

QUESTIONS ON WILD $\mathbb{Z}/p\mathbb{Z}$ -QUOTIENT SINGULARITIES IN DIMENSION 2

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1. SOME QUESTIONS

Let A denote a regular local ring of dimension 2 with maximal ideal \mathcal{M}_A . Let p be prime, and let $H = \mathbb{Z}/p\mathbb{Z}$ act on A . Let $\mathcal{Z} := \text{Spec}(A^H)$. Assume that the action of H on $\text{Spec}(A)$ is free off the closed point, and that \mathcal{M}_A^H is the only singular point of \mathcal{Z} . When the residue characteristic of A/\mathcal{M}_A is equal to p , \mathcal{M}_A^H is called a *wild cyclic quotient singularity*.

Let $f : \mathcal{X} \rightarrow \mathcal{Z}$ be a resolution of the singularity, minimal with the property that the irreducible components of $f^{-1}(\mathcal{M}_A^H)$ are smooth with normal crossings. Attached to this resolution are two natural objects that we now describe, the *intersection matrix* N , and the *resolution graph* G . The exceptional divisor $f^{-1}(\mathcal{M}_A^H)$ consists in n irreducible components C_i , $i = 1, \dots, n$. Denote by $N := ((C_i \cdot C_j)_{\mathcal{X}})$ the associated symmetric matrix. The matrix N is negative-definite and, in particular, $\det(N) \neq 0$. Let G denote the graph whose vertices are the n irreducible components of $f^{-1}(\mathcal{M}_A^H)$, and where two vertices C and D are linked by $(C \cdot D)_{\mathcal{X}}$ edges.

For future reference, recall that the *degree* of a vertex C in a graph G is the number of edges connected to C , and a vertex of degree at least 3 on a graph is called a *node*. A vertex of degree 1 is a *terminal vertex*. The closure in G of a connected component of $G \setminus \{\text{all nodes of } G\}$ is called a *chain* of G . If the chain contains a terminal vertex, it is called a *terminal chain*.

Wild $\mathbb{Z}/p\mathbb{Z}$ -quotient singularities of surfaces are expected to have resolution graphs which are trees, with associated intersection matrices N satisfying $\det(N) = p^s$ for some $s \geq 0$. The Smith group $\Phi_N := \mathbb{Z}^n / \text{Im}(N)$ is expected to be killed by p ([1], 2.6). Finally, we should expect that the fundamental cycle Z of N has self-intersection $|Z^2| \leq p$. This latter combinatorial result follows from the algebraic result that the multiplicity of the singularity is expected to be at most p ([1], 2.3).

Question 1.1 Classify the matrices N which can occur as intersection matrices associated with minimal resolutions of wild $\mathbb{Z}/p\mathbb{Z}$ -quotient singularities in dimension 2. This question is probably too broad to be useful. Here are some more focused sub-questions.

Minimal resolutions with graphs having only one node are exhibited in [2], 6.6.

- a) Can you produce, for each p , examples of intersection matrices N associated with resolutions of wild $\mathbb{Z}/p\mathbb{Z}$ -quotient singularities in dimension 2 whose graphs $G(N)$ do not have a node?
- b) Can you produce, for each p , examples of intersection matrices N associated with minimal resolutions of wild $\mathbb{Z}/p\mathbb{Z}$ -quotient singularities in dimension 2 whose graphs $G(N)$ have more than one node? (One such example is exhibited in mixed characteristic 2 in [1], 4.10. See also question 1.2 (a)). Is there a bound on the possible number of nodes that such a graph can have?

- c) Can you produce, for each p and each $s \geq 0$, examples of intersection matrices N associated with minimal resolutions of wild $\mathbb{Z}/p\mathbb{Z}$ -quotient singularities in dimension 2 with $|\det(N)| = p^s$ (examples with $s - 1 > 0$ and coprime to p are given in [3], 3.12)?

Question 1.2 Is there a difference between the set of intersection matrices associated with the minimal resolutions of wild $\mathbb{Z}/p\mathbb{Z}$ -quotient singularities $\text{Spec } A^H$ in dimension 2 when A is of equicharacteristic p , and the set of intersection matrices associated with the minimal resolutions of wild $\mathbb{Z}/p\mathbb{Z}$ -quotient singularities $\text{Spec } A^H$ in dimension 2 when A is of mixed characteristic $(0, p)$? One related such difference is found in [1], 4.1. A second instance, related also to 1.1 (b), is in [1], 4.10, and prompts the following question:

- (a) Produce discrete valuations fields of residue characteristic p , and curves X/K with potentially good reduction after an extension L/K of degree p , such that 1) the special fiber of the smooth model of X_L/L has p -rank 0, and 2) the graph of the special fiber of the regular model of X/K has more than one node. The case of elliptic curves E/K in equicharacteristic 2 with reduction of type I_n^* with $n > 0$ is open.
- (b) Is it possible to exhibit the Dynkin diagrams D_n (having n vertices) with $n \equiv 2 \pmod{4}$ as the resolution graphs of $\mathbb{Z}/2\mathbb{Z}$ -quotients singularities in dimension 2 when A is of equicharacteristic 2?

Question 1.3 Is there a difference between the set of intersection matrices associated with the minimal resolutions of wild $\mathbb{Z}/p\mathbb{Z}$ -quotient singularities in dimension 2, and the set of intersection matrices associated with the minimal resolutions of wild $\mathbb{Z}/p\mathbb{Z}$ -quotient singularities occurring on the normal quotient \mathcal{Z} , model of a curve X/K with potentially good reduction after a degree p extension, as in [2], 5.3.

We suggest in 6.2 of [2] some extra structure that one may be able to attach to the intersection matrix in the case of a model of a curve. In particular, can the intersection matrix exhibited in [2], 6.14, occur as the intersection matrix associated with the minimal resolution of a wild $\mathbb{Z}/5\mathbb{Z}$ -quotient singularity in dimension 2.

Question 1.4 Consider an intersection matrix N , and assume that for some prime p , it satisfies all known conditions that would have to be satisfied if this intersection matrix was associated with the resolution of a $\mathbb{Z}/p\mathbb{Z}$ -singularity: its graph $G(N)$ is a tree, $\det(N)$ is a power of p and the Smith group is killed by p , and the fundamental cycle Z has $|Z^2| \leq p$.

If $\det(N) = 1$ and $G(N)$ is a tree, then the above conditions are satisfied for every prime at least equal to $|Z^2|$. In particular, when $\det(N) = 1$, the matrix N could potentially be associated with the resolution of a $\mathbb{Z}/p\mathbb{Z}$ -singularity for infinitely many primes p . Can this actually happen? A related question: consider the existence of examples of such N occurring as $\mathbb{Z}/p\mathbb{Z}$ -quotient singularities for more than one prime. See [2], 6.13, for a related discussion.

REFERENCES

- [1] D. Lorenzini, *Wild quotient singularities of surfaces*, Preprint.
 [2] D. Lorenzini, *Wild models of curves*, Preprint.
 [3] D. Lorenzini, *Wild quotients of products of curves*, Preprint.

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