Exam 3 - Practice Questions

1. Evaluate the given line integral.
   (a) $\int_C yz^2\, ds$, where C is the line segment from $(-1, 1, 3)$ to $(0, 3, 5)$
   (b) $\int_C x^3 z\, ds$, where $C:\ x = 2\sin t,\ y = t,\ z = 2\cos t,\ 0 \le t \le \pi/2$
   (c) $\int_C x^3 y\, dx - x\, dy$, where C is the circle $x^2 + y^2 = 1$ with counterclockwise orientation
   (d) $\int_C x \sin y\, dx + xyz\, dz$, where C is given by $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $0 \le t \le 1$
   (e) $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = x^2 y\,\mathbf{i} + e^y\,\mathbf{j}$ and C is given by $\mathbf{r}(t) = t^2\,\mathbf{i} - t^3\,\mathbf{j}$, $0 \le t \le 1$
   (f) $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle x + y, z, x^2 y\rangle$ and C is given by $\mathbf{r}(t) = \langle 2t, t^2, t^4\rangle$, $0 \le t \le 1$

2. Find the work done by the force field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ in moving a particle from the point $(3, 0, 0)$ to the point $(0, \pi/2, 3)$
   (a) along a straight line
   (b) along the helix $x = 3\cos t,\ y = t,\ z = 3\sin t$

3. Show that $\mathbf{F}$ is a conservative vector field and find a potential function f such that $\mathbf{F} = \nabla f$.
   (a) $\mathbf{F}(x, y) = \sin y\,\mathbf{i} + (x\cos y + \sin y)\,\mathbf{j}$
   (b) $\mathbf{F}(x, y, z) = (2xy^3 + z^2)\,\mathbf{i} + (3x^2 y^2 + 2yz)\,\mathbf{j} + (y^2 + 2xz)\,\mathbf{k}$

4. Show that $\mathbf{F}$ is a conservative vector field and use this fact to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve.
   (a) $\mathbf{F}(x, y) = (2x + y^2 + 3x^2 y)\,\mathbf{i} + (2xy + x^3 + 3^2)\,\mathbf{j}$, where C is the arc of the curve $y = x\sin x$ from $(0, 0)$ to $(\pi, 0)$
   (b) $\mathbf{F}(x, y, z) = yz(2x + y)\,\mathbf{i} + xz(x + 2y)\,\mathbf{j} + xy(x + y)\,\mathbf{k}$, where C is given by $\mathbf{r}(t) = (1 + t)\,\mathbf{i} + (1 + 2t^2)\,\mathbf{j} + (1 + 3t^3)\,\mathbf{k}$, $0 \le t \le 1$

5. Verify that Green's Theorem is true for the line integral
$$\int_C xy\, dx + x^2\, dy$$
   where C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$.

6. Use Green's Theorem to evaluate
$$\int_C (1 + \tan x)\, dx + (x^2 + e^y)\, dy$$
   where C is the positively oriented boundary of the region enclosed by the curves $y = \sqrt{x}$, $x = 1$, and $y = 0$.

7. Find the counterclockwise circulation and outward flux of the field $\mathbf{F}(x, y) = (-\sin x)\,\mathbf{i} + x\cos y\,\mathbf{j}$ around and over the square cut from the first quadrant by the lines $x = \pi/2$ and $y = \pi/2$.

8. Use Green's Theorem to find the work done by the force $\mathbf{F}(x, y) = x(x + y)\,\mathbf{i} + xy^2\,\mathbf{j}$ in moving a particle from the origin along the x-axis to $(1, 0)$, then along the line segment to $(0, 1)$, and then back to the origin along the y-axis

9. Use Green's Theorem to find the area of the regions bounded by the curves with the following vector equation.

   (a) $\mathbf{r}(t) = \cos^3 t\,\mathbf{i} + \sin^3 t\,\mathbf{j}$, $0 \le t \le 2\pi$

   (b) $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin^3 t\,\mathbf{j}$, $0 \le t \le 2\pi$

10. Evaluate
$$\int_C y^2\,dx + 3xy\,dy$$
where C is the positively oriented boundary of the semi-annular region R in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

11. Find the area of the part of the surface $z = x^2 + 2y$ that lies above the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$.

12. Evaluate the given surface integral.

   (a) $\iint_S z\,d\sigma$, where S is the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$

   (b) $\iint_S (x^2 z + y^2 z)\,d\sigma$, where S is the part of the plane $z = 4 + x + y$ that lies inside the cylinder $x^2 + y^2 = 4$

   (c) $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} - 2y\,\mathbf{j} + 3x\,\mathbf{k}$ and S is the sphere $x^2 + y^2 + z^2 = 4$ with outward orientation

13. Verify Stokes' Theorem is true for the vector field $\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$, where S is the part of the plane $x + y + z = 1$ that lies in the first octant, orientated upwards.

14. Use Stokes' Theorem to evaluate
$$\iint_S \operatorname{curl} \mathbf{F}\cdot\mathbf{n}\,d\sigma$$
where $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + yz^2\,\mathbf{j} + z^3e^{xy}\,\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 5$ that lies above the plane $z = 1$ and is oriented upward.

15. Use Stokes' Theorem to evaluate
$$\int_C \mathbf{F}\cdot d\mathbf{r}$$
where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, oriented counterclockwise as viewed from above.

16. Verify that the Divergence Theorem is true for the vector field $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, where D is the unit ball $x^2 + y^2 + z^2 \le 1$.

17. Use the Divergence Theorem to calculate the surface integral
$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$$
where $\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ and S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2$.

18. Compute the outward flux of
$$\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$$
through

   (a) the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$

   (b) the cube bounded by the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$

19. Find curl $\mathbf{F}$ and div $\mathbf{F}$ if $\mathbf{F}(x, y, z) = x^2z\,\mathbf{i} + 2x\sin y\,\mathbf{j} + 2z\cos y\,\mathbf{k}$.

20. Show that there is no vector field $\mathbf{G}$ such that curl $\mathbf{G} = 2x\,\mathbf{i} + 3yz\,\mathbf{j} + xz^2\,\mathbf{k}$.