \mathcal{G})$ of $f_i \in H^0(Z, \mathcal{F})$. Choosing a basis for $H^0(Z, \mathcal{G})$, we find that the map α is defined by a matrix with coefficients in R, and the rows of the matrix are the relations describing the kernel of α. We use these same relations to define a closed subscheme of $\mathbb{A}^N_S$ that we denote by T, and such that $T(R) = \operatorname{Ker}(\alpha)$. We claim that T satisfies the conditions of the proposition for the morphism $Z \to S$. Indeed, let $s \in S$. Consider the natural commutative diagram

(1)



The map β_s is an isomorphism by construction. The map γ_s is injective because $\mathcal{G}_s$ is torsion-free. The support of $\mathcal{F}_{\text{tors}}$ is a strict closed subset of Z. By further restricting S, we can suppose that $\operatorname{Supp}(\mathcal{F}_{\text{tors}}) \cap Z_s \neq Z_s$ for all $s \in S$. Hence the kernel of $\mathcal{F}_s \to \mathcal{G}_s$ has support in a strict closed subset of Z_s. This implies that δ_s is an isomorphism. It follows now easily that $T(k(s)) = \ker \alpha_s = \ker \mu_s = \Sigma(s)$.

We note for use at the end of the proof of 3.7 that $\dim T_s$ is equal to the dimension of the vector space $\ker \mu_s$. □

The following proposition is an essential part of our method for producing interesting closed subschemes of projective schemes X/S. In our applications of this proposition in 3.9 and in 5.3, Condition (1) below will be satisfied with $V = S$, and in 3.9 the set B is empty.

Proposition 2.3. *Let $S = \operatorname{Spec} R$ be a noetherian affine scheme. Let T be a constructible subset of $\mathbb{A}^N_S$. Let B be a subset of $\mathbb{A}^N_S$ (in general, not constructible) with B_s a proper closed subset of the fiber $\mathbb{A}^N_{k(s)}$ for all $s \in S$, and $B_s = \emptyset$ for all but finitely many $s \in S$. Suppose that:*

(1) *There exists an open subset $V \subset S$ with finite complement such that*
$$\dim(T \cap \mathbb{A}^N_V) < N.$$
(2) *For all $s \in S$, there exists a $k(s)$-rational point in $\mathbb{A}^N_{k(s)}$ which does not belong to $T_s \cup B_s$.*

Then there exists a section σ of $\pi : \mathbb{A}^N_S \to S$ such that $\sigma(S) \cap (T \cup B) = \emptyset$.

Proof. Let us start by recording two uses of the noetherian hypothesis on R. First, we claim that:

(a) *There exists* $\delta \geq 1$ *such that for all* $s \in V$, $T_s \cup B_s$ *is contained in a hypersurface of degree* $\leq \delta$. Indeed, as $\dim T \cap \mathbb{A}^N_V < N$, no fiber T_s, $s \in V$, is dense in $\mathbb{A}^N_{k(s)}$. Every point $s \in V$ has an open neighborhood W such that T_W is contained in a hypersurface of $\mathbb{A}^N_W$ of some degree δ_s. As V is quasi-compact since S is noetherian, there exists an upper bound for the δ_s's. Since B_s is a proper closed subset and there are only finitely many non-empty B_s, we can increase δ so that for all $s \in S$, $T_s \cup B_s$ is contained in a hypersurface of degree $\leq \delta$.

The noetherian hypothesis on R is also used when applying the following lemma. A proof of this lemma in the affine case is given in [33], Proposition 13. We provide here an alternate proof.

Lemma 2.4. *Let* S *be any scheme. Let* $c \in \mathbb{N}$. *Then the subset* $\{s \in S \mid \mathrm{Card}(k(s)) \leq c\}$ *is closed in* S *and has dimension* 0. *When* S *is noetherian, this subset is then finite.*

Proof. It is enough to prove that when S is a scheme over a finite prime field $\mathbb{F}_p$, and q is a power of p, the set $\{s \in S \mid \mathrm{Card}(k(s)) = q\}$ is closed of dimension 0.

Let $\mathbb{F}_q$ be a field with q elements. Then any point $s \in S$ with $\mathrm{Card}(k(s)) = q$ is the image by the projection $S_{\mathbb{F}_q} \to S$ of a rational point of $S_{\mathbb{F}_q}$. Therefore we can suppose that S is a $\mathbb{F}_q$-scheme and we have to show that $S(\mathbb{F}_q)$ is closed of dimension 0. Let Z be the Zariski closure of $S(\mathbb{F}_q)$ in S, endowed with the reduced structure. Let U be an affine open subset of Z. Let $f \in \mathcal{O}_Z(U)$. For any $x \in U(\mathbb{F}_q)$, $(f^q - f)(x) = 0$ in $k(x)$, hence $x \in V(f^q - f)$. As $U(\mathbb{F}_q)$ is dense in U and U is reduced, we have $f^q - f = 0$. For any irreducible component Γ of U, this identity then holds on $\mathcal{O}(\Gamma)$, so Γ is just a rational point. Hence $U = U(\mathbb{F}_q)$ and $\dim U = 0$. Consequently, $Z = S(\mathbb{F}_q)$ is closed and has dimension 0. ☐

The key to the proof of Proposition 2.3 is the following assertion:

(b) *There exist* $t := t_1 + a_1 \in R[t_1, \dots, t_N]$, $a_1 \in R$, *and an open subset* $U \subseteq S$ *with finite complement, such that the closed subscheme* $H := V(t) \to S$ *of* $\mathbb{A}^N_S$ *has the following properties:*

(i) $\dim(H_U \cap T) < N - 1$,
(ii) $\dim(H_s \cap B) < N - 1$ *for all* $s \in U$,
(iii) *for all* $s \in S$, *there exists a* $k(s)$*-rational point in* H_s *which does not belong to* $T_s \cup B_s$.

Using (b), we can conclude the proof of our proposition as follows. First, note that when $N = 1$, (b) implies that the closed subset H is disjoint from $T \cup B$. Indeed, (i) and (ii) imply that $H_U \cap T = \emptyset = H_s \cap B_s$ for all $s \in U$. As H_s contains exactly one $k(s)$-rational point, when $s \notin U$, (iii) shows that this point is not in $T_s \cup B_s$.

When $N > 1$, we apply (b) repeatedly to obtain a sequence of closed sets

$$\mathbb{A}^N_S \supset V(t_1 + a_1) \supset \cdots \supset V(t_1 + a_1, t_2 + a_2, \dots, t_N + a_N).$$

The latter set is the image of the desired section, as we can use the case $N = 1$ on $V(t_1 + a_1, \dots, t_{N-1} + a_{N-1})$ to show that

$$V(t_1 + a_1, \dots, t_{N-1} + a_{N-1}, t_N + a_N) \cap (T \cup B) = \emptyset.$$

Proof of (b): A constructible subset T of a noetherian scheme X has the following property: there exists a finite set of points $\xi_1, \dots, \xi_\rho$ of T such that $T \subseteq \cup_i \overline{\{\xi_i\}}$.

Indeed, T is a the disjoint union of finitely many locally closed subsets $F_i \cap U_i$, $i = 1, \ldots, m$, with F_i closed and U_i open. We can take $\{\xi_1, \ldots, \xi_\rho\}$ to be the set of generic points of the irreducible components of the closed subset $\cup_i F_i$ which belong to T.

Let now $X = \mathbb{A}^N_S$ and let V be the open set as in (1), with $\dim(T \cap \mathbb{A}^N_V) < N$. Apply the above discussion to the constructible set $T \cap \mathbb{A}^N_V$, to obtain the associated set $\{\xi_1, \ldots, \xi_\rho\}$. When B is not empty, we will abuse notation and also denote by $\{\xi_1, \ldots, \xi_\rho\}$ the union of the generic points of $T \cap \mathbb{A}^N_V$ with the union of the generic points of the finitely many non-empty closed sets B_s of $\mathbb{A}^N_{k(s)}$. If W is any locally closed subset of X, to show that $\dim(T \cap W) < \dim(T)$ and $\dim(B_s \cap W) < \dim B_s$ when $B_s \neq \emptyset$, it suffices to show that $\xi_i \notin W$ for all $i = 1, \ldots, \rho$.

Upon renumbering the points of $\{\xi_1, \ldots, \xi_\rho\}$ if necessary, we can assume that the image of ξ_i under $\pi : \mathbb{A}^N_S \to S$ has finite residue field if and only if $i > r$ for some $r \leq \rho$. Let Z be the union of $S \setminus V$ with $\{\pi(\xi_{r+1}), \ldots, \pi(\xi_\rho)\}$ and with the finite subset of the closed points of S satisfying $\mathrm{Card}(k(s)) \leq \delta$ (that this set is finite follows from 2.4. We will later set δ to be as in (a)). For each $s \in Z$, we can use 2.3 (2) and fix a $k(s)$-rational point $x_s \in (\mathbb{A}^N_S)_s$ which does not belong to $T_s \cup B_s$.

Since every point of Z is closed in S, the Chinese Remainder Theorem implies that the canonical map $R \to \prod_{s \in Z} k(s)$ is surjective. Let $a \in R$ be such that $a \equiv t_1(x_s)$ in $k(s)$, for all $s \in Z$. Replacing t_1 by $t_1 - a$, we can assume that $t_1(x_s) = 0$ for all $s \in Z$.

Let $\mathfrak{p}_j \subset R[t_1, \ldots, t_N]$ be the prime ideal corresponding to ξ_j. Let $\mathfrak{m}_s \subset R$ denote the maximal ideal of R corresponding to s. Let $I := \cap_{s \in Z} \mathfrak{m}_s$. We claim that

$$I + t_1 \not\subseteq \cup_{1 \leq j \leq r} \mathfrak{p}_j.$$

Indeed, the intersection $(I + t_1) \cap \mathfrak{p}_j$ is either empty, or contains $a_j + t_1$ for some $a_j \in I$. In the latter case, $(I + t_1) \cap \mathfrak{p}_j = t_1 + a_j + (\mathfrak{p}_j \cap I)$. If $I + t_1 \subseteq \cup_{1 \leq j \leq r} \mathfrak{p}_j$, then every $t_1 + i$ with $i \in I$ belongs to some $t_1 + a_j + (\mathfrak{p}_j \cap I)$. Let $\mathfrak{q}_j := R \cap \mathfrak{p}_j$. It follows that

$$I \subseteq \cup_j (a_j + \mathfrak{q}_j)$$

where the union possibly runs only over a subset of $\{1, \ldots, r\}$. Since the domains $R/\mathfrak{q}_j$ are all infinite when $j \leq r$, Lemma 2.5 below implies that I is contained in some $\mathfrak{q}_{j_0}$ for $1 \leq j_0 \leq r$. As $I = \cap_{s \in Z} \mathfrak{m}_s$, we find that $\mathfrak{q}_{j_0} = \mathfrak{m}_s$ for some $s \in Z$. This is a contradiction, since for $j \leq r$, $\pi(\xi_j)$ does not belong to Z because the residue field of $\pi(\xi_j)$ is infinite and $\pi(\xi_j) \notin S \setminus V$.

Now that the claim is proved, we can choose $t \in (I + t_1) \setminus \cup_{1 \leq j \leq r} \mathfrak{p}_j$. Let $U := S \setminus Z$ and let $H := V(t)$. Clearly, t is of the form $t = t_1 + a_1$ for some $a_1 \in R$, as desired. We have $H \simeq \mathbb{A}^{N-1}_S$.

To prove that (b)(i) holds, that is, that $\dim(H_U \cap T) < N - 1$, we use the hypothesis (1) that $\dim(\mathbb{A}^N_V \cap T) < N$. By construction, $t \notin \cup_{1 \leq j \leq r} \mathfrak{p}_j$, so $V(t)$ does not contain any generic point ξ_j such that $\pi(\xi_j) \in U$. It follows that $\dim(H \cap \mathbb{A}^N_U \cap T) < \dim(\mathbb{A}^N_V \cap T)$.

To prove that (b)(ii) holds, that is, that $\dim(H_s \cap B) < N - 1$ for all $s \in U$, we use the hypothesis that B_s is a proper closed subset of $\mathbb{A}^N_{k(s)}$, so that $\dim(B_s) \leq N - 1$. Again, $V(t)$ does not contain any generic point ξ_j such that $\pi(\xi_j) \in U$. It follows that $\dim(H \cap B_s) < \dim(B_s)$.

It remains to prove (b)(iii). Let $s \in Z$. Since $a_1 \in I$ and $t_1(x_s) = 0$ for all $s \in Z$, we find that x_s is a $k(s)$-rational point of $V(t_1 + a_1)$. The point x_s was chosen above such that it does not belong to $T_s \cup B_s$. Let now $s \notin Z$. Then $|k(s)| \geq \delta + 1$ by construction. Choose now δ with the property in (a). Then, since t has degree 1, we find that $H_s \cap (T \cup B)$ is contained in a hypersurface $V(f)$ of H_s with $\deg(f) \leq \delta$. We conclude that H_s contains a $k(s)$-rational point that does not belong to $B_s \cup T_s$ using the following claim:

Assume that k is either an infinite field or that $|k| = q \geq \delta + 1$. Let $f \in k[T_1, \dots, T_\ell]$ with $\deg(f) \leq \delta$, $f \neq 0$. Then $V(f)(k) \subsetneq \mathbb{A}^\ell(k)$.

Indeed, when k is a finite field, an easy induction on ℓ shows that $\mathrm{Card}(V(f)(k)) \leq q^\ell - (q-\delta)^\ell < q^\ell$. When k is infinite, the k-rational points are dense in $\mathbb{A}^\ell/k$. □

Our next lemma follows from [23], Theorem 5. We provide here a more direct proof using the earlier reference [28].

Lemma 2.5. *Let R be a commutative ring, and let $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ be (not necessarily distinct) prime ideals of R with infinite quotients $R/\mathfrak{q}_i$ for all $i = 1, \dots, r$. Let I be an ideal of R and suppose that there exist $a_1, \dots, a_r \in R$ such that*

$$I \subseteq \cup_{1 \leq i \leq r}(a_i + \mathfrak{q}_i).$$

Then I is contained in the union of those $a_i + \mathfrak{q}_i$ with $I \subseteq \mathfrak{q}_i$. In particular, I is contained in at least one $\mathfrak{q}_i$.

Proof. We have $I = \cup_i((a_i + \mathfrak{q}_i) \cap I)$. If $(a_i + \mathfrak{q}_i) \cap I \neq \emptyset$, then it is equal to $\alpha_i + (\mathfrak{q}_i \cap I)$ for some $\alpha_i \in I$. Hence

$$I = \cup_i(\alpha_i + (\mathfrak{q}_i \cap I))$$

where the union runs on part of $\{1, \dots, r\}$. By [28], I is the union of those $\alpha_i + (\mathfrak{q}_i \cap I)$ with $I/(\mathfrak{q}_i \cap I)$ finite. For any such i, the ideal $(I + \mathfrak{q}_i)/\mathfrak{q}_i$ of $R/\mathfrak{q}_i$ is finite and, hence, equal to (0) because $R/\mathfrak{q}_i$ is an infinite domain. □

Remark 2.6 The hypothesis in 2.3 (1) is needed. Indeed, let $S = \operatorname{Spec} \mathbb{Z}$, and $N = 1$. Consider the closed subset $V(t^3 - t)$ of $\operatorname{Spec} \mathbb{Z}[t] = \mathbb{A}^1_{\mathbb{Z}}$. Let T be the constructible subset of $\mathbb{A}^1_{\mathbb{Z}}$ obtained by removing from $V(t^3 - t)$ the maximal ideals $(2, t-1)$ and $(3, t-1)$. Then, for all $s \in S$, the fiber T_s is distinct from $\mathbb{A}^1_{k(s)}(k(s))$, and $\dim T_s = 0$. However, $\dim T = 1$, and we note now that there exists no section of $\mathbb{A}^1_{\mathbb{Z}}$ disjoint from T. Indeed, let $V(t-a)$ be a section. If it is disjoint from T, then $a \neq 0, 1, -1$, and $6 \mid a - 1$. So there exists a prime $p > 3$ with $p \mid a$, and $V(t-a)$ meets T at the point (p, t).

