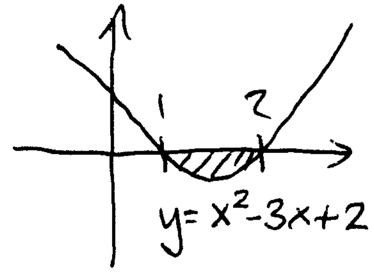
Exam 1 - Practice Questions

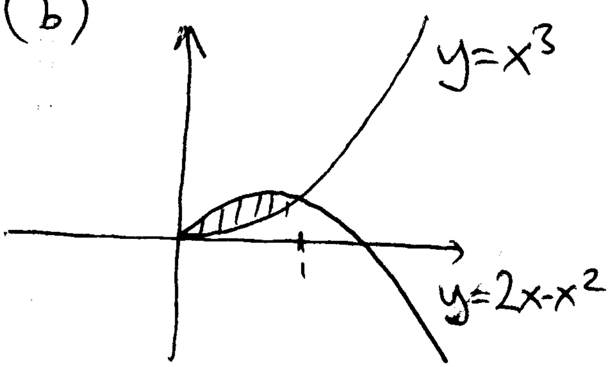
"Short Solutions"/Answers (many details missing)

Q1 (a) Using "shells":

$$V=\int_1^2 2\pi x(-x^2+3x-2)\,dx=\frac{\pi}{2}$$



(b)



(i) Using "washers":

$$V=\pi\int_0^1\left[(2x-x^2)^2-(x^3)^2\right]dx$$

(ii) Using "washers":

$$V=\pi\int_0^1\left[(2-x^3)^2-(2-2x+x^2)^2\right]dx$$

(iii) Using "shells":

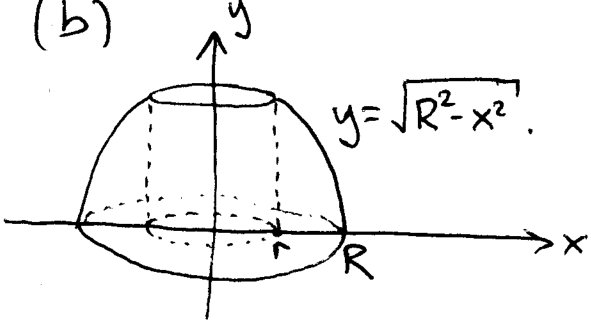
$$V=2\pi\int_0^1 x\left[(2x-x^2)-x^3\right]dx$$

(iv) Using "shells":

$$V=2\pi\int_0^1 (x+2)\left[(2x-x^2)-x^3\right]dx.$$

Q2 (a) Surprisingly (?) they both have same amount of wood!! Lets see why!

(b)



By symmetry, the volume of napkin ring obtained by drilling a hole of radius r through a sphere of radius R is twice the volume obtained by rotating the area above x-axis and below $y=\sqrt{R^2-x^2}$ between $x=r$ & $x=R$, about the y-axis. Using "shells":

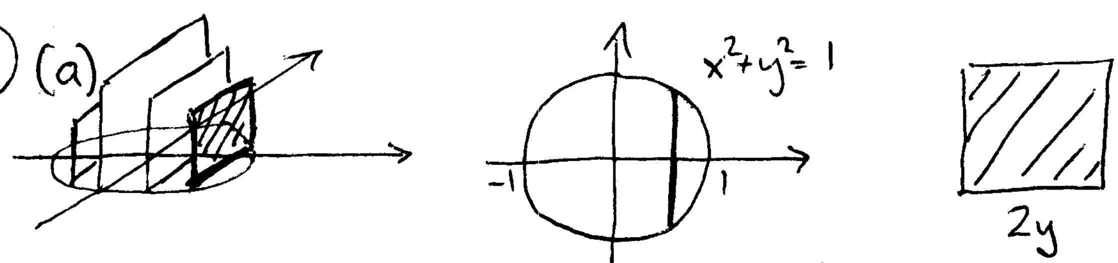
$$V=2\cdot 2\pi\int_r^R x\sqrt{R^2-x^2}\,dx=\frac{4}{3}\pi(R^2-r^2)^{3/2}$$

But by Pythagorean Theorem $R^2 - r^2 = (\frac{1}{2}h)^2$ & hence

$$V = \frac{4}{3}\pi\left(\frac{1}{2}h\right)^3 = \underline{\underline{\frac{\pi}{6}h^3}}$$

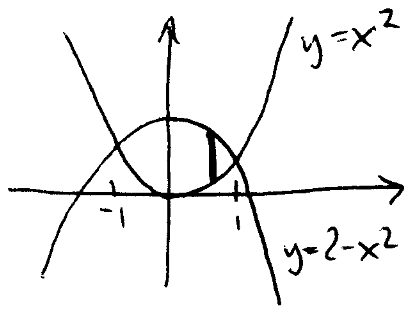
which is independent of both R & r; that is, the amount of wood in a napkin ring of height h is the same regardless of the size of the sphere used to create it.

