Exam 1 - Practice Questions

Standard Questions

1. Evaluate the following limits.
   (a) $\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{2+h} - \frac{1}{2}\right)$
   (b) $\lim_{x\to\infty} \frac{2-3x}{4x-5}$
   (c) $\lim_{p\to 0} \frac{\sqrt{2-p}-\sqrt{2}}{p}$
   (d) $\lim_{x\to 2} \frac{x-5+\frac{6}{x}}{x-3+\frac{2}{x}}$
   (e) $\lim_{y\to 3^-} \frac{y-2}{y^2-4y+3}$ and $\lim_{y\to 3^+} \frac{y-2}{y^2-4y+3}$

2. Evaluate the following limits.
   (a) $\lim_{x\to 0} \dfrac{\sin 3x}{2x}$
   (b) $\lim_{x\to 0} \dfrac{\tan 4x}{\sin 5x}$
   (c) $\lim_{x\to 0} \dfrac{1-\cos 2x}{\sin x}$
   (d) $\lim_{x\to 0} \dfrac{x}{\sin\sqrt{x}}$
   (e) $\lim_{x\to 0} \dfrac{1}{x^2}\sin^2\dfrac{3x}{2}$

3. For what values of a are the following functions continuous everywhere.
   (a) $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ ax^2 & \text{if } x \geq 1 \end{cases}$
   (b) $g(x) = \begin{cases} ax+1 & \text{if } x \leq 3 \\ ax^2-1 & \text{if } x > 3 \end{cases}$
   (c) $h(x) = \begin{cases} x^2-a^2 & \text{if } x < 4 \\ ax+20 & \text{if } x \geq 4 \end{cases}$

4. Find $f(1)$ given that f is continuous at all reals and
$$f(x) = \frac{x^2-1}{1-x} \quad \text{for } x \neq 1.$$

5. For each of the following functions use the definition of the derivative find $f'(x)$.
   (a) $f(x) = x^{-2}$
   (b) $f(x) = \sin 2x$
   (c) $f(x) = \frac{1}{\sqrt{x}}$
   (d) $f(x) = \frac{x-2}{3-x}$
   (e) $f(x) = ax^3 + bx^2 + cx + d$

6. (a) Find the rate at which the area of a circle increases with respect to its circumference.
   (b) According to Boyle's Law, if the temperature of a confined gas is held fixed, then the product of the pressure P and the volume V is a constant. Suppose that, for a certain gas, $PV = 800$, where P is measured in pounds per square inch and V is measured in cubic inches.
      i. Find the average rate of change of P as V increases from 200in^2 to 250in^2.
      ii. Express V as a function of P and show that the instantaneous rate of change of V with respect to P is proportional to the inverse square of P.

7. Find $\frac{dy}{dx}$ for the following.
   (a) $y = x^{1/2} - \dfrac{1}{x^{1/2}}$
   (b) $y = \cos^2 x$
   (c) $y = \sin(x^2)$
   (d) $y = \dfrac{2-x}{2x-3}$
   (e) $y = (x^2+1)\sqrt{x^4+1}$

8. (a) Find an equation for the tangent line to the curve $y = (2-x)^{2/3}$ at the point $(1,1)$.
   (b) Find an equation for the tangent line to the curve $y = (x^2+1)^3(x^4+1)^2$ at the point $(1,32)$.
   (c) Find all points where the tangent to the curve $y = x\sqrt{4-x^2}$ is horizontal.

9. Each of the following cubic equations has 3 real solutions, use the Intermediate Value Theorem to find the best integer approximation for each.
   (a) $x^3 - 3x - 1 = 0$
   (b) $2x^3 + x^2 - 3x - 1 = 0$
   (c) $2x^3 + 5x^2 + x - 1 = 0$

10. Find the derivative of the following functions.
   (a) $f(x) = \dfrac{1}{(x^2 + \frac{1}{x^2})^3}$
   (b) $F(t) = \sqrt{t\sqrt{t\sqrt{t}}}$
   (c) $g(u) = \left[u - (1 - \frac{1}{u})^{-1}\right]^2$
   (d) $G(y) = \dfrac{\sin^2 y}{\cos^2 y}$
   (e) $h(z) = \sin^2(\cos(1 - z^2))$

More Challenging Questions

11. (a) Prove that if f is continuous at a then so is $|f|$.
   (b) Is the converse to this statement true? If so, prove it. If not, find a counterexample

12. Let f be a continuous at 0, and define g by $g(x) = xf(x)$. Prove that the derivative of g exists at 0 and that $g'(0) = f(0)$.

13. Investigate the differentiability of the following functions.

   (a) $f(x) = x \cdot |x|$ (b) $g(x) = x + |x|$