For a more geometric example, let k be any infinite field. Let $S = \operatorname{Spec} k[u]$ and $\mathbb{A}^1_S = \operatorname{Spec} k[u, t]$. When $T := V(t^2 - u) \subset \mathbb{A}^1_S$, then $\mathbb{A}^1_S \setminus T$ does not contain any section $V(t - g(u))$ of $\mathbb{A}^1_S$. Indeed, otherwise $(t^2 - u, t - g(u)) = (1)$, and $g(u)^2 - u$ would be an element of k^*.

3. Local complete intersection subschemes

The theorem below is the key to reducing the proof of the Moving Lemma 7.2 to the case of relative dimension 1. This section and the next are devoted to its proof, which follows the outline given in 1.4.

Theorem 3.1. *Let S be an affine noetherian scheme of dimension* 1*, and let $X \to S$ be a quasi-projective morphism. Let C be a closed subscheme of X, finite over S, and such that:*
(1) *$C \to X$ is a regular immersion and C has pure codimension $d > 0$ in X;*
(2) *For all non-empty C_s,* $\operatorname{codim}(C_s, X_s) \geq d$.
Let F be a closed subset of X. Fix an ample invertible sheaf $\mathcal{O}_X(1)$ on X. Then there exist $n > 0$ and a section f of $\mathcal{O}_X(n)$ such that:
(i) *C is a closed subscheme of H_f, $C \to H_f$ is a regular immersion, and C has pure codimension $d-1$ in H_f;*
(ii) *For all non-empty C_s,* $\operatorname{codim}(C_s, (H_f)_s) \geq d-1$;
(iii) *For all $s \in S$, any irreducible component Γ of $F \cap X_s$ is such that* $\dim(\Gamma \cap (H_f)_s) \leq \max(\dim(\Gamma) - 1, 0)$.

Furthermore, there exists a closed subscheme Y of X containing C, such that C is the support of a Cartier divisor on Y, and such that for all $s \in S$, any irreducible component Γ of $F \cap X_s$ is such that $\dim(\Gamma \cap Y_s) \leq \max(\dim(\Gamma) - (d-1), 0)$. *In particular, if the codimension of F_s in X_s is at least* 1 *in a neighborhood of C_s, then $F \cap Y_s$ has dimension at most* 0 *in a neighborhood of C_s.*

Remark 3.2 The hypothesis that $C \to X$ is a regular immersion is equivalent to the condition that $C \to X$ is a local complete intersection morphism (see, e.g., [21], 6.3.21). If in addition C is connected, then the hypothesis that C has pure codimension $d > 0$ in X implies that $C \to X$ is a regular immersion of codimension d (see, e.g., [21], 6.3.9).

Remark 3.3 Keep the hypotheses of 3.1 above. The following are equivalent:
a) Every irreducible component of X_s which meets C_s has dimension at least d.
b) Condition (2) holds.

This is immediate since by hypothesis, C_s is the union of finitely many closed points of X_s. Condition 3.1(2) is satisfied when 3.1(1) holds and C is flat over S.

Consider now any closed subscheme H_f in X containing C. Fix $s \in S$. Let Γ be an irreducible component of $(H_f)_s$ which meets C_s. Then Γ is contained in an irreducible component Γ' of X_s which meets C_s. Thus, since 3.1 (2) is satisfied, $\dim(\Gamma') \geq d$. Then Γ is an irreducible component of $(H_f)_s \cap \Gamma'$. Since $(H_f)_s$ is defined in X_s by a single equation, we find that $\dim(\Gamma) \geq \dim(\Gamma') - 1 \geq d - 1$. It follows that 3.1 (ii) is satisfied.

Remark 3.4 Let us note that if X/S and C satisfy the hypotheses of the version of 3.1 given in the introduction, then they satisfy the hypotheses of the above version of 3.1. In other words, assume that C/S is finite and surjective, with C irreducible. We need to show that Condition 3.1(2) holds.

Under these additional hypotheses, S is irreducible, and we let η denote its generic point. Let Δ denote an irreducible component of X containing C. Then Δ_η is an irreducible component of X_η containing the closed point C_η. By hypothesis, C has codimension at least d in X, so C_η has codimension at least d in X_η. Hence, Δ_η has dimension at least d. Using [11], IV.13.1.1, we find that for all $s \in S$, the irreducible components of Δ_s have dimension at least d. It follows that for all $s \in S$, every irreducible component of X_s which meets C_s has dimension at least d. As noted in 3.3, this latter condition is equivalent to 3.1(2).

3.5 *Proof of Theorem 3.1.* We first give a complete proof of 3.1 in the case where $X \to S$ is projective. The proof of 3.1 when X/S is only assumed to be quasi-projective is in 3.10.

Let us start our discussion with hypotheses slightly more general than in 3.1. Fix the following notation. Let $S = \operatorname{Spec} R$ be a noetherian affine scheme, not necessarily of dimension 1. Consider a projective morphism $X \to S$. Fix a very ample sheaf $\mathcal{O}_X(1)$ on X relative to S. Let $C \subset X$ be a closed subscheme. Let $\mathcal{J}$ be the ideal sheaf defining C. If $\mathcal{F}$ is any sheaf on X and $s \in S$, let $\mathcal{F}_s$ denote the pull-back of $\mathcal{F}$ on a fiber X_s. For $n \geq 1$, set $\mathcal{J}(n) := \mathcal{J} \otimes \mathcal{O}_X(n)$, and for $s \in S$, let $\mathcal{J}_s(n) := \mathcal{J}_s \otimes \mathcal{O}_{X_s}(n)$. Let $\bar{\mathcal{J}}_s$ denote the image of $\mathcal{J}_s \to \mathcal{O}_{X_s}$.

To prove Theorem 3.1, we will show the existence of $f \in H^0(X, \mathcal{J}(n))$, for some n sufficiently large, such that the associated closed subscheme $H_f \subset X$ satisfies the conclusion of the theorem. Note that when X/S is projective, the subscheme denoted by H_f in the introduction is often denoted by $V_+(f)$.

To enable us to use the results of the previous section to produce the desired f, we introduce the following notation. Let n be big enough such that $\mathcal{J}(n)$ is generated by its global sections. Fix a system of generators $f_1, \dots, f_N$ of $H^0(X, \mathcal{J}(n))$. Denote by $\overline{f}_{i,s}$ the image of f_i in $\bar{\mathcal{J}}_s(n)$. For $s \in S$, consider the following sets:

- Let $\Sigma_1(s)$ denote the set of $(\alpha_1, \dots, \alpha_N) \in k(s)^N$ such that there exists some $x \in C \cap X_s$ with $\sum_{i=1}^N \alpha_i (f_i|_{X_s}) = 0$ in $\mathcal{J}_s(n)/\mathcal{J}_s^2(n) \otimes k(x)$.
- Let F be a closed subset of X. Let $\Sigma_2(s)$ denote the set of $(\alpha_1, \dots, \alpha_N) \in k(s)^N$ such that the closed subset $V_+(\sum_{i=1}^N \alpha_i \overline{f}_{i,s})$ in X_s, defined by the section $\sum_{i=1}^N \alpha_i \overline{f}_{i,s}$ of $\mathcal{O}_{X_s}(n)$, contains at least one irreducible component of F_s of positive dimension.

We will use the fact that if $f \in H^0(X, \mathcal{J}(n))$ and $\overline{f}_s$ is its image in $\bar{\mathcal{J}}_s(n)$, then $V_+(f) \cap X_s = V_+(\overline{f}_s)$. Assume now that $C \to S$ is as in 3.1. Note then that by the hypothesis 3.1 (2), C_s does not contain any isolated point of X_s since such a point would be an irreducible component of X_s, and in a neighborhood of C_s, all components of X_s have dimension at least d. Therefore, for any $x \in C_s$, $(\bar{\mathcal{J}}_s)_x \neq 0$ and, hence, both $\bar{\mathcal{J}}_s(n)/\bar{\mathcal{J}}_s^2(n) \otimes k(x)$ and $\mathcal{J}_s(n)/\mathcal{J}_s^2(n) \otimes k(x)$ are non-zero.

Our goal is to prove, using 2.2 and for an appropriate choice of n, the existence of constructible subsets T_1 and T_2 associated with the sets $\Sigma_1(s)$ and $\Sigma_2(s)$. Great care will be needed in 3.6 and 3.7 to insure that the dimension of the fibers of T_1 and T_2 are controlled, so that Proposition 2.3 can be used to produce the desired $f \in H^0(X, \mathcal{J}(n))$. For this, we will use results which depends on properties of m-regular sheaves on a projective variety. These properties and results are discussed in 4.1-4.4. The proof of Theorem 3.1 when $X \to S$ is projective ends in 3.9.

For use in our next lemma, we now assume that n is large enough so that $H^1(X, \mathcal{J}_s^2(n)) = (0)$ for all $s \in S$. That such an n exists follows from 4.4.

Lemma 3.6. *Keep the above notation and assume that C is as in 3.1. Then there exists a constructible subset T_1 of $\mathbb{A}_S^N$ such that for all $s \in S$, $\dim(T_1)_s \leq N - d$, and $\Sigma_1(s)$ is exactly the set of $k(s)$-rational points of $(\mathbb{A}_S^N)_s$ contained in $(T_1)_s$.*

Proof. We apply Proposition 2.2 to the following Z and $\mathcal{F}$. Let $Z := C$, and let $\mathcal{F}$ denote the sheaf $\mathcal{J}(n)/\mathcal{J}^2(n)$ considered as a coherent sheaf on C. Its support is C. As C_s is finite, $\Sigma_1(s)$ is nothing but the set $\Sigma(s)$ considered in 2.2.

By hypothesis, for each $s \in S$, we have an isomorphism $H^0(X, \mathcal{F}) \otimes k(s) \to H^0(X_s, \mathcal{F}_s) \cong H^0(C_s, \mathcal{F}_s) \cong \oplus_{x \in C_s} (\mathcal{F}_s)_x$. It follows that for each $x \in C_s$, the natural map $H^0(C_s, \mathcal{F}_s) \to (\mathcal{F}_s)_x$ is surjective. By hypothesis, $(\mathcal{F}_s)_x$ is free of rank d, so we find using 2.2 that $\dim(T_{1,s}) \leq N - d$. □

Lemma 3.7. *Keep the notation of 3.5. In particular, S is an affine noetherian scheme, C is a closed subscheme of X (not necessarily finite over S, or l.c.i. in X), and F a closed subset of X. Assume also that for all $s \in S$, no irreducible component of F_s of positive dimension is contained in C_s. Let $c \in \mathbb{N}$.*

Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for any choice $\{f_1, \ldots, f_N\}$ of generators of $H^0(X, \mathcal{J}(n))$, there exists a constructible subset T_2 of $\mathbb{A}^N_S$ such that for all $s \in S$, $\Sigma_2(s)$ is exactly the set of $k(s)$-rational points of $(\mathbb{A}^N_S)_s$ contained in $(T_2)_s$, and such that $\dim(T_2)_s \leq N - c$.

Proof. As $F \to S$ is proper, Zariski's Main Theorem implies that the subset of the points of F which are isolated in their fibers is open in F. Let Z denote the complement of this open subset in F, endowed with the structure of reduced closed subscheme of X. The fibers of $Z \to S$ have no isolated points and, hence, have no zero-dimensional irreducible components. We plan to apply Proposition 2.2 to the scheme $Z \to S$ and the coherent sheaf $\mathcal{F} := \mathcal{J}(n)|_Z$. Note that $\operatorname{Supp} \mathcal{F} = Z$ since the fibers of $Z \to S$ have no isolated points.

We recall below the stratification of S introduced in the proof of 2.2 which does not depend on $\mathcal{F}$. First, S is written as a finite set-theoretically disjoint union $S = \coprod_i S_i$ of integral schemes S_i locally closed in S, and we also require now that $C \times_S S_i \to S_i$ is flat. Second, for each i, there exists a finite surjective morphism $S'_i \to S_i$ such that $(Z \times_S S'_i)_{\mathrm{red}} \to S'_i$ has good properties. In order to facilitate the notation, let us denote by S' any of the schemes S'_i. Then we require that the irreducible components $Z_1, \ldots, Z_r$ of $(Z \times_S S')_{\mathrm{red}} \to S'$ have integral fibers and such that for all $s' \in S'$, $(Z_1)_{s'}, \ldots, (Z_r)_{s'}$ are exactly the irreducible components of $Z_{s'}$. Let Γ be any of the irreducible components of $(Z \times_S S')_{\mathrm{red}}$. We prove the lemma in three steps.

First step. We claim that there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ and for all $s' \in S'$, the natural map

$$H^0(X, \mathcal{J}(n)) \otimes_{k(s)} k(s') \longrightarrow H^0(\Gamma_{s'}, ((\mathcal{J}(n) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})|_\Gamma)_{s'})$$

is surjective. Indeed, we have closed immersions $Z \to X$ and $\Gamma \to (Z \times_S S')_{\mathrm{red}} \to Z \times_S S'$. Since $\mathcal{O}_X(1)$ is very ample relatively to $X \to S$, there exists n_1 such that for all $n \geq n_1$, both maps

$$H^0(X, \mathcal{J}(n)) \longrightarrow H^0(Z, \mathcal{J}(n)|_Z) \tag{2}$$

and

$$H^0(Z_{S'}, (\mathcal{J}(n) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})|_{Z_{S'}}) \longrightarrow H^0(\Gamma, (\mathcal{J}(n) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})|_\Gamma) \tag{3}$$

are surjective. We use now properties of m-regular sheaves as in 4.4, and find that after increasing n_1 if necessary, we can assume that for all $n \geq n_1$, and for all $s \in S$,

$$H^0(Z, \mathcal{J}(n)|_Z) \otimes k(s) \longrightarrow H^0(Z_s, (\mathcal{J}(n)|_Z)_s) \tag{4}$$

is an isomorphism, and for all $s' \in S'$ lying over s,

$$(5) \qquad H^0(\Gamma, (\mathcal{J}(n) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})|_\Gamma) \otimes k(s') \longrightarrow H^0(\Gamma_{s'}, ((\mathcal{J}(n) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})|_\Gamma)_{s'})$$

is an isomorphism. Hence, tensoring (2) and (4) by $k(s')$, we obtain the surjection

$$(6) \quad H^0(X, \mathcal{J}(n)) \otimes k(s') \longrightarrow H^0(Z, \mathcal{J}(n)|_Z) \otimes k(s') \longrightarrow H^0(Z_s, (\mathcal{J}(n)|_Z)_s) \otimes k(s').$$

Recall now that there is a natural isomorphism

$$(7) \qquad H^0(Z_s, (\mathcal{J}(n)|_Z)_s) \otimes k(s') \longrightarrow H^0((Z_{S'})_{s'}, ((\mathcal{J}(n) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})|_{Z_{S'}})_{s'}).$$

Consider the commutative square

$$\begin{array}{ccc} H^0(Z_{S'}, (\mathcal{J}(n) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})|_{Z_{S'}}) \otimes k(s') & \longrightarrow & H^0((Z_{S'})_{s'}, ((\mathcal{J}(n) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})|_{Z_{S'}})_{s'}) \\ \beta\big\downarrow & & \delta\big\downarrow \\ H^0(\Gamma, (\mathcal{J}(n) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})|_\Gamma) \otimes k(s') & \xrightarrow{\gamma} & H^0(\Gamma_{s'}, ((\mathcal{J}(n) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})|_\Gamma)_{s'}). \end{array}$$

The right vertical map δ is surjective because tensoring (3) by $k(s')$ and composing with (5) shows that the map $\gamma \circ \beta$ is surjective. Composing (6), (7), and δ produces the desired surjection.