(Q3) (a)



$$A(x) = (2y)^2 = 4(1-x^2) \Rightarrow V = \int_{-1}^{1} 4(1-x^2)\,dx = \underline{\underline{\frac{16}{3}}}$$

(b)



(i)

$$A(x) = \frac{\pi}{8}D^2 = \frac{\pi}{8}(2-2x^2)^2$$

$$\Rightarrow V = \frac{\pi}{2}\int_{-1}^{1}(1-2x^2+x^4)\,dx = \underline{\underline{\frac{8}{15}\pi}}$$

(ii)

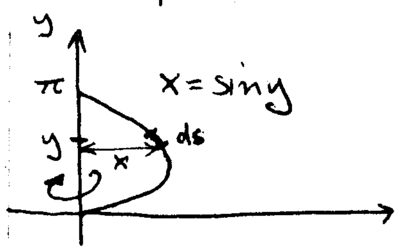
$$A(x) = \frac{D^2}{4} \Rightarrow V = \int_{-1}^{1}(1-2x^2+x^4)\,dx = \underline{\underline{\frac{16}{15}}}$$

(Q4) (a) Since $\frac{dy}{dx} = 3x^2 \Rightarrow L = \int_0^1 \sqrt{1+9x^4}\,dx$

(b) Since $\frac{dy}{dx} = e^x(\cos x - \sin x) \Rightarrow \left(\frac{dy}{dx}\right)^2 = e^{2x}(1-\sin 2x)$

$$\Rightarrow L = \int_0^{\pi/2} \sqrt{1+e^{2x}(1-\sin 2x)}\,dx.$$

(c) $\frac{dx}{dy} = \cos y \Rightarrow ds = \sqrt{1+\cos^2 y}\, dy$



$$S = 2\pi \int_0^{\pi} \sin y \sqrt{1+\cos^2 y}\, dy.$$

Q5 (a) Since $\frac{dx}{dt} = t$ & $\frac{dy}{dt} = \sqrt{2t+1}$

$$\Rightarrow ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{t^2+2t+1} = t+1$$

$$\Rightarrow L = \int_0^4 (t+1)\, dt = \left[\frac{t^2}{2} + t\right]_0^4 = 12$$

(b) (i) $\frac{dy}{dx} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right) \Rightarrow ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)$

$$\Rightarrow L = \frac{1}{2}\int_1^2 \left(x^2 + \frac{1}{x^2}\right) dx = \frac{17}{12}$$

(ii) $S = \int_1^2 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 2\pi \left(\frac{x^3}{6} + \frac{1}{2x}\right)\frac{1}{2}\left(x^2 + \frac{1}{x^2}\right) dx = \frac{47\pi}{16}$

(c) $\frac{dy}{dx} = 3x^2 \Rightarrow S = \int_0^2 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_0^2 2\pi x^3 \sqrt{1+9x^4}\, dx$$

let $u = 1+9x^4$
$du = 36x^3 dx$

$$= \frac{2\pi}{36} \int_1^{145} \sqrt{u}\, du$$

$$= \frac{\pi}{27}\left(145\sqrt{145} - 1\right)$$

(Q6) Let L denote the natural length of the spring in meters.
Then,

$$6 = \int_{0.10-L}^{0.12-L} kx\,dx = \tfrac{1}{2}k\left[(0.12-L)^2-(0.10-L)^2\right]$$

and

$$10 = \int_{0.12-L}^{0.14-L} kx\,dx = \tfrac{1}{2}k\left[(0.14-L)^2-(0.12-L)^2\right]$$

In other words,

$$12 = k(0.0044-0.04L) \quad \& \quad 20 = k(0.0052-0.04L)$$

Subtracting 1st eqn from the second gives:

$$8 = 0.0008k \Rightarrow \underline{\underline{k = 10{,}000}}$$

Now 2nd eqn gives:

$$20 = 52 - 400L \Rightarrow L = \frac{32}{400}\,m = \underline{\underline{8\,cm}}$$

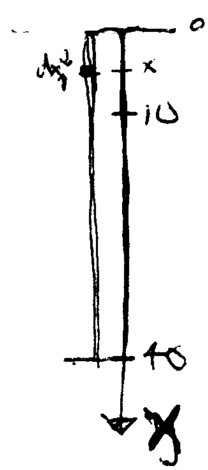
(Q7) The cable weighs 1.5 lb/ft $= \frac{3}{2}$ lb/ft.