Second step. For simplicity, let us write $\mathcal{J}' := (\mathcal{J} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})|_\Gamma$. The pull-back of $\mathcal{O}_X(1)$ to Γ will be denoted by $\mathcal{O}_\Gamma(1)$. It is clear that $\mathcal{J}'(n) := \mathcal{J}' \otimes \mathcal{O}_\Gamma(1)$ is isomorphic to $(\mathcal{J}(n) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})|_\Gamma$. Let $\mathcal{H}$ denote the image of the natural map $\mathcal{J}' \to \mathcal{O}_\Gamma$. The ideal sheaf $\mathcal{H}$ defines the closed scheme $C' := C_{S'} \times_{X_{S'}} \Gamma$ of Γ.

We claim that there exists $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$ and for all $s' \in S'$,

$$\dim_{k(s')} H^0(\Gamma_{s'}, \mathcal{J}'(n)_{s'}) \geq c.$$

Indeed, recall that for all $s' \in S'$, $\Gamma_{s'}$ is integral. We note that for all $s' \in S'$, $C'_{s'}$ is a strict subset of $\Gamma_{s'}$ and, hence, has smaller dimension than $\Gamma_{s'}$. The former fact follows from our hypothesis that for all $s \in S$, no irreducible component of F_s of positive dimension is contained in C_s. It follows that the Hilbert polynomial $P_{C'_{s'}}$ of $C'_{s'}$ satisfies $\deg P_{C'_{s'}} < \deg P_{\Gamma_{s'}}$. As the set of Hilbert polynomials $P_{\Gamma_{s'}}$ and $P_{C'_{s'}}$, $s' \in S'$, is finite ([27], p. 58, (ii)) and such polynomials have positive leading coefficient ([13], III.9.10), there exists n_2 such that for all $n \geq n_2$,

$$P_{\Gamma_{s'}}(n) - P_{C'_{s'}}(n) \geq c.$$

We use now properties of m-regular sheaves as in 4.4, and find that after increasing n_2 if necessary, we can assume that for all $n \geq n_2$, for all $i \geq 1$ and for all $s' \in S'$,

$$H^i(\Gamma_{s'}, \mathcal{O}_{\Gamma_{s'}}(n)) = (0) = H^i(C'_{s'}, \mathcal{O}_{C'_{s'}}(n)).$$

Since $P_{\Gamma_{s'}}(n) = \chi(\mathcal{O}_{\Gamma_{s'}}(n))$ for all $n > 0$, the first vanishing above implies that for $n \geq n_2$ and for all s',

$$P_{\Gamma_{s'}}(n) = \dim H^0(\Gamma_{s'}, \mathcal{O}_{\Gamma_{s'}}(n)).$$

Similarly, the second vanishing implies that $P_{C'_{s'}}(n) = \dim H^0(C'_{s'}, \mathcal{O}_{C'_{s'}}(n))$.

Denote by $\bar{\mathcal{H}}_{s'}$ the image of $\mathcal{H}_{s'} \to \mathcal{O}_{\Gamma_{s'}}$. The ideal $\bar{\mathcal{H}}_{s'}$ is the defining ideal of $C'_{s'}$ in $\Gamma_{s'}$. Increasing n_2 if necessary, we can assume that for all $n \geq n_2$, $H^1(\Gamma, \mathcal{H}(n)) = (0)$, and that for all s', $H^0(C', \mathcal{O}_{C'}(n)) \otimes k(s') \to H^0(C'_{s'}, \mathcal{O}_{C'_{s'}}(n))$ is an isomorphism.

This implies that $H^0(\Gamma_{s'}, \mathcal{O}_{\Gamma_{s'}}(n)) \to H^0(C'_{s'}, \mathcal{O}_{C'_{s'}}(n))$ is surjective. Hence, for all $n \geq n_2$ and for all $s' \in S'$,

$$\dim H^0(\Gamma_{s'}, \bar{\mathcal{H}}_{s'}(n)) = \dim H^0(\Gamma_{s'}, \mathcal{O}_{\Gamma_{s'}}(n)) - \dim H^0(C'_{s'}, \mathcal{O}_{C'_{s'}}(n)) \geq c.$$

Final step. Now let $n_0 = \max\{n_1, n_2\}$. Choose and fix $n \geq n_0$, and then fix a system of generators $f_1, \ldots, f_N$ of $H^0(X, \mathcal{J}(n))$. This allows us to define the set $\Sigma_2(s)$ as in 3.5:

> $\Sigma_2(s)$ *denotes the set of* $(\alpha_1, \ldots, \alpha_N) \in k(s)^N$ *such that the closed subset* $V_+(\sum_{i=1}^N \alpha_i \overline{f}_{i,s})$ *in* X_s *contains at least one irreducible component of* F_s *of positive dimension.*

Consider the scheme Z, along with the coherent sheaf $\mathcal{F} := \mathcal{J}(n)|_Z$ on Z and the sections $f_{1|Z}, \ldots, f_{N|Z}$ of $H^0(X, \mathcal{J}(n)|_Z)$. As in 2.2, we define

$$\Sigma(s) := \left\{ (\alpha_1, \ldots, \alpha_N) \in k(s)^N \;\middle|\; \sum_i \alpha_i f_{i,s} \text{ vanishes at some generic point of } Z_s \right\}.$$

Let ξ be any generic point of Z_s. As no irreducible component of F_s of positive dimension is contained in C, $\xi \notin C$. Hence, $(\mathcal{J}(n)|_{Z_s})_\xi = \mathcal{O}_{Z_s,\xi}$. Let $f \in H^0(X, \mathcal{J}(n))$ with image $f|_{Z_s}$ in $H^0(Z_s, \mathcal{J}(n)|_{Z_s})$. It follows that $V_+(f)$ contains ξ if and only if $f|_{Z_s}$ vanishes at ξ, and that $\Sigma_2(s) = \Sigma(s)$.

We now apply 2.2 to Z and to $\mathcal{F}$ to obtain the existence of a constructible set $T_2 \subset \mathbb{A}^N_S$ such that for all $s \in S$, $\Sigma_2(s)$ is exactly the set of $k(s)$-rational points of $(\mathbb{A}^N_S)_s$ contained in $(T_2)_s$.

As the proof of 2.2 shows, to prove that $\dim(T_2)_s \leq N - c$ for all $s \in S$, it is enough to show a similar statement for all the constructible subsets T' associated with the morphisms $\Gamma \to S'$ and with the associated data consisting of the sheaf $\mathcal{J}'(n) := (\mathcal{J}(n) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})|_\Gamma$ and the sections $\{f_1|_\Gamma, \ldots, f_N|_\Gamma\}$ (where $f_j|_\Gamma$ is the canonical image of f_j in $H^0(\Gamma, \mathcal{J}'(n))$, and $\Gamma \to S'$ is as at the beginning of the proof). We can further argue on each piece of the stratification of $S' = \sqcup S'_i$ implicitly appearing in the discussion in the last part of the proof of 2.2.

Consider the composition

$$k(s')^N \longrightarrow H^0(X, \mathcal{J}(n)) \otimes k(s') \longrightarrow H^0(\Gamma_{s'}, \mathcal{J}'(n)_{s'}),$$

which first sends the standard basis vector e_i of $k(s')^N$ to $f_i \otimes 1$ in $H^0(X, \mathcal{J}(n)) \otimes k(s')$. Step 1 shows that the composition of these two maps is surjective for all $n \geq n_0$, and for all $s' \in S'$. The reader will check that this composition is nothing but the map corresponding in our situation to the map μ in the diagram (1) in the proof of 2.2. Let ξ denote the generic point of $\Gamma_{s'}$. The dimension of $(T_2)_{s'}$ is given by the dimension over $k(s')$ of the kernel of the natural map ν is the diagram below:

$$\begin{array}{ccccc} k(s')^N & \overset{\mu}{\twoheadrightarrow} & H^0(\Gamma_{s'}, \mathcal{J}'(n)_{s'}) & \overset{\rho}{\twoheadrightarrow} & H^0(\Gamma_{s'}, \bar{\mathcal{H}}(n)_{s'}) \\ & \searrow^{\nu} & \downarrow & \swarrow^{\theta} & \\ & & \mathcal{J}'(n)_{s'} \otimes k(\xi). & & \end{array}$$

Increasing n_0 if necessary to apply Lemma 3.8 to $\Gamma \to S'$ and C', we find that the map ρ is surjective for all $s' \in S'$. To conclude the proof of the lemma, it suffices to note that the map θ is injective. Indeed, then Step 2 shows that the dimension of $H^0(\Gamma_{s'}, \mathcal{J}(n)_{s'})$ is bounded by c. Hence, the kernel of μ has dimension at most $N - c$. $\square$

Lemma 3.8. *Let $X \to S$ be a projective scheme over an affine noetherian scheme S. Let C be a closed subscheme of X with ideal sheaf $\mathcal{J}$. Let $\bar{\mathcal{J}}_s$ denote the image of $\mathcal{J}_s$ in $\mathcal{O}_{X_s}$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $s \in S$, the canonical map $\mathcal{J}_s(n) \to \bar{\mathcal{J}}_s(n)$ induces a surjection*

$$H^0(X_s, \mathcal{J}(n)_s) \longrightarrow H^0(X_s, \bar{\mathcal{J}}_s(n)).$$

Proof. Using the flattening stratification of S induced by the sheaf $\mathcal{O}_X/\mathcal{J}$, we deduce the existence of finitely many irreducible locally closed subsets U_i of S such that $S = \cup U_i$ (set theoretically), and when each U_i is endowed with the structure of integral scheme, then $C \times_S U_i \to U_i$ is flat.

Denote by U one of the integral schemes U_i. Denote by $\mathcal{K}$ and $\mathcal{J}'$ the kernel and image of the natural morphism $\mathcal{J} \otimes_{\mathcal{O}_S} \mathcal{O}_U \to \mathcal{O}_{X_U}$, with associated exact sequence of sheaves on X_U

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{J} \otimes_{\mathcal{O}_S} \mathcal{O}_U \longrightarrow \mathcal{J}' \longrightarrow 0.$$

For all $n \in \mathbb{Z}$, we then have the exact sequence

$$0 \to \mathcal{K}(n) \to \mathcal{J}(n) \otimes_{\mathcal{O}_S} \mathcal{O}_U \to \mathcal{J}'(n) \to 0.$$

Since $X_U \to U$ is projective, we can find $n_0 \geq 1$ such that $H^1(X_U, \mathcal{K}(n)) = (0)$ for all $n \geq n_0$. Increasing n_0 if necessary and using properties of m-regular sheaves, we can assume that for all $n \geq n_0$ and for all $s \in U$,

$$H^0(X_U, \mathcal{J}'(n)) \otimes k(s) \longrightarrow H^0(X_s, \mathcal{J}'(n)_s)$$

is an isomorphism. Since $C \times_S U \to U$ is flat, the sequence $0 \to \mathcal{J}'_s \to \mathcal{O}_{X_s} \to \mathcal{O}_{C_s} \to 0$ is exact for all $s \in U$. It follows that $\mathcal{J}'(n)_s = \bar{\mathcal{J}}_s(n)$. For any $s \in U$ and for $n \geq n_0$, consider the commutative diagram:

$$\begin{array}{ccc}
H^0(X_U, \mathcal{J}(n) \otimes \mathcal{O}_U) \otimes k(s) & \twoheadrightarrow & H^0(X_U, \mathcal{J}'(n)) \otimes k(s) \\
\downarrow & & \downarrow \\
H^0(X_s, \mathcal{J}(n)_s) & \longrightarrow & H^0(X_s, \bar{\mathcal{J}}_s(n)).
\end{array}$$

The top horizontal map is surjective because $H^1(X_U, \mathcal{K}(n)) = (0)$, and the right vertical arrow is an isomorphism by construction. Thus, we see that $H^0(X_s, \mathcal{J}(n)_s) \to H^0(X_s, \bar{\mathcal{J}}_s(n))$ is surjective for all $n \geq n_0$ and all $s \in U$. $\square$

3.9 We proceed now to prove Theorem 3.1 in $d - 1$ steps, under the assumption that $X \to S$ is projective. When $d = 1$, there is nothing to prove.

Fix n large enough so that 3.6, 3.7, and 4.4, can be applied to our situation. Let T_1 and T_2 be the constructible subsets of $\mathbb{A}^N_S$, pertaining to $\Sigma_1(s)$ and $\Sigma_2(s)$, and whose existence is proved in 3.6 and 3.7. Let $T := T_1 \cup T_2$. For each $s \in S$, the element $f_{s,n} \in H^0(X_s, \mathcal{J}_s(n))$ exhibited in Lemma 4.4 defines a $k(s)$-rational point of $\mathbb{A}^N_S$ not contained in $T_{1,s} \cup T_{2,s}$ because $(\bar{\mathcal{J}}_s(n)/\bar{\mathcal{J}}^2_s(n)) \otimes k(x) \neq 0$ for all $x \in C_s$.

It follows that for all $s \in S$, there exists a $k(s)$-rational point of $\mathbb{A}^N_S$ not contained in T_s. We also have when $d \geq 2$, that $\dim T_s \leq N - 2$ for all $s \in S$ (3.6 and

3.7). Since we assume that $\dim(S) = 1$, we find that $\dim(T) \leq N-1$. We can thus apply Proposition 2.3 to find a section $(a_1, \dots, a_N) \in \mathbb{A}^N_S(S)$ such that for all $s \in S$, $(a_1(s), \dots, a_N(s))$ is not a $k(s)$-rational point of $(\mathbb{A}^N_S)_s$ contained in T_s. Let $f := \sum_{i=1}^N a_i f_i$, and consider the closed subscheme $V_+(f) \subset X$. By definition of f, we have $C \subset V_+(f)$, and by definition of T, we have:

(i) for all $x \in C$, the image of f in $(\mathcal{J}(n)/\mathcal{J}^2(n)) \otimes k(x)$ is non-zero;
(ii) for all $s \in S$, $V_+(f)$ does not contain any irreducible component of F_s of positive dimension.

The first condition implies that C is l.c.i. in $V_+(f)$, pure of codimension $d-1$. Indeed, this is a local question. Fix $x \in C$. Let $I = \mathfrak{p}\mathcal{O}_{X,x}$ and let $g \in I$ correspond to the section f. Since the image of g in $I/I^2 \otimes k(\mathfrak{p})$ is not zero, the image of g in the free $\mathcal{O}_{C,x}$-module I/I^2 can be completed into a basis of I/I^2, and it is then well-known that g belongs to a regular sequence generating I.

The second condition implies that for all $s \in S$, and for all irreducible components Γ of $F \cap X_s$, we have $\dim(\Gamma \cap V_+(f)_s) \leq \max(\dim(\Gamma) - 1, 0)$.

The closed subscheme Y whose existence is asserted at the end of Theorem 3.1 is obtained by repeating the above arguments $(d-2)$ times, starting with $X' := V_+(f)$, $F' := F \cap V_+(f)$, and $C \subset X'$. This concludes the proof of 3.1 when $X \to S$ is projective.

3.10 Assume now that $X \to S$ is quasi-projective. Using a power of the given ample sheaf for $X \to S$, we can consider a projective morphism $\overline{X} \to S$ and an open immersion $X \to \overline{X}$ of S-schemes, along with a very ample sheaf $\mathcal{O}_{\overline{X}}(1)$ which restrict on X to some power of the initial ample sheaf on X. Then $C \to S$ is a closed subset of $\overline{X}$ of pure codimension d, and $C \to \overline{X}$ is a regular immersion. The subset C of $\overline{X}$ also satisfies Condition (2) of 3.1 for the morphism $\overline{X} \to S$.

Let $F \subset X$ be the given closed subset, and denote by $\overline{F}$ its closure in $\overline{X}$. The conclusion of 3.1 in the case of C, $\overline{X} \to S$, and $\overline{F}$, implies the existence of a closed subscheme H_f of $\overline{X}$ containing C such that $C \to H_f$ is a regular immersion, and such that for all $s \in S$, any irreducible component Γ of $\overline{F} \cap \overline{X}_s$ is such that $\dim(\Gamma \cap (H_f)_s) \leq \max(\dim(\Gamma) - 1, 0)$.

We claim that the closed subset $H := H_f \cap X$ of X satisfies the conclusion of 3.1 for the morphism $X \to S$ and the closed subset F. Clearly, since C is closed in X, C is contained in H and $C \to H$ is a regular immersion. Suppose now that Θ is an irreducible component of F_s. We need to show that $\dim(\Theta \cap H_s) \leq \max(\dim(\Theta) - 1, 0)$. This will follow if the closure of Θ in $(\overline{F})_s$ is an irreducible component of $(\overline{F})_s$. That this is indeed the case follows from the fact that F is open in $\overline{F}$. □

4. m-REGULAR SHEAVES

It remains to prove various assertions used in the proof of Theorem 3.1.

4.1 We start by recalling the definition and properties of m-regular sheaves needed in our next lemmas. Let X be a projective variety over a field k, with a fixed very ample sheaf $\mathcal{O}_X(1)$. Let $\mathcal{F}$ be a coherent sheaf on X, and let $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_X(n)$. Let $m \in \mathbb{Z}$. Recall ([27], Lecture 14, p. 99) that $\mathcal{F}$ is called *m-regular* if $H^i(X, \mathcal{F}(m-i)) = 0$ for all $i \geq 1$.

Assume that $\mathcal{F}$ is m-regular. Then it is known (see, e.g., [34], Proposition 4.1.1) that for all $n \geq m$,

(a) $\mathcal{F}$ is n-regular,
(b) $H^i(X, \mathcal{F}(n)) = 0$ for all $i \geq 1$,
(c) $\mathcal{F}(n)$ is generated by its global sections, and
(d) The canonical homomorphism

$$H^0(X, \mathcal{F}(n)) \otimes H^0(X, \mathcal{O}_X(1)) \longrightarrow H^0(X, \mathcal{F}(n+1))$$

is surjective.

Lemma 4.2. *Let X be a projective variety over a field k with a fixed very ample sheaf $\mathcal{O}_X(1)$. Let C be a finite closed subscheme of X. Let $\Gamma_1, \dots, \Gamma_r$ be irreducible closed subsets of X not contained in C. Let $\mathcal{J}$ denote the ideal sheaf of C in X, and assume that $\mathcal{J}$ is m-regular for some $m \geq 0$.*

(a) *If $\mathrm{Card}(k) > r + \mathrm{Card}(C)$, then for all $n \geq m$, there exists a section $f_n \in H^0(X, \mathcal{J}(n))$ such that $V_+(f_n)$ does not contain any Γ_i, and such that the image of f_n in $(\mathcal{J}(n)/\mathcal{J}^2(n)) \otimes k(x)$ is non-zero for all $x \in C$ such that $(\mathcal{J}(n)/\mathcal{J}^2(n)) \otimes k(x) \neq (0)$.*
(b) *There exists an integer $n_0 > 0$ such that for all $n \geq n_0$, there exists a section $f_n \in H^0(X, \mathcal{J}(n))$ as in (a).*

Proof. (a) Let $x \in C$ and $n \geq m$, and consider the k-linear map $H^0(X, \mathcal{J}(n)) \to (\mathcal{J}(n)/\mathcal{J}^2(n))_x$. Since $\mathcal{J}(n)$ is generated by its global sections (4.1 (c)), this map is not zero for all $x \in C$ such that $(\mathcal{J}(n)/\mathcal{J}^2(n))_x \neq (0)$. Let H_x denote its kernel. Then $H_x \neq H^0(X, \mathcal{J}(n))$ for all $x \in C$ such that $(\mathcal{J}(n)/\mathcal{J}^2(n))_x \neq (0)$.

Use now a closed embedding $X \to \mathbb{P}^d_k$ associated with the very ample sheaf $\mathcal{O}_X(1)$ to identify X with $\mathrm{Proj}\, B$, where $B = \oplus_{n \geq 0} B(n)$ is a graded k-algebra isomorphic to $k[x_0, \dots, x_d]$ modulo a homogeneous ideal. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the homogeneous prime ideals of B defining $\Gamma_1, \dots, \Gamma_r$. Under the appropriate identification, $\oplus_{n \geq 0} H^0(X, \mathcal{J}(n))$ is identified with a homogeneous ideal $\oplus_{n \geq 0} J(n)$ of B corresponding to the closed subscheme C. Neither $\mathfrak{p}_i$ nor $\oplus_{n \geq 0} J(n)$ contain $B(1)$. Since Γ_i is not contained in C, we find that $\oplus_{n \geq 0} J(n)$ is not contained in $\mathfrak{p}_i$. Then there exists $n_0 \geq 0$ such that each $J(n)$ with $n \geq n_0$ is not contained in $\mathfrak{p}_i$ (4.3 (a)). We claim that for each $i \leq r$, and for each $n \geq m$, the preimage in $H^0(X, \mathcal{J}(n))$ of $J(n) \cap \mathfrak{p}_i$ is a proper subvector space of $H^0(X, \mathcal{J}(n))$. Indeed, if $J(n) \cap \mathfrak{p}_i = J(n)$ for some $n \geq m$, then the surjectivity of the map in 4.1 (d) implies that $J(n') \cap \mathfrak{p}_i = J(n')$ for all $n' \geq n$. This would contradict the existence of n_0.

For each $n \geq m$, we have thus constructed $r + \mathrm{Card}(C)$ subvector spaces of $H^0(X, \mathcal{J}(n))$. Any element f_n in the complement of the union of these subvector spaces satisfies the desired properties. Since $r + \mathrm{Card}(C) < \mathrm{Card}(k)$ by hypothesis, the union cannot be the whole space $H^0(X, \mathcal{J}(n))$, and (a) follows.

(b) Choose $m \geq 0$ large enough such that both $\mathcal{J}$ and $\mathcal{J}^2$ are m-regular. As $H^1(X, \mathcal{J}^2(n)) = (0)$ for $n \geq m$ by 4.1 (b), the map

$$H^0(X, \mathcal{J}(n)) \longrightarrow H^0(X, \mathcal{J}(n)/\mathcal{J}^2(n)) = \oplus_{x \in C} (\mathcal{J}(n)/\mathcal{J}^2(n))_x$$

is surjective for all $n \geq m$. Let then $f \in H^0(X, \mathcal{J}(m))$ be a section such that its image in $(\mathcal{J}(m)/\mathcal{J}^2(m)) \otimes k(x)$ is non-zero for each $x \in C$ such that $(\mathcal{J}(m)/\mathcal{J}^2(m)) \otimes k(x) \neq (0)$. Let $I := \oplus_{n \geq 0} H^0(X, \mathcal{J}^2(n))$. Keep the notation introduced in (a), and identify I with a homogeneous ideal of B. Then $I \not\subseteq \mathfrak{p}_i$ for all $i \leq r$, since otherwise $\oplus_{n\geq 0} H^0(X, \mathcal{J}(n)) \subseteq \mathfrak{p}_i$, which contradicts the hypothesis that Γ_i is not contained in C. Let $\mathfrak{p}_{r+1}, \dots, \mathfrak{p}_s$ be the homogeneous ideals of B corresponding to the (reduced) points of C.

Lemma 4.3 (b) below implies then the existence of $n_0 \geq 0$ such that for all $n \geq n_0$, there exist $x_n \in H^0(X, \mathcal{J}^2(n))$ and $g_n \in H^0(X, \mathcal{O}_X(n-m)) \setminus \cup_{1\leq i\leq s}\mathfrak{p}_i$ such that $fg_n + x_n \notin \cup_{1\leq i\leq r}\mathfrak{p}_i$. Clearly, $f_n := fg_n + x_n \in H^0(X, \mathcal{J}(n))$. As g_n is invertible in $\mathcal{O}_X(n-m)_x$, f_n is non-zero in $(\mathcal{J}(n)/\mathcal{J}^2(n)) \otimes k(x)$ for all $x \in C$. □

The following Prime Avoidance Lemma for graded rings is needed in the proof of 4.2. (For related statements, see [36], Theorem A.1.2., or [3], III, 1.4, Prop. 8, page 161.)

Lemma 4.3. *Let $B = \oplus_{n\geq 0} B(n)$ be a graded ring. Let $I = \oplus_{n\geq 0} I(n)$ be a homogeneous ideal of B. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be homogeneous prime ideals of B not containing $B(1)$ and not containing I.*

(a) *Then there exists an integer $n_0 \geq 1$ such that for all $n \geq n_0$, $I(n) \not\subseteq \cup_{1\leq i\leq r}\mathfrak{p}_i$.*
(b) *Let $f \in B(m)$ for some $m \geq 1$. Assume in addition that $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ do not contain $fB + I$. Let $s \geq r$, and when $s > r$, let $\mathfrak{p}_{r+1}, \dots, \mathfrak{p}_s$ be additional homogeneous prime ideals not containing $B(1)$. Then there exists $n_0 \geq 1$ such that for all $n \geq n_0$, there are $g_n \in B(n-m) \setminus \cup_{1\leq i\leq s}\mathfrak{p}_i$ and $x_n \in I(n)$ with $fg_n + x_n \notin \cup_{1\leq i\leq r}\mathfrak{p}_i$.*

Proof. (a) We proceed by induction on r. If $r = 1$, choose $t \in B(1) \setminus \mathfrak{p}_1$ and a homogeneous element $\alpha \in I \setminus \mathfrak{p}_1$, say of degree n_0. Then $t^{n-n_0}\alpha \in I(n) \setminus \mathfrak{p}_1$ for all $n \geq n_0$, as desired. Let $r \geq 2$ and suppose that the lemma is true for $r-1$. We can suppose that $\mathfrak{p}_i$ is not contained in $\mathfrak{p}_r$ for all $i \neq r$, so that $I\mathfrak{p}_1 \cdots \mathfrak{p}_{r-1} \not\subseteq \mathfrak{p}_r$. Similarly, we can suppose that $\mathfrak{p}_r$ is not contained in $\mathfrak{p}_i$ for all $i \neq r$, so that $I\mathfrak{p}_r \not\subseteq (\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_{r-1})$. Hence, we can apply the case $r = 1$ and the induction hypothesis to obtain that there exists n_0 such that for all $n \geq n_0$, there are homogeneous elements $f_n \in I\mathfrak{p}_1 \cdots \mathfrak{p}_{r-1} \setminus \mathfrak{p}_r$ and $g_n \in I\mathfrak{p}_r \setminus (\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_{r-1})$ of degree n. It is easy to check that $f_n + g_n \in I(n) \setminus \cup_{1\leq i\leq r}\mathfrak{p}_i$, as desired.

(b) As $\mathfrak{p}_i$ does not contain $B(1)$ for all $i \leq s$, we can find $n_1 \geq 0$ such that for all $n \geq n_1$, there exists $g_n \in B(n-m) \setminus \cup_{1\leq i\leq s}\mathfrak{p}_i$ (use (a) with $I = B$). We proceed by induction on r. First suppose that $r = 1$. If $f \notin \mathfrak{p}_1$, then the property is true with $x_n = 0$. Suppose then that $f \in \mathfrak{p}_1$. Then $I \not\subseteq \mathfrak{p}_1$ since $\mathfrak{p}_1$ does not contain $fB + I$ by hypothesis. Choose $t \in B(1) \setminus \mathfrak{p}_1$, and choose a homogeneous element $\alpha \in I \setminus \mathfrak{p}_1$, of some degree n_2. Set $n_0 := \max(n_1, n_2)$. Then $fg_n + t^{n-n_2}\alpha \notin \mathfrak{p}_1$ for all $n \geq n_0$.

Assume now that $r \geq 2$. We can suppose that there is no inclusion relation between $\mathfrak{p}_1, \dots, \mathfrak{p}_r$. By induction hypothesis, there exists $n_1 \geq 0$ such that for all $n \geq n_1$, there exists $x_n \in I(n)$ such that $fg_n + x_n \notin \cup_{i\leq r-1}\mathfrak{p}_i$. If n is such that $fg_n + x_n \notin \mathfrak{p}_r$, we are done. Suppose that n is such that $fg_n + x_n \in \mathfrak{p}_r$. Then $I \not\subseteq \mathfrak{p}_r$ since $\mathfrak{p}_r$ does not contain $fB + I$. It follows that $I\mathfrak{p}_1 \cdots \mathfrak{p}_{r-1} \not\subseteq \mathfrak{p}_r$. Hence, there exists $n_2 \geq 0$ such that for each $n \geq n_2$, there exists $y_n \in (I\mathfrak{p}_1 \cdots \mathfrak{p}_{r-1})(n) \setminus \mathfrak{p}_r$ (use (a)). Then for any $n \geq \max(n_1, n_2)$, $fg_n + (x_n + y_n) \notin \cup_{1\leq i\leq r}\mathfrak{p}_i$, as desired. □

Keep the notation introduced in 3.5.

Lemma 4.4. *Let S be a noetherian affine scheme, and let $X \to S$ be projective. Let $\mathcal{O}_X(1)$ be a relatively very ample sheaf for $X \to S$. Let $C := V(\mathcal{J})$ be a closed subscheme of X, finite over S. Then there exists $n_0 \geq 0$ such that for all $n \geq n_0$ and for all $s \in S$, the sheaves $\mathcal{J}_s(n)$ and $\mathcal{J}_s^2(n)$ are n-regular, and the canonical maps*

$$H^0(X, \mathcal{J}(n)) \otimes k(s) \longrightarrow H^0(X_s, \mathcal{J}_s(n)) \longrightarrow H^0(X_s, \bar{\mathcal{J}}_s(n))$$

are surjective.

Let F be a closed subscheme of X. Then, for each $s \in S$ and $n \geq n_0$, there exists $f_{s,n} \in H^0(X, \mathcal{J}(n))$, whose image in $(\bar{\mathcal{J}}_s(n)/\bar{\mathcal{J}}_s^2(n)) \otimes k(x)$ is non-zero for all $x \in C$ with $(\bar{\mathcal{J}}_s(n)/\bar{\mathcal{J}}_s^2(n)) \otimes k(x) \neq (0)$, and whose zero locus $V_+(f_{s,n})$ in X does not contain any irreducible component of F_s of positive dimension.

Proof. The following results are found in [34], step 3 in the proof of Theorem 4.2.11. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r_S$. Then there exists an integer N such that for all $n \geq N$ and for every $s \in S$, $H^j(\mathbb{P}^r_{k(s)}, \mathcal{F}_s(n)) = (0)$ for $j \geq 1$. Moreover, letting $p : \mathbb{P}^r_S \to S$ denote the canonical projection map, then the natural map of $k(s)$-vector spaces $p_*(\mathcal{F}(n)) \otimes k(s) \to H^0(\mathbb{P}^r_{k(s)}, \mathcal{F}_s(n))$ is an isomorphism.

We use these facts in the following context. Let $\mathcal{G}$ be any coherent sheaf on X. Associated with the very ample sheaf $\mathcal{O}_X(1)$ is an embedding $i : X \to \mathbb{P}^r_S$ for some r. Since $H^i(X_s, \mathcal{G}_s) = (0)$ for all $i \geq \dim(X)$, we find from the above facts applied to $\mathcal{F} := i_*\mathcal{G}$ that $\mathcal{G}_s$ is $(r+N)$-regular for all $s \in S$. It follows that there exists m such that for all $n \geq m$ and for all $s \in S$, $\mathcal{J}_s(n)$ and $\mathcal{J}_s^2(n)$ are n-regular, and $H^0(X, \mathcal{J}(n)) \otimes k(s) \to H^0(X_s, \mathcal{J}_s(n))$ is surjective.

Let $s \in S$ and recall that $\bar{\mathcal{J}}_s$ is the image of $\mathcal{J}_s \to \mathcal{O}_{X_s}$. As the kernel $\mathcal{K}_s$ of $\mathcal{J}_s \to \bar{\mathcal{J}}_s$ is coherent and supported on the affine (finite) subscheme C_s, we find that $H^j(X_s, \mathcal{K}_s(n)) = (0)$ for $j \geq 1$, and $H^j(X_s, \mathcal{J}_s(n)) \to H^j(X_s, \bar{\mathcal{J}}_s(n))$ is surjective for all n and all $j \geq 0$. It follows that since $\mathcal{J}_s$ is m-regular, then so is $\bar{\mathcal{J}}_s$.