<u>Note</u>: Each part of the top 10 ft of cable is lifted a distance equal to it distance from the top.
* The remaining 30ft of cable is lifted 10ft *.

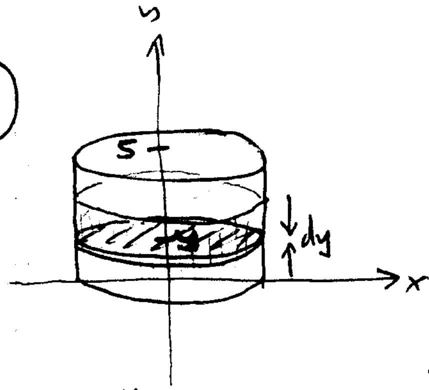
Thus,

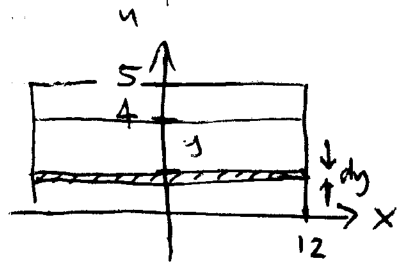
$$W = \int_0^{10} \frac{3}{2}x\,dx + \int_{10}^{40} \frac{3}{2}(10)\,dx$$

$$= 75 + 450 = \underline{\underline{525 \text{ ft-lb}}}$$



Q8





Volume of slab of water $= \pi (12)^2 dy$
$= 144\pi\, dy.$

* Since density of water is 62.5 lb/ft³ *

Weight of slab $= (62.5)144\pi\, dy$

Distance slab lifted $= 5-y.$

$\Rightarrow \; dW = (62.5)144\pi(5-y)\,dy$

$\Rightarrow \; W = \int_0^4 dW = \underline{\underline{108{,}000\,\pi}}\ \underline{\underline{\text{ft-lb}}}$

Q9 (a) $\dfrac{dy}{dx} = \dfrac{\ln x}{x}\cdot\dfrac{1}{y}\;,\quad y(1)=2$

$\int y\,dy = \int \dfrac{\ln x}{x}\,dx$ let $u=\ln x$, $du = \frac{1}{x}dx$

$\dfrac{y^2}{2} = \dfrac{(\ln x)^2}{2} + C$

$\Rightarrow \; \underline{\underline{y^2 = (\ln x)^2}} + C$ ← General Solution

Since $y(1)=2 \Rightarrow 4 = (\ln 1)^2 + C \Rightarrow C = 4$

Hence $\underline{\underline{y^2 = (\ln x)^2 + 4}}$ ← Particular Solution

(b) $\int 2y\,dy = -\int \frac{x\,dx}{\sqrt{x^2+1}} \Rightarrow \underline{y^2 = -\sqrt{x^2+1} + C}$

let $u = x^2+1 \ldots$

Since $y(0) = 1$

$\Rightarrow (-1)^2 = -1 + C \Rightarrow C = 2$

and hence

$$\underline{\underline{y^2 = 2 - \sqrt{x^2+1}}}$$

Q10 (a) Let $y(t)$ = amount of bacteria after t hours

$$\frac{dy}{dt} = ky \Leftrightarrow y(t) = Ae^{kt}$$

Since $y(0) = 4000 \Rightarrow \underline{A = 4000}$

Since $y(\tfrac{1}{2}) = 12000 \Rightarrow 3 = e^{30k} \Rightarrow k = \frac{\ln 3}{(1/2)} = \underline{2\ln 3}$

Hence $\underline{\underline{y(t) = 4000e^{(2\ln 3)t}}}$

(b) $y(\tfrac{1}{3}) = 4000e^{(2\ln 3)/3} \approx \underline{\underline{8320}}$

(c) $4000e^{(2\ln 3)t} = 20000$

$\Rightarrow (2\ln 3)t = \ln 5$

$\Rightarrow \underline{\underline{t = \frac{\ln 5}{2\ln 3}}} \approx 0.73\text{h} \approx \underline{44 \text{ mins}}$