We see by [11], IV.9.7.8, that there exists a positive integer c_0 such that for all $s \in S$, C_s and F_s each have at most c_0 irreducible components. Let

$$Z_0 = \{s \in S \mid \operatorname{Card}(k(s)) \leq 2c_0\}.$$

Lemma 2.4 shows that Z_0 is a finite set.

Let $s \notin Z_0$, and let $\Gamma_1, \dots, \Gamma_{r_s}$ denote the irreducible components of F_s of positive dimension. Then the existence for all $n \geq m$ of the desired $f_{s,n}$ follows from Lemma 4.2(a) applied to $\bar{\mathcal{J}}_s$ and the fact that $H^0(X, \mathcal{J}(n)) \otimes k(s) \to H^0(X_s, \bar{\mathcal{J}}_s(n))$ is surjective. For the finitely many $s \in Z_0$, the existence of $f_{s,n}$ for all n large enough follows from 4.2(b). □

5. Hypersurfaces

Our main result in this section is Theorem 5.3, a variant on Theorem 3.1. Let S be an affine scheme and let $X \to S$ be a quasi-projective morphism. Let $\mathcal{O}_X(1)$ denote a very ample invertible sheaf on X relatively to $X \to S$. Let $n \geq 1$, and let f be a global section of $\mathcal{O}_X(n)$.

For convenience, we will call the closed subscheme H_f of X defined by the ideal sheaf $\mathcal{I} := f\mathcal{O}_X \otimes \mathcal{O}_X(-n)$ a *hypersurface* when no irreducible component of positive dimension of X_s is contained in H_f, for all $s \in S$. If, moreover, the ideal sheaf $\mathcal{I}$ is invertible, we will say that the hypersurface H_f is *locally principal.* Note that when $\mathcal{I}$ is invertible, H_f is the support of an effective Cartier divisor on X.

Lemma 5.1. *Let S be an affine scheme. Let $X \to S$ be a quasi-projective morphism and let $\mathcal{O}_X(1)$ be a relatively very ample sheaf on X. Let H be a hypersurface of X as above.*

(1) *If $\dim X_s \geq 1$, then either H_s is empty, or $\dim H_s \leq \dim X_s - 1$. If, moreover, $X \to S$ is projective, then H_s meets every irreducible component of positive dimension of X_s, and in particular $\dim H_s = \dim X_s - 1$.*
(2) *If $X \to S$ is finitely presented, then so is $H \to S$.*
(3) *Assume that S is noetherian and that $X \to S$ is flat. Then H is locally principal and flat over S if and only if for all $s \in S$, H does not contain any embedded point of X_s.*
(4) *Assume S noetherian. If H does not contain any associated point of X, then H is locally principal.*

Proof. (1) By hypothesis, H_s does not contain any irreducible component of X_s of positive dimension. Since H_s is locally defined by one equation, we obtain that $\dim H_s \leq \dim X_s - 1$. The strict inequality may occur for instance in case $\dim X_s \geq 2$, and H_s does not meet any component of X_s of maximal dimension. Assume now that X_s is projective. Let f denote the section of $\mathcal{O}_X(n)$ defining the open set X_f such that $H = X \setminus X_f$. Then $X_f \cap X_s$ is affine and, thus, can only contain an irreducible component of X_s of dimension 0.

(2) Results from the fact that H is locally defined by a single equation in X.

(3) See [11], IV.11.3.8, and use the fact that in a noetherian ring R, an element is regular if and only if it is not contained in any associated prime. Since by hypothesis, H is a hypersurface, it does not contain any generic point of any component of any fiber X_s.

(4) Let $f \in H^0(X, \mathcal{O}_X(n))$ be such that $H = H_f$. Cover X by affine open subsets U_i such that $\mathcal{O}_X(n)|_{U_i}$ is free for all $i = 1, \ldots, m$. Choose a basis e_i of $\mathcal{O}_X(n)|_{U_i}$ for each i, and write $\mathcal{O}_X(n)|_{U_i} = e_i\mathcal{O}_X$. Then $f|_{U_i} = e_i h_i$ for some $h_i \in \mathcal{O}_X(U_i)$, and $H \cap U_i = V(h_i)$. The hypothesis $H \cap \mathrm{Ass}(X) = \emptyset$ implies that h_i is a regular element of $\mathcal{O}_X(U_i)$ ([24], (7.B), Corollary 2, where the noetherian hypothesis is used). The system $\{(U_i, h_i)\}_i$ defines an effective Cartier divisor D whose support is H, and the ideal sheaf $\mathcal{I}$ defining the closed subscheme H is isomorphic to the ideal sheaf defined on each U_i by $\mathcal{O}_X(U_i)h_i$. In other words, $\mathcal{I} \simeq \mathcal{O}_X(-D)$. □

Theorem 5.2. *Let S be an affine scheme, and let $X \to S$ be a quasi-projective and finitely presented morphism. Let C be a closed subscheme of X, proper and finitely presented over S. Let A be a finite subset of X such that $A \cap C = \emptyset$. Let F be a closed subset of X defined by an ideal sheaf of finite presentation. Assume that for all $s \in S$, C does not contain any irreducible component of positive dimension of F_s and of X_s.*

Fix an ample invertible sheaf $\mathcal{O}_X(1)$ on X. Then there exist $n > 0$ and a global section f of $\mathcal{O}_X(n)$ such that the hypersurface H_f of X contains C as a closed subscheme, $A \cap H_f = \emptyset$, and H_f does not contain any irreducible component of positive dimension of F_s.

Proof. Without loss of generality, we may assume that $\mathcal{O}_X(1)$ is a relatively very ample invertible sheaf on X. Let us first show that it suffices to prove the theorem when the base S is noetherian and of finite dimension. For this, we will use the machinery developed in [11], IV.8 (see in particular the *scholie* IV.8.8.3). Using [11], IV.8.9.1 and IV.8.10.5, we find the existence of an affine scheme S_0 of finite type over $\mathbb{Z}$, and of a morphism $S \to S_0$ such that all the objects of Theorem 5.2 descend to S_0. More precisely, there exists a quasi-projective scheme $X_0 \to S_0$ such that X is isomorphic to $X_0 \times_{S_0} S$. We will denote by $p : X \to X_0$ the associated 'first projection' morphism. There also exists an invertible relatively very ample sheaf $\mathcal{O}_{X_0}(1)$ on X_0 whose pull-back to X is $\mathcal{O}_X(1)$. There exists a closed subscheme C_0 of X_0, proper over S_0, such that C is isomorphic to $C_0 \times_{S_0} S$. Finally, there exists a closed subset F_0 of X_0 such that F is isomorphic to $F_0 \times_{S_0} S$. Let $A_0 := p(A)$.

Since S_0 is of finite type over $\mathbb{Z}$, S_0 is noetherian and of finite dimension. Suppose the theorem proven for X_0, $\mathcal{O}_{X_0}(1)$, F_0, C_0, and A_0, and let H_0 denote the hypersurface of X_0 as in the conclusion of the theorem. Let $H := H_0 \times_{X_0} X$. Then H is a hypersurface in X satisfying the required properties with respect to $\mathcal{O}_X(1)$, F, C, and A. (Note that for all $s \in S$, if $s_0 \in S_0$ is the image of s, then the image of an irreducible component of F_s in X_0 is an irreducible component of $(F_0)_{s_0}$ of the same dimension.)

We suppose from now on that S is noetherian and of finite dimension. Using the same arguments as in the proof of Theorem 3.1 in 3.10, we reduce further to the case where $X \to S$ is projective.

Assume now that $X \to S$ is projective. Let $\mathcal{J} \subseteq \mathcal{O}_X$ denote the ideal sheaf of C. Let $n \geq 1$, and fix generators $f_1, \dots, f_N$, for the finitely generated $\mathcal{O}_S(S)$-module $H^0(X, \mathcal{J}(n))$. As in the proof of Theorem 3.1, we will exhibit an integer n and a section $f = \sum_{i=1}^N a_i f_i$, $a_i \in \mathcal{O}_S(S)$, producing H_f with the required properties, using Proposition 2.3. The element $(a_1, \dots, a_N)$ will correspond to a section of $\mathbb{A}^N_S/S$ which does not meet both a constructible set T and an auxiliary set B, sets which we now describe.

As in the proof of 3.7, we note that since $F \to S$ is proper, Zariski's Main Theorem implies that the subset of the points of F which are isolated in their fibers is open in F. Let $\mathbf{F}$ denote the complement of this open subset in F. Then the fibers of $\mathbf{F} \to S$ have no isolated points and, hence, have no zero-dimensional irreducible components. Consider the coherent sheaf $\mathcal{F} := \mathcal{J}(n)|_{\mathbf{F}}$ on $\mathbf{F}$, and note that the support of $\mathcal{F}$ is $\mathbf{F}$. Define for each $s \in S$ as in 2.1 the set $\Sigma_1(s)$ associated with $Z := \mathbf{F}$ and $\mathcal{F} := \mathcal{J}(n)|_{\mathbf{F}}$:

$$\Sigma_1(s) := \left\{ (\alpha_1, \dots, \alpha_N) \in k(s)^N \;\middle|\; \sum_i \alpha_i f_{i,s} \text{ vanishes at some generic point of } Z_s \right\}.$$

When $Z_s = \emptyset$, we set $\Sigma_1(s) := \emptyset$. Let $c := \dim(S) + 2$ in Lemma 3.7, and apply 3.7 to $Z := \mathbf{F}$ and $\mathcal{F} := \mathcal{J}(n)|_{\mathbf{F}}$. It follows that for all $n \geq n_0$, there exists a constructible subset T_1 of $\mathbb{A}^N_S$ such that for all $s \in S$, $\Sigma_1(s)$ is exactly the set of $k(s)$-rational points of $(\mathbb{A}^N_S)_s$ contained in $(T_1)_s$, and such that $\dim(T_1)_s \leq N - 2 - \dim S$. In particular, $\dim(T_1) \leq N - 1 < N$.

We now repeat the construction above with the set X instead of F, obtaining first $\mathbf{X}$ with the sheaf $\mathcal{J}(n)|_{\mathbf{X}}$, and then applying 3.7 to obtain the existence for all

$n \geq n_1 \geq n_0$ of a constructible subset $T_2 \in \mathbb{A}^N_S$ with $\dim(T_2) < N$ associated with the collection of sets $\Sigma_2(s)$. We let $T := T_1 \cup T_2$.

Let us now define the subset $B = \cup_{s \in S} B_s$. Let $s \in S$. If $X_s \cap A = \emptyset$, set $B_s := \emptyset$. If $X_s \cap A$ is not empty, consider the set D_s of all $(\alpha_1, \dots, \alpha_N) \in k(s)^N$ such that $\sum_{1 \leq i \leq N} \alpha_i f_{i,s}$ vanishes at some point of $X_s \cap A$. Our hypothesis that $C_s \cap A = \emptyset$ implies that D_s is contained in a proper closed hypersurface of $\mathbb{A}^N_{k(s)}$, since for any point $x \in A_s \setminus C_s$, the sheaf $(\mathcal{J}(n)|_{X_s})_x$ is not trivial and, hence, $\mathcal{J}(n)|_{X_s}$ is not trivial in an open set of X_s containing x. We denote by B_s such a hypersurface.

By construction, the sets T and B (for all n large enough) satisfy Condition (1) in the statement of Proposition 2.3. To be able to apply 2.3, we show now that Condition (2) is also satisfied for some n large enough, namely, that for all $s \in S$, $T_s \cup B_s$ does not contain all the $k(s)$-rational points of $\mathbb{A}^N_{k(s)}$. We proceed as in 4.4. (The arguments that follow also give a direct proof of the theorem when S is the spectrum of a field, using a variation on the Avoiding Lemma 4.3(a).)

By [11], IV.9.7.8, there exists a positive integer c_0 such that for all $s \in S$, F_s, X_s, and A_s each have at most c_0 irreducible components. Let

$$Z_0 = \{s \in S \mid \mathrm{Card}(k(s)) \leq 3c_0\}.$$

Lemma 2.4 shows that Z_0 is a finite set.

Let $s \notin Z_0$, and let $\Gamma_1, \dots, \Gamma_{r_s}$ denote the irreducible components of F_s and of X_s of positive dimension, and the closures in X_s of the points of A_s. Then the existence for all $n \geq m$ of a $k(s)$-rational point of $\mathbb{A}^N_{k(s)}$ not contained in $T_s \cup B_s$ follows from a variation on Lemma 4.2(a) applied to $\bar{\mathcal{J}}_s$ and the fact that $H^0(X, \mathcal{J}(n)) \otimes k(s) \to H^0(X_s, \bar{\mathcal{J}}_s(n))$ is surjective. For the finitely many $s \in Z_0$, the existence of the desired $k(s)$-rational point for all n large enough follows from a variation on 4.2(b).

We can thus apply Proposition 2.3 to find, for some n large enough, a section $(a_1, \dots, a_N) \in \mathbb{A}^N_S(S)$ such that for all $s \in S$, $(a_1(s), \dots, a_N(s))$ is not a $k(s)$-rational point of $(\mathbb{A}^N_S)_s$ contained in $T_s \cup B_s$. Let $f := \sum_{i=1}^N a_i f_i \in H^0(X, \mathcal{J}(n))$, and consider the closed subscheme $V_+(f) \subset X$. By definition of f, we have $C \subset V_+(f)$, and by definition of T_1, T_2, and B, we have:

(i) for all $s \in S$, $V_+(f)$ does not contain any irreducible component of F_s of positive dimension,
(ii) for all $s \in S$, $V_+(f)$ does not contain any irreducible component of X_s of positive dimension,
(iii) $V_+(f)$ does not contain any point of A,

as desired. □

Theorem 5.3. *Let S be an affine scheme and let $X \to S$ be a quasi-projective and finitely presented morphism. Let C be a closed subscheme of X, proper and finitely presented over S. Let F be a closed subset of X, such that when F is endowed with the induced reduced structure of scheme, the morphism $F \to S$ is finitely presented. Assume that for all $s \in S$, C_s does not contain any irreducible component of positive dimension of F_s and of X_s.*

Fix an ample invertible sheaf $\mathcal{O}_X(1)$ on X. Then there exist $n > 0$ and a global section f of $\mathcal{O}_X(n)$ such that

(1) *C is a closed subscheme of H_f,*

(2) *H_f does not contain any irreducible component of positive dimension of F_s and of X_s, for all $s \in S$.*

Assume in addition that S is noetherian, and that $C \cap \mathrm{Ass}(X) = \emptyset$. Then there exists such a hypersurface H_f which is locally principal.

Proof. Apply 5.2 with $A = \mathrm{Ass}(X)$. The fact that H_f is locally principal when $H_f \cap \mathrm{Ass}(X) = \emptyset$ is noted in 5.1, (4). □

Remark 5.4 It is classical that if X/k is a projective scheme over a field, $C \subset X$ a proper closed subset, and $\xi_1, \dots, \xi_r$ are points of X not contained in C, then there exists a hypersurface H in X such that $C \subset H$ and $\xi_1, \dots, \xi_r \notin H$.

Let S be a noetherian scheme, and let X/S be a projective morphism. Let $F_1, \dots, F_r$ be finitely many irreducible closed subsets of X, finite and surjective over S. It is not always possible to find a hypersurface H in X which does not intersect $\cup_{i=1}^r F_i$. The reader can use 7.7 to produce an example of the form $X = \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})$ with $\mathcal{L}$ a non-trivial invertible sheaf of finite order. This shows that in general Condition (2) in 5.3 cannot be strengthened by asserting that $H_f \cap F_s = \emptyset$ for all $s \in S$.

Recall that a locally noetherian scheme Z is (S_ℓ) for some integer $\ell \geq 0$ if for all $z \in Z$, the depth of $\mathcal{O}_{Z,z}$ is at least equal to $\inf\{\ell, \dim \mathcal{O}_{Z,z}\}$ ([11], IV.5.7.2).

Corollary 5.5. *Let S be an affine scheme, and let $X \to S$ be a quasi-projective and finitely presented morphism. Let C be a closed subscheme of X, proper and finitely presented over S. Assume that for all $s \in S$, C does not contain any irreducible component of positive dimension of X_s. Suppose that for some $\ell \geq 1$, X_s is (S_ℓ) for all $s \in S$.*

Then there exists a hypersurface $H \subset X$ such that C is a closed subscheme of H, and the fibers of $H \to S$ are $(\mathrm{S}_{\ell-1})$. In particular, if the fibers of $X \to S$ are Cohen-Macaulay, then the same is true of the fibers of $H \to S$.

Assume that S is noetherian and $\mathrm{Ass}(X) \cap C = \emptyset$. Then we may in addition assume H to be locally principal, and if $X \to S$ is flat, then $H \to S$ can be assumed to be flat.

Proof. We apply Theorem 5.2 to $X \to S$ and C, with $F = X$, and with $A = \mathrm{Ass}(X)$ if the latter is finite and $A = \emptyset$ otherwise. Let H be a hypersurface in X as given by 5.2. For all $s \in S$, H_s does not contain any irreducible component of X_s. Since X_s is (S_ℓ) with $\ell \geq 1$, X_s has no embedded points. It follows that H_s is locally generated everywhere by a regular element and, thus, H_s is $(\mathrm{S}_{\ell-1})$. This also implies that $H \to S$ is flat when $X \to S$ is flat (5.1, (3) and (4)). □

Remark 5.6 Let S be an affine scheme and let $X \to S$ be projective and smooth. Fix an ample invertible sheaf $\mathcal{O}_X(1)$ on X as in Theorem 5.3. It is not possible in general to find $n > 0$ and a global section $f \in \mathcal{O}_X(n)$ such that $H_f \to S$ is smooth. Examples of N. Fakhruddin illustrating this point can be found in [30], 5.14 and 5.15.

Corollary 5.7. *Let S be an affine noetherian irreducible scheme of dimension* 1. *Let $X \to S$ be quasi-projective and flat, and assume that its generic fiber is* (S_1). *Then there exists a locally principal hypersurface $H \subset X$, flat over S.*

Proof. We have to find a hypersurface H which avoids both the finite set $\mathrm{Ass}(X)$ (5.3), and the set M of all $x \in X$ such that x is an embedded point of $X_{\pi(x)}$ (5.1 (4)). For all $s \in S$, the set of embedded points of X_s is finite. By [11], IV.9.9.2(viii), M is constructible, and by hypothesis, it does not meet the generic fiber of $X \to S$. Since the image of a constructible set is constructible, and S is irreducible of dimension 1, the image of M is a finite set. Therefore, M is finite. Now apply 5.2 with $F = C = \emptyset$ and $A = M \cup \mathrm{Ass}(X)$. □

6. Finite quasi-sections

Let $X \to S$ be a surjective morphism. We call a closed subscheme T of X a *finite quasi-section* when $T \to S$ is finite and surjective. We establish in 6.2 the existence of a finite quasi-section for certain types of projective morphisms. The existence of quasi-finite quasi-sections for flat or smooth morphisms is discussed in [11], 17.16.

When $\dim S = 1$ and $X \to S$ is proper, the existence of a finite quasi-section T is well-known and easy to establish. It suffices to take T to be the Zariski closure of a closed point of the generic fiber of $X \to S$. When $\dim S > 1$, the process of taking the closure of a closed point of the generic fiber does not always produce a closed subset *finite* over S, as the simple example below shows.

Example 6.1 Let $S = \operatorname{Spec} A$ with A a noetherian integral domain, and let $K = \mathrm{Frac}(A)$. Let $X = \mathbb{P}^1_A$. Choose coordinates and write $X = \mathrm{Proj}\ A[t_0, t_1]$. Let $P \in X_K(K)$ be given as $(a : b)$, with $a, b \in A \setminus \{0\}$. When $(bt_0 - at_1)$ is a prime ideal in $A[t_0, t_1]$, then $T := V_+(bt_0 - at_1)$ is the Zariski closure of P in X. When in addition $aA + bA \neq A$, T is not finite over S. For a concrete example with S regular of dimension 2, take k a field and $A = k[t, s]$, with $a = t$, and $b = s$. (Note that when $\dim(A) = 1$ and $aA + bA \neq A$, the ideal $(bt_0 - at_1)$ is never prime in $A[t_0, t_1]$).

More generally, to produce K-rational points on the generic fiber of $\mathbb{P}^n_S \to S$ for some $n > 1$ whose closure is not finite over S, we can proceed as follows. Let $T \to S$ be the blowing-up of S with respect to a coherent sheaf of ideals I, and choose I so that $T \to S$ is not finite. Then $T \to S$ is a projective morphism, and we can choose $T \to \mathbb{P}^n_S$ to be a closed immersion over S for some $n > 0$. Let ξ denote the generic point of the image of T in $X := \mathbb{P}^n_S$. Then ξ is a closed point of the generic fiber of $X \to S$, and the closure of ξ in X is not finite over S.

Theorem 6.2. *Let S be an affine scheme and let $X \to S$ be a projective, finitely presented morphism. Suppose that all fibers of $X \to S$ are of the same dimension $d \geq 0$. Let C be a finitely presented closed subscheme of X, with $C \to S$ finite but not necessarily surjective. Then there exists a finite quasi-section T which contains C. Assume now that S is noetherian. Then:*

1. *If C and X are irreducible, then there exists an irreducible finite quasi-section T which contains C.*
2. *If $X \to S$ is flat with Cohen-Macaulay fibers (e.g., if S is regular and X is Cohen-Macaulay), then there exists a finite quasi-section containing C which is flat over S.*

(3) *If $X \to S$ is flat and a local complete intersection morphism*[3]*, then there exists a finite quasi-section T containing C with $T \to S$ flat and a local complete intersection morphism.*

Proof. To prove the first conclusion of the theorem, it suffices to show that X/S has a finite quasi-section T. Then $T \cup C$ is a finite quasi-section which contains C. If $d = 0$, then $X \to S$ itself is finite. Suppose $d \geq 1$. It follows from Theorem 5.2, with $A = \emptyset$ and $F = X$, that there exists a hypersurface H in X. By definition of a hypersurface, for all $s \in S$, H_s does not contain any irreducible component of X_s of positive dimension. Lemma 5.1(1) and our hypotheses show that every fiber H_s has dimension $d-1$. Lemma 5.1(2) shows that H/S is also finitely presented. Repeating this process another $d-1$ times produces the desired quasi-section.

Assume now S noetherian, and let us prove (1). Note that since X is irreducible and the fibers of $X \to S$ are all not empty by hypothesis, then $X \to S$ is surjective and S is irreducible. When $d = 0$, $X \to S$ is then an irreducible finite quasi-section, and contains C. Assume now $d \geq 1$. Then we can find a hypersurface H_f which contains C. Using 6.3 below and the assumption that C is irreducible, we can find an irreducible component Γ of H_f which contains C, dominates S, and such that all fibers of $\Gamma \to S$ have dimension $d-1$. If $d-1 > 0$, we repeat the process.

Part (2) is proved similarly to the first statement of the theorem, using Corollary 5.5. Note that when $d = 0$, the statement is obvious. When $d > 0$, we find that since $X \to S$ is flat with Cohen-Macaulay fibers of positive dimension, the depth formula [11], IV.6.3.1, implies that $\mathrm{Ass}(X)$ consists of the generic points of X_s for each $s \in \mathrm{Ass}(S)$. Since $C \to S$ is finite, we find that $\mathrm{Ass}(X) \cap C = \emptyset$.

To prove (3), note that as $X \to S$ is flat and l.c.i, the fibers X_s are Cohen-Macaulay and, hence, have no embedded points. Let H be a locally principal hypersurface in X containing C, as in 5.5. By 5.1(3), H is flat over S. Then H_s does not contain any associated point of X_s. By 5.1(4), H_s is locally principal in X_s and, hence, l.c.i. over $k(s)$. Therefore, $H \to S$ is flat and l.c.i. Repeating this process another $d-1$ times produces the desired quasi-section. □

Lemma 6.3. *Let S be noetherian and irreducible. Let $\pi : X \to S$ be a projective morphism with $\dim(X_s) > 0$ for all $s \in S$. Let $H \subset X$ be a hypersurface. If all irreducible components of X dominate S, then all irreducible components of H dominate S. Assume in addition that for each irreducible component Δ of X, the fibers of $\Delta \to S$ all have dimension $d \geq 1$. Then the fibers of any irreducible component Γ of H all have dimension $d-1$.*

Proof. Let η denote the generic point of S. Let Γ be an irreducible component of H, and let Z denote its image in S. Let t be the generic point of Z. Fix any closed point x of Γ_t, and denote also by x its image in Γ. Assume that x does not belong to any other irreducible components of H, so that $\dim \mathcal{O}_{\Gamma,x} = \dim \mathcal{O}_{H,x}$. We have $\dim \mathcal{O}_{\Gamma,x} = \dim \mathcal{O}_{\Gamma_t,x}$. Because H is a hypersurface and the fibers have positive dimension, we have $\dim \mathcal{O}_{\Gamma_t,x} = \dim \mathcal{O}_{X_t,x} - 1$. We also have $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{\Gamma,x} + 1$. Indeed, for each $x \in H$, $\dim \mathcal{O}_{H,x} \geq \dim \mathcal{O}_{X,x} - 1$, with equality if and only if H does not contain an irreducible component Δ of X passing through

[3]Since the morphism $X \to S$ is flat, it is a local complete intersection morphism if and only if every fiber is a local complete intersection morphism (see, e.g., [21], 6.3.23).

x. But the latter is not possible, since $\Delta_\eta \neq \emptyset$ by hypothesis, and thus Δ_η is an irreducible component of X_η. It follows that

$$\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{\Gamma,x} + 1 = \dim \mathcal{O}_{X_t,x}.$$

Since $\dim \mathcal{O}_{X_t,x} \leq \dim \mathcal{O}_{\pi^{-1}(Z),x}$, and $\dim \mathcal{O}_{X,x} = \operatorname{codim}(\overline{\{x\}}, X)$, we find that

$$\operatorname{codim}(\overline{\{x\}}, X) = \operatorname{codim}(\overline{\{x\}}, \pi^{-1}(Z)).$$

Thus, one irreducible component of X is contained in the closed set $\pi^{-1}(Z)$. Since all irreducible components of X dominate S, we find that $Z = S$.

Now that we know that $\Gamma \to S$ is surjective, we can apply [11], IV.13.1.6, and find that for all $s \in S$, $\dim \Gamma_s \geq \dim \Gamma_\eta$. Suppose now that $\dim X_s = d \geq 1$ for all $s \in S$. Our hypothesis that H is a hypersurface implies that for all $s \in S$, H_s does not contain any irreducible component of X_s of positive dimension. Thus, for all $s \in S$, we have

$$d - 1 = \dim X_s - 1 \geq \dim \Gamma_s \geq \dim \Gamma_\eta.$$

By hypothesis, we find that all irreducible components of X_η have dimension d. Hence, the equality $\dim \mathcal{O}_{\Gamma_t,x} = \dim \mathcal{O}_{X_t,x} - 1$ proved above implies that $\dim \Gamma_\eta = d - 1$. Thus, $\dim \Gamma_s = d - 1$. □

Remark 6.4 Let S be an affine scheme, and let $X \to S$ be a *smooth*, projective, and surjective, morphism. We may ask whether $X \to S$ always admits a *finite étale* quasi-section. A positive answer to this question over $S = \operatorname{Spec} \mathbb{Z}$ would imply that any smooth, projective, surjective, morphism $X \to \operatorname{Spec} \mathbb{Z}$ has a generic fiber which has a $\mathbb{Q}$-rational point. Note that the existence of a quasi-finite étale quasi-section is proved in [11], 17.16.3 (ii).

Let K be a number field, let $B \subset \mathbb{Z}$ be a finite set of primes, and denote by $\mathcal{O}_{K,B} = \mathcal{O}_K[1/p, p \in B]$ the ring of B-integers in K. Let $S := \operatorname{Spec} \mathcal{O}_{K,B}$. Let again $X \to S$ be smooth, projective, and surjective. A weaker question would be to ask whether the generic fiber $X_K/\operatorname{Spec} K$ of $X \to S$ always has a point in a finite extension L/K such that any ramified prime of L/K lies over a prime in B. In other words, we may ask whether the morphism $X \to S$ always has a finite quasi-section T such that the normalisation $\overline{T} \to T$ is étale over S.

Let now S be a smooth affine geometrically irreducible curve over a finite field. Let $X \to S$ be a *smooth* and surjective morphism, with geometrically irreducible generic fiber. Then X/S has a *finite étale* quasi-section ([37], Theorem (0.1)).

Corollary 6.5. *Let A be a commutative ring. Let M be a finitely generated projective A-module of constant rank $r \geq 1$. Then there exist an A-algebra B, finite and faithfully flat over A, with B a local complete intersection over A, and locally free B-modules $L_1, \dots, L_r$ of rank 1 such that $M \otimes_A B$ and $L_1 \oplus \cdots \oplus L_r$ are isomorphic B-modules.*

Proof. Let us first consider the case where A is noetherian. Let $S := \operatorname{Spec} A$. Assume $r > 1$. Let $\mathcal{M}$ denote the locally free $\mathcal{O}_S$-module of rank r associated with M. Let $X := \mathbb{P}(\mathcal{M})$. Then the natural map $X \to S$ is projective, smooth, and its fibers all have dimension $r - 1$. We are thus in a position to apply Theorem 6.2 (3) to obtain the existence of a finite flat quasi-section $T \to S$ as in (3). In particular, $T = \operatorname{Spec} B$ for some finite and faithfully flat A-algebra B, with B a local complete intersection over A. Moreover, the existence of an S-morphism $g : T \to X$ corresponds to the existence of an $\mathcal{O}_T$-invertible sheaf $\mathcal{L}_1$ and of a surjective morphism $g^*\mathcal{M} \to \mathcal{L}_1$. Let $\mathcal{M}_1$ denote the kernel of this morphism. The $\mathcal{O}_T$-module $\mathcal{M}_1$ is locally free of rank $r-1$, and $g^*\mathcal{M} \cong \mathcal{L}_1 \oplus \mathcal{M}_1$. We may thus proceed as above and use Theorem 6.2 (3) another $r-2$ times to obtain the conclusion of the corollary.

We now show how to reduce the general case to the case where A is noetherian. The following holds: *Let A be a commutative ring, and let M be a projective finitely generated A-module. Then there exist a subring A_1 of A, with A_1 finitely generated over $\mathbb{Z}$ (and, hence, noetherian), and a finitely generated projective A_1-module M_1, such that $M_1 \otimes_{A_1} A \simeq M$.*

Indeed, first note that M is finitely presented since it is projective and finitely generated. By [11], IV.8.9.1, there exist a subring A_0 of A and a finitely generated projective module M_1 over A_0 such that A_0 is finitely generated over $\mathbb{Z}$ and such that $M_0 \otimes_{A_0} A \simeq M$. We claim that there exists an extension $A_0 \subseteq A_1 \subset A$ such that $M_1 := M_0 \otimes_{A_0} A_1$ is locally free over A_1. As M is locally free, $\operatorname{Spec} A$ is covered by finitely many principal open subsets $D(f_i)$ such that M_{f_i} is free over A_{f_i} for all $i \in I$. Replacing A_0 by $A_0[f_i, i \in I]$ if necessary, we assume that $f_i \in A_0$ for all $i \in I$.

We are thus reduced to consider the case where the module M is free of rank r over A, and there exist a subring A_0 and an A_0-module M_0 such that $M_0 \otimes_{A_0} A \simeq M$. Write down an isomorphism $A^r \to M$. Increasing A_0 if necessary, we may assume that there exists a set of r elements of M_0 which becomes a basis of $M_0 \otimes_{A_0} A$. In other words, the given isomorphism $A^r \to M$ descends to a surjective linear map $A_0^r \to M_0$. We therefore have an exact sequence of the form $A_0^q \to A_0^r \to M_0$, such that the first map $A_0^q \to A_0^r$ becomes the 0-map when tensored by A. It is therefore already the 0-map, and M_0 is free.

To conclude the proof in the non-noetherian case, we apply the noetherian case to the ring A_1 and the module M_1, and tensor everything back with A to obtain the result for M. □

Remark 6.6 The corollary strengthens, in the affine case, the classical splitting lemma for vector bundles ([8], V.2.7). When A is of finite type over an algebraically closed field k, and is regular, it is shown in [35], 3.1, that it is possible to find a finite faithfully flat *regular* A-algebra B over which M splits.

Let S be a scheme and let $U \subset S$ be an open subset. Given a family $C \to U$ of stable curves over U, conditions are known (see, e.g., [16]) to insure that this family extends to a family of stable curves over S. It is natural to consider the analogous problem of extending a given family $D \to Z$ of stable curves over a closed subset Z of S. For this, we may use the existence of finite quasi-sections in appropriate moduli spaces, as in the proposition below.

Proposition 6.7. *Let S be a noetherian affine scheme. Let Z be a closed subscheme of S, and let $D \to Z$ be a stable curve of genus $g \geq 2$. Then there exist a finite surjective morphism $T \to S$, a stable curve $\mathcal{D} \to T$ of genus g, a finite surjective morphism $Z' \to Z$, a closed S-immersion $Z' \to T$, and a morphism $D \times_Z Z' \to \mathcal{D}$*

such that the diagram below commutes and the top square in the diagram is cartesian:

$$\begin{array}{ccc} D \times_Z Z' & \hookrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ Z' & \hookrightarrow & T \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & S \end{array}$$

Proof. Let $\mathcal{F} := \mathcal{M}_g$ be the Deligne-Mumford stack of stable curves of genus g over S (see [6], 5.1). The general theory ([18], 1.3 (1)) implies the existence of a coarse moduli space M for $\mathcal{F}$, with M an algebraic space, and a proper S-morphism $\mathcal{F} \to M$. In the case of $\mathcal{M}_g$ over $\operatorname{Spec}\mathbb{Z}$, it is known that the coarse moduli space M_g is a scheme which is projective over $\operatorname{Spec}\mathbb{Z}$, and that the fibers of $M_g \to \operatorname{Spec}\mathbb{Z}$ all have the same dimension ([14], 2.1, 3.8). The curve $D \to Z$ corresponds to an element in the set $\mathcal{F}(Z)$, which in turn corresponds to a closed immersion of Z into $\mathcal{F}$, since $F \to S$ is proper and Z is a closed subscheme of S. Our proposition follows then from the following 'stacky' version:

Let $\mathcal{F}$ be a separated Deligne-Mumford stack over a noetherian affine scheme S. Let M be the coarse moduli space for $\mathcal{F}$, with M an algebraic space, and associated proper S-morphism $\mathcal{F} \to M$. Assume the following: There exist a scheme M' and a surjective S-morphism $M' \to M$ such that the morphism $M' \to S$ is projective and surjective, and its fibers all have the same dimension. Let $\mathcal{Z}$ be a closed substack of $\mathcal{F}$, finite over S.

Then there exist: a scheme T with an S-morphism of stacks $T \to \mathcal{F}$ and such that $T \to S$ is finite and surjective, and a closed scheme Z' of T with a finite surjective morphism $Z' \to \mathcal{Z}$, making the following diagram commute:

$$\begin{array}{ccccc} Z' & \hookrightarrow & T & & \\ \downarrow & & \downarrow & & \\ \mathcal{Z} & \longrightarrow & F & \longrightarrow & S. \end{array}$$

Let $\mathcal{F}' := \mathcal{F} \times_M M'$. Since $\mathcal{F}'$ is a separated Deligne-Mumford stack with M' as moduli space (or algebraic stack in the terminology of [39]), there exists a finite surjective morphism from a scheme X to $\mathcal{F}'$ ([39], 2.6). It follows from the construction in the proof of [39], 2.6, that the composition $X \to M'$ is a finite surjective morphism of schemes. Thus X/S is projective since S is affine and $M' \to S$ is projective. Moreover, for each $s \in S$, $\dim X_s = \dim M'_s$ because $X \to M'$ is finite and surjective. Consider the stack $Z' := (\mathcal{Z} \times_M M') \times_{F'} X$. Since $\mathcal{Z} \to \mathcal{F}$ is a closed immersion, Z' is a closed substack of the scheme X and, thus, is a closed subscheme of X. The situation is represented in the following diagram:

$$\begin{array}{ccccc}
Z' := (\mathcal{Z} \times_M M') \times_{\mathcal{F}'} X & \longrightarrow & X & & \\
\downarrow & & \downarrow & & \\
\mathcal{Z} \times_M M' & \longrightarrow & \mathcal{F}' := \mathcal{F} \times_M M' & \longrightarrow & M' \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{Z} & \longrightarrow & \mathcal{F} & \longrightarrow & M.
\end{array}$$

By Theorem 6.2, there exists a finite quasi-section T of X/S containing Z'. Since $X \to \mathcal{F}'$ is finite surjective and $M' \to M$ is surjective, then $Z' \to \mathcal{Z}$ is surjective. Since $\mathcal{Z}/S$ is finite and Z'/S is finite because Z' is a closed subscheme of T which is finite over S, we find that $Z' \to \mathcal{Z}$ is finite.

Let us apply the above considerations to the Deligne-Mumford stack $\mathcal{F} := \mathcal{M}_g$. The stable curve $\mathcal{D} \to T$ whose existence is asserted in the statement of the proposition corresponds to the element of $\mathcal{F}(T)$ associated with the morphism $T \to \mathcal{F}$ whose existence has just been demonstrated. □

Some condition on the dimension of the fibers of $X \to S$ in 6.2 is necessary for a finite quasi-section to exist, as the following easy facts show.

6.8 *Let S be noetherian. Let $f : X \to S$ be a projective morphism between integral schemes, and let T be a finite quasi-section.*

(a) *Assume that f is birational. Then f is finite, and* $\dim X_s = 0$ *for all $s \in S$.*
(b) *Assume that S is affine, that X is regular, and that the generic fiber of $X \to S$ has dimension* 1*. Then* $\dim X_s = 1$ *for all $s \in S$.*

Proof. (a) Since f is birational, X and S have the same dimension. Since $T \to S$ is finite, T and S have the same dimension. Thus, T and X have the same dimension, and since T is closed in X, it must contain the generic point of X. Hence, $T = X$.

(b) Since T contains an irreducible component which surjects onto S, replacing T by this component if necessary, we can suppose T irreducible. Replacing T by T_{red} if necessary, we may also assume that T is reduced. Since the fibers of $T \to S$ are of dimension 0, the closed set T satisfies the hypothesis of Theorem 5.3, and we find a locally principal hypersurface H of X containing T, and such that for all $s \in S$, $\dim H_s = \dim X_s - 1$. In particular the generic fiber of H/S has dimension 0. Therefore, the preimage of the generic point of S in H contains only generic points of H. It follows that T coincides (as closed subsets) with an irreducible component of H. Since X is regular, we find that T is then also a Cartier divisor on X, locally defined by one equation in X. Thus, $0 = \dim T_s \geq \dim X_s - 1$, and if follows that $\dim X_s = 1$ for all s. □

A slight improvement to Theorem 6.2 in a case of interest in arithmetic geometry is as follows.

6.9 *Let S be a noetherian excellent regular integral scheme, with function field K. Let $X \to S$ be a projective morphism such that every fiber X_s, $s \in S$, is an abelian variety or, more generally, does not contain a rational curve. Then the closure T of any K-rational point of the generic fiber of $X \to S$ is finite over S.*

Proof. Consider the projective birational morphism $T \to S$, and denote by $\overline{T} \to T$ the normalisation of T. Then $f : \overline{T} \to S$ is a birational morphism of finite type. Denote by $E(f)$ the exceptional set of f, that is, the set of points $x \in \overline{T}$ such that f is not a local isomorphism at x. Since S is assumed to be regular, we can apply a theorem of Abhyankar ([1], see also [20], Appendix, VI.1.3) to $f : \overline{T} \to S$ and obtain that every irreducible component of E is ruled over its image in S. It follows that if E is not empty, then there exists $s \in S$ such that E_s contains a rational curve. Since the morphism $\overline{T} \to T$ is finite, it follows that X_s contains a rational curve, and this is a contradiction. So E is empty and the morphism $T \to S$ is an isomorphism. □

7. Moving lemma for 1-cycles

We review below the basic notation needed to state our moving lemma. Let X be a noetherian scheme. Let $\mathcal{Z}(X)$ denote the free abelian group on the set of closed integral subschemes of X. An element of $\mathcal{Z}(X)$ is called a cycle, and if Y is an integral closed subscheme of X, we denote by $[Y]$ the associated element in $\mathcal{Z}(X)$.

Let $\mathcal{K}_X$ denote the sheaf of meromorphic functions on a noetherian scheme X (see [19], top of page 204 or [21], Definition 7.1.13). Let $f \in \mathcal{K}_X^*(X)$. Its associated principal Cartier divisor is denoted by $\operatorname{div}(f)$ and defines a cycle on X:

$$[\operatorname{div}(f)] = \sum_x \operatorname{ord}_x(f_x)[\overline{\{x\}}]$$

where x ranges through the points of codimension 1 in X, and $\operatorname{ord}_x : \mathcal{K}_{X,x}^* \to \mathbb{Z}$ is defined, for a regular element of $g \in \mathcal{O}_{X,x}$, to be the length of the $\mathcal{O}_{X,x}$-module $\mathcal{O}_{X,x}/(g)$.

A cycle Z is *rationally equivalent to* 0 or *rationally trivial*, if there are finitely many integral closed subschemes Y_i and principal Cartier divisors $\operatorname{div}(f_i)$ on Y_i, such that $Z = \sum_i [\operatorname{div}(f_i)]$. Two cycles Z and Z' are rationally equivalent in X if $Z - Z'$ is rationally equivalent to 0.

A proper morphism of schemes $\pi : X \to Y$ induces by *push forward of cycles* a group homomorphism $\pi_* : \mathcal{Z}(X) \to \mathcal{Z}(Y)$. If Z is any integral subscheme of X, then $\pi_*(Z) := [k(Z) : k(\pi(Z))][\pi(Z)]$, with the convention that $[k(Z) : k(\pi(Z))] = 0$ if $\dim(Z) > \dim(\pi(Z))$.

7.1 Assume now that both X/S and Y/S are schemes of finite type over a scheme S which is universally catenary and equidimensional at every point, and let $f : X \to Y$ be a proper morphism of S-schemes. Let C and C' be two cycles on X which are rationally equivalent. Then $f_*(C)$ and $f_*(C')$ are rationally equivalent on Y ([38], Proposition 6.5, and 3.11).

We are now ready to state the main theorem of this section.

Theorem 7.2. *Let R be a Dedekind domain satisfying Condition* (T*) *in 1.2, and let $S := \operatorname{Spec} R$. Let $X \to S$ be a flat and quasi-projective morphism, with X integral and regular. Let C be a horizontal* 1*-cycle on X. Let F be a closed subset of X such that for all $s \in S$, $F \cap X_s$ has codimension at least* 1 *in X_s. Then some positive multiple mC of C is rationally equivalent to a horizontal* 1*-cycle C' on X whose support does not meet F. When R is semi-local, we can take $m = 1$.*

Let us briefly introduce facts about contraction morphisms needed in the proof of Theorem 7.2.

Proposition 7.3. *Let R be a Dedekind domain, and $S := \mathrm{Spec}(R)$. Let $X \to S$ be a projective morphism of relative dimension 1, with X integral. Let C be an effective Cartier divisor on X, flat over S. Then*

(a) *There exists $m_0 \geq 0$ such that the invertible sheaf $\mathcal{O}_X(mC)$ is generated by its global sections for all $m \geq m_0$.*
(b) *The morphism $X' := \mathrm{Proj}\,(\oplus_{m\geq 0} H^0(X, \mathcal{O}_X(mC))) \longrightarrow S$ is projective, with X' integral, and the canonical morphism $u : X \to X'$ is projective, with $u_*\mathcal{O}_X = \mathcal{O}_{X'}$ and connected fibers.*
(c) *For any vertical prime divisor Γ on X, $u|_\Gamma$ is constant if $\Gamma \cap \mathrm{Supp}\, C = \emptyset$, and is finite otherwise.*
(d) *Let Z be the union of the vertical prime divisors of X disjoint from $\mathrm{Supp}\, C$. Then u induces an isomorphism $X \setminus Z \to X' \setminus u(Z)$.*

Proof. In [2], Theorem 1 in 6.7, a similar statement is proved, with R local, and X normal. (The normality is not assumed in [7] and [29]. A global base is considered in [21], 8.3.30.) We leave it to the reader to check that the proof of [2], 6.7/1, can be used *mutatis mutandis* to prove 7.3. Part (a) follows from the first part of the proof of 6.7/1. Part (b) follows from 6.7/2. Part (c) follows from the second part of the proof of 6.7/1. We now give a proof of (d). The morphism u is birational because it induces an isomorphism $X_\eta \to X'_\eta$ over the generic point η of S, since C_η is ample, being effective of positive degree. It follows that Z is the union of finitely many prime divisors of X. As u has connected fibers, it follows from (c) that $Z = u^{-1}(u(Z))$. The restriction $v : X \setminus Z \to X' \setminus u(Z)$ of u is thus projective and quasi-finite. Therefore, v is finite and, hence, affine. As $\mathcal{O}_{X'\setminus\pi(Z)} = v_*\mathcal{O}_{X\setminus Z}$, v is an isomorphism. □

We are now ready to prove Theorem 7.2. The end of our proof below is reminiscent of 4.3 in the proof of Rumely's Local-Global Principle given in [25].

Proof of Theorem 7.2. It suffices to prove the theorem in the case where C is integral. As in the proof of Theorem 3.3 in [9], we reduce the proof of 7.2 to the case where $C \to X$ is l.c.i. as follows.

Proposition 4.2 in [9] shows the existence of a finite morphism $D \to C$ such that the composition $D \to C \to S$ is an l.c.i. morphism. Since C is affine, there exists for some $N \in \mathbb{N}$ a closed immersion $D \to C \times_S \mathbb{P}^N_S \subseteq X \times_S \mathbb{P}^N_S$. We claim that it suffices to prove the theorem for the 1-cycle D and the closed subset $\mathbf{F} := F \times_S \mathbb{P}^N_S$ in the regular scheme $X \times_S \mathbb{P}^N_S$. Indeed, let D' be a horizontal 1-cycle whose existence is asserted by the theorem in this case. In particular, $\mathrm{Supp}(D') \cap \mathbf{F} = \emptyset$. Consider the projection $p : X \times_S \mathbb{P}^N_S \to X$, which is a projective morphism. Then $p_*(D) = C$ because $D \to C$ is birational. It follows from 7.1 that $C = p_*(D)$ is rationally equivalent to the horizontal 1-cycle $C' := p_*(D')$ on X. Moreover, $\mathrm{Supp}(C') \cap F = \emptyset$. Since D/S is l.c.i., each local ring $\mathcal{O}_{D,x}$, $x \in D$, is an absolute complete intersection ring, and the closed immersion $D \to X \times_S \mathbb{P}^N_S$ is a regular immersion ([11], IV.19.3.2).

Let us now assume that $C \to X$ is l.c.i. Let d denote the codimension of C in X. Since S is universally catenary and $C \to S$ is surjective, we find that X_s is pure of dimension d for all s. We can thus apply Theorem 3.1 (as stated in section 3) and obtain a closed subscheme Y of X such that C is the support of a Cartier divisor on Y and such that $F \cap Y_s$ is finite for all $s \in S$. Clearly, C is also the support of a Cartier divisor on Y_{red}, and on any irreducible component of Y_{red} passing through C. Thus, we are reduced to proving the theorem when X is integral of dimension 2 and F is quasi-finite over S.

Let $\overline{X}$ be an integral projective compatification of X. Let $\overline{F}$ be the Zariski closure of F in $\overline{X}$ (it is finite over S). Let $u : \overline{X} \to X'$ be the contraction morphism associated to C in 7.3. Let Z denote the union of the irreducible components E of the fibers of $\overline{X} \to S$ such that $E \cap C = \emptyset$. Let $U = X \setminus (Z \cap X)$. Then $C \subset U$, and $u|_U$ is an isomorphism onto its image. Let $F' = u(\overline{F} \cup Z \cup (\overline{X} \setminus X)) \cup C$. Then $X' \setminus F' \subseteq U$, and F' is finite over S. We endow F' with the structure of reduced closed subscheme of X'.

Now suppose that R satisfies Condition (T*). Then $\mathrm{Pic}(F')$ is a torsion group because F' is the disjoint union of a closed subscheme finite flat over S, and some isolated points. So, fix $n > 0$ such that $\mathcal{O}_{X'}(nC)|_{F'}$ is trivial. Since C meets every irreducible component of every fiber of $X' \to S$, the sheaf $\mathcal{O}_{X'}(C)$ is relatively ample for $X' \to S$ ([11], III.4.1.7). Let $\mathcal{I}$ denote the ideal sheaf of F' in X'. Then there exists a multiple m of n such that $H^1(X', \mathcal{I} \otimes \mathcal{O}_{X'}(mC)) = (0)$. It follows that a trivialization of $\mathcal{O}_{X'}(mC)|_{F'}$ lifts to a section $f \in H^0(X', \mathcal{O}_{X'}(mC))$.

Recall that by definition, $\mathcal{O}_{X'}(mC)$ is a subsheaf of $\mathcal{K}_{X'}$. We thus consider $f \in H^0(X', \mathcal{O}_{X'}(mC)) \subset \mathcal{K}_X(X)$ as a rational function. The support of the divisor $\mathrm{div}_{X'}(f) + mC$ is disjoint from F' by construction. In particular, it is contained in U and horizontal, and $\mathrm{div}_{X'}(f)$ has also its support contained in U. Considering the pull back of the divisors under $X \to \overline{X} \to X'$ shows that the divisor $\mathrm{div}_X(f) + mC$ contained in U is disjoint from F, horizontal and linearly equivalent to mC.

When R is semi-local, the set $F' \subset X'$ is a finite set of points. Thus we may apply Proposition 6.1 of [9] directly to the Cartier divisor whose support is C to find a Cartier divisor D linearly equivalent to C and whose support does not meet F'. □

Example 7.4 Let R be any Dedekind domain. Let $S = \mathrm{Spec}\, R$. Our next example shows that Theorem 7.2 can hold only if R has the property that $\mathrm{Pic}(S')$ is a torsion group for all $S' := \mathrm{Spec}(R')$ such that R' is the integral closure of R in a finite extension of the field of fractions of R, and R' is finite over R.

Indeed, choose an invertible sheaf $\mathcal{L}$ over S, and consider the projective scheme $X := \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})$, with the associated projective morphism $\pi : X \to S$. Denote by C_0 and C_∞ the images of the two natural sections of π obtained from the projections $\mathcal{O}_S \oplus \mathcal{L} \to \mathcal{O}_S$ and $\mathcal{O}_S \oplus \mathcal{L} \to \mathcal{L}$. Let $C := C_0 + C_\infty$. Let $F := \mathrm{Supp}(C)$.

If Theorem 7.2 holds, then a multiple of C can be moved, and there exists a 1-cycle C' of X such that $\mathrm{Supp}(C') \cap F = \emptyset$. Since the fibers of π are projective lines and, hence, irreducible, every irreducible component in $\mathrm{Supp}(C')$ dominates S. Since π is proper, we find the existence of an integral subscheme Y of X, finite over S, and disjoint from F. Let $S' \to Y$ denote the normalization of Y and let $g : S' \to S$ denote the composition $S' \to Y \to S$. Let $X' := X \times_S S'$, with projection $\pi' : X' \to S'$. Clearly, π' corresponds to the natural projection $\mathbb{P}(\mathcal{O}_{S'} \oplus g^*\mathcal{L}) \to S'$. We find that the morphism π' has now three pairwise disjoint sections, corresponding to three homomorphisms from $\mathcal{O}_{S'} \oplus g^*\mathcal{L}$ to lines bundles, two of them being the obvious projection maps.

7.5 We claim that three such pairwise disjoint sections can exist only if $\mathcal{L}' := g^*\mathcal{L}$ is the trivial invertible sheaf. Let $\mathcal{N} \subset \mathcal{O}_{S'} \oplus \mathcal{L}'$ be the submodule corresponding to the third section ([11], II.4.2.4). For any $s' \in S'$, $\mathcal{N} \otimes k(s')$ is different from $\mathcal{L}' \otimes k(s')$ because in the fiber above s', the section defined by $\mathcal{N}$ is different from the section defined by the projection to $\mathcal{O}_{S'}$, so the image of $\mathcal{N} \otimes k(s')$ in the quotient $k(s')$ is non zero. Therefore the canonical map $\mathcal{N} \to \mathcal{O}_{S'}$ is surjective, hence isomorphic. Similarly, the canonical map $\mathcal{N} \to \mathcal{L}'$ is an isomorphism. Therefore $\mathcal{L}' \simeq \mathcal{O}_{S'}$.

Lemma 7.6 shows that $\mathcal{L}$ is a torsion element in $\mathrm{Pic}(S)$. Thus, for Theorem 7.2 to hold, it is necessary that $\mathrm{Pic}(S)$ be a torsion group. Repeating the same argument starting with any invertible sheaf $\mathcal{L}'$ over any S' (which is regular, and finite and flat over S) and considering the map $\mathbb{P}(\mathcal{O}_{S'} \oplus \mathcal{L}') \to S' \to S$ we find that for Theorem 7.2 to hold, it is necessary that $\mathrm{Pic}(S')$ be a torsion group.

Lemma 7.6. *Let L/K be a finite extension of degree d. Let R be a Dedekind domain with field of fractions K, and let R' denote a subalgebra of L, integral over R. Then the kernel of* $\mathrm{Pic}(R) \to \mathrm{Pic}(R')$ *is killed by d.*

Proof. When R' is finite over R, this is well-known (see, e.g., [12], 2.1). (The hypothesis that R is Dedekind is used here to insure that the ring R' is flat over R.) In general, let M be a locally free R-module of rank 1 such that $M \otimes_R R'$ is isomorphic as R'-module to R'. Consider then an isomorphism $f' : R' \to M \otimes_R R'$ and its inverse. Then there exist a finite R-algebra A contained in R' and two morphisms of A-modules $f : A \to M \otimes_R A$ and $g : M \otimes_R A \to A$ such that $(g \circ f) \otimes_A R'$ is an isomorphism. Let $h := g \circ f : A \to A$. This morphism is determined by $h(1)$, and $h(1)$ is a unit in R'. Since R' is integral over R, we find that $h(1)$ is a unit in A and, hence, g is an isomorphism. It follows that $M^{\otimes d}$ is trivial in $\mathrm{Pic}(R)$, since A/R is finite. □

Example 7.7 Keep the notation introduced in Example 7.4, and choose a non-trivial line bundle $\mathcal{L}$ of finite order in $\mathrm{Pic}(S)$. Let $X := \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})$. Let $C := C_0$, and let $F := C_0 \cup C_\infty$. Then C itself cannot be moved away from F. Indeed, otherwise, $X \to S$ would have a third section C_1 disjoint from C_0 and C_∞. But as Example 7.4 shows, this can only happen if $\mathcal{L}$ is trivial.

8. Finite morphisms to $\mathbb{P}^d_S$.

Theorem 8.1. *Let R be a Dedekind domain satisfying Condition* (T*) *in 1.2, and let $S := \mathrm{Spec}\, R$. Then, for any $d \geq 0$, and for any projective morphism $X \to S$ such that* $\dim X_s = d$ *for all $s \in S$, there exists a finite surjective S-morphism $X \to \mathbb{P}^d_S$.*

Proof. Fix an ample invertible sheaf $\mathcal{O}_X(1)$ on X. We first apply Theorem 5.3 to $X \to S$ with $C = \emptyset$ and $F = X$. We then find $n_0 > 0$ and $f_0 \in H^0(X, \mathcal{O}_X(n_0))$ such that $H_{f_0} \to S$ has fibers of dimension $d-1$. We apply again Theorem 5.3, this time to $H_{f_0} \to S$ and the sheaf $\mathcal{O}_X(n_0)|_{H_{f_0}}$. We find an integer n_1 and a section $f_1 \in H^0(H_{f_0}, \mathcal{O}_X(n_0 n_1)|_{H_{f_0}})$ whose associated hypersurface $H_{f_1} \subset H_{f_0}$ has fibers over S of dimension $d-2$. We continue this process $d-2$ additional times, to find a sequence of hypersurfaces $H_{f_{d-1}} \subset \ldots \subset H_{f_1} \subset H_{f_0}$, where $H_{f_{d-1}} \to S$ has all its fibers of dimension 0 and, hence, is finite (and surjective, 5.1 (1)) over S since it is projective.

Note that replacing f_0 by a positive power of f_0 does not change the topological properties of the closed set H_{f_0}. We can thus use the vanishing properties of the H^1 groups to obtain the following: there exist $n > 0$ and sections $g_0, \ldots, g_{d-1} \in \mathcal{O}_X(n)$ such that $Y := H_{g_0} \cap \ldots \cap H_{g_{d-1}}$ has all its fibers of dimension 0 and, thus, $Y \to S$ is finite and surjective.

By Condition (T*), $\mathrm{Pic}(Y)$ is a torsion group. So there exists $m \geq 1$ such that $\mathcal{O}_X(nm)|_Y \simeq \mathcal{O}_Y$. Therefore, we can find k large enough such that a trivialization of $\mathcal{O}_X(nmk)|_Y$ lifts to a section $h_d \in H^0(X, \mathcal{O}_X(nmk))$. Consider now $h_0 := g_0^{mk}, \ldots, h_{d-1} := g_{d-1}^{mk} \in H^0(X, \mathcal{O}_X(nmk))$. By construction, $H_{h_0} \cap \ldots \cap H_{d-1} \cap H_d = \emptyset$.

Consider the homomorphism $\rho : (\mathcal{O}_S^{d+1}) \otimes_{\mathcal{O}_S} \mathcal{O}_X \to \mathcal{O}_X(nmk)$ which maps the ith element of the canonical basis of $\mathcal{O}_S^{d+1}$ to h_i. By construction, ρ is surjective. It then induces a morphism $r : X \to \mathbb{P}(\mathcal{O}_S^{d+1}) = \mathbb{P}^d_S$ such that $r^*\mathcal{O}(1) \simeq \mathcal{O}_X(nmk)$. This morphism is finite because it is proper and fiberwise finite (see [17], Lemma 3). □

Remark 8.2 The morphism r obtained in the above theorem is often flat. Indeed, suppose that S is a noetherian regular scheme, and that X is Cohen-Macaulay. As $\mathbb{P}^d_S$ is regular since S is, and r is finite and surjective, r is flat ([11], IV.15.4.2).

Remark 8.3 One may wonder whether Theorem 8.1 can be generalized to other situations. Let S be the spectrum of a complete discrete valuation domain R with field of fractions K. A smooth projective geometrically connected curve Y_K/K of genus $g > 0$ (in short, a *curve*) has a 'canonical' model Y/S, namely, its minimal regular model. Theorem 8.1 shows that given any curve X_K/K with minimal regular model X/S, there exists a finite S-morphism $X \to \mathbb{P}^1_S$. One may wonder, given a curve X_K/K which admits a finite K-morphism $X_K \to Y_K$ to another curve Y_K, whether there exists a finite S-morphism of minimal regular models $X \to Y$. This question has a negative answer in general (see, e.g., section 6 of [22]), even when X and Y are allowed to be any regular models of their generic fibers.

We can prove a partial converse to Theorem 8.1.

Proposition 8.4. *Let R be a Dedekind domain with field of fractions K, and let $S := \mathrm{Spec}\, R$. Suppose that for any $d \geq 0$, and for any projective morphism $X \to S$ such that $\dim X_s = d$ for all $s \in S$, there exists a finite surjective S-morphism $X \to \mathbb{P}^d_S$. Then $\mathrm{Pic}(R')$ is a torsion group for any Dedekind domain R' obtained as the integral closure of R in a finite extension L/K and such that R' is finite over R.*

Proof. Let us show first that $\mathrm{Pic}(S)$ is a torsion group. Assume that it is not, and choose a line bundle $\mathcal{L}$ of infinite order in $\mathrm{Pic}(S)$. Let $X := \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})$. We claim that there exists no finite S-morphism $X \to \mathbb{P}^1_S$. Indeed, since $\mathbb{P}^1_S$ has three pairwise disjoint sections, such a morphism would induce three pairwise disjoint finite quasi-sections on X. After a finite flat base change $S' \to S$, the morphism $X \times_S S' \to S'$ would have three pairwise disjoint sections. But we saw in 7.5 that this is only possible when $\mathcal{L}$ has finite order in $\mathrm{Pic}(S)$.

Let L/K be any finite extension, and let R' denote the integral closure of R in L, assumed to be finite over R. Let $S' := \mathrm{Spec}\, R'$. Since the morphism $S' \to S$ is finite, we find that for any $d \geq 0$, and for any projective morphism $X' \to S'$ such that $\dim X'_s = d$ for all $s \in S'$, there exists a finite surjective S'-morphism $X' \to \mathbb{P}^d_{S'}$. Thus, our argument above applies and $\mathrm{Pic}(S')$ is a torsion group. □

Remark 8.5 We thank Robert Varley for the following example of a Dedekind domain R with infinitely many maximal ideals, satisfying Condition (T*), and where Condition (T)(b) in 1.1 does not hold. Consider the ring $\mathbb{C}[x]$ and let S denote the multiplicative subset of all polynomials which do not vanish on $\mathbb{Z}$ or, more generally, on a given countable subset of $\mathbb{C}$. Then $R := S^{-1}(\mathbb{C}[x])$ is a principal ideal domain satisfying Condition (T*).

More generally, let $A/\overline{\mathbb{Q}}$ be a Dedekind domain, finitely generated as a $\overline{\mathbb{Q}}$-algebra. Let S denote the multiplicative subset of all elements of $A \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ which do not vanish at any maximal ideal M of $A \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ restricting under the inclusion $A \to A \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ to a maximal ideal of A. Then $R := S^{-1}(A \otimes_{\overline{\mathbb{Q}}} \mathbb{C})$ is a principal ideal domain satisfying Condition (T*).

Example 8.6 Let K be a number field, and let $S = \mathrm{Spec}(\mathcal{O}_K)$. Consider a class of smooth projective varieties X_K/K which admit a 'canonical' model X/S. To fix ideas, let us say that X_K/K is a smooth projective geometrically connected curve of genus $g > 0$, and that X/S is its minimal regular model over S. For any open subset U of S, we can define an invariant $d(X_K/K, U)$ as the smallest degree of a finite U-morphism $X_U \to \mathbb{P}^1_U$. Note that in general, this invariant depends on the reduction type of the curve X_K at various places of bad reduction. Indeed, the degree of any finite morphism $X_U \to \mathbb{P}^1_U$ is at least equal to the number of irreducible components of any closed fiber X_s, $s \in U$.

Suppose that X_K/K is either an elliptic curve or a hyperelliptic curve, and that its minimal regular model $X \to S$ has only integral fibers. If $\mathrm{Pic}(S) = (0)$, then X admits a finite morphism $X \to \mathbb{P}^1_S$ of degree 2 and, thus, $d(X_K/K, S) = 2$.

Proof. By definition, a hyperelliptic curve X_K/K is endowed with a K-involution $\sigma : X_K \to X_K$, with quotient $X_K/\langle\sigma\rangle$ isomorphic to $\mathbb{P}^1_K$. Since the model X is minimal, this involution extends to an S-involution $\overline{\sigma} : X \to X$. The quotient $X/\langle\overline{\sigma}\rangle$ is a normal model of $\mathbb{P}^1_K$ with integral fibers, hence is smooth over S. It is isomorphic to $\mathbb{P}^1_S$ because $\mathrm{Pic}(S) = (0)$. The same is true for elliptic curves by considering the involution multiplication-by-(-1). □

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