Math 3100 **Fall 2005**

Exam 1

1. (15 points) Let $\{a_n\}$ be the sequence which is given recursively by $a_1 = 1$, and $a_{n+1} = a_n + (2n+1)$. Use induction to prove that $a_n = n^2$ for all $n \in \mathbb{N}$.

2. (15 points) Evaluate the following limits. Justify your answers, but **do not** use l'Hôpital's rule.

   (a) $\lim\limits_{n\to\infty} \dfrac{n! + n}{2^n + 3n!}$

   (b) $\lim\limits_{n\to\infty} \dfrac{2^n \cos(n)}{3^n}$

   (c) $\lim\limits_{n\to\infty} \ln\left(\frac{n+\ln(n)}{n+1}\right)$

3. (20 points)

   (a) Carefully state the definition of the convergence of a sequence $\{a_n\}$ to a real number L.

   (b) Using the definition of convergence, prove that

   $$\lim_{n\to\infty} \frac{5n+4}{2n-7} = 5/2.$$

4. (15 points) Given any sequence $\{a_n\}$, prove that

   $$\lim_{n\to\infty} a_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} |a_n| = 0.$$

5. (20 points) Assume that $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} b_n = L$. Using only the definition of convergence prove the following two statements;

   (a) There exists $M > 0$ such that $|b_n| \le M$ for all $n \in \mathbb{N}$.

   (b) $\lim\limits_{n\to\infty} a_n b_n = 0$

6. (15 points) Prove that if $\{a_n\}$ is an increasing sequence with $\lim\limits_{n\to\infty} a_n = L$, then $a_n \le L$ for all $n \in \mathbb{N}$.

7. **(Bonus points)** Let $\{a_n\}$ be the Fibonacci sequence given recursively by $a_1 = 1$ $a_2 = 1$ and $a_{n+1} = a_n + a_{n-1}$ for $n > 1$. We now construct a new sequence $\{b_n\}$ by setting $b_n = a_{n+1}/a_n$ for all $n \in \mathbb{N}$.

   (a) Show that $\{b_n\}$ satisfies the recursive formula $b_{n+1} = 1 + 1/b_n$ with $b_1 = 1$, and that $1 \le b_n \le 2$ for all $n \in \mathbb{N}$.

   (b) It follows from the formula in (a) that $b_{n+2} = 1 + \frac{b_n}{1+b_n}$ for all $n \in \mathbb{N}$, using this and induction prove that

      i. the subsequence $\{b_{2n}\}$ is decreasing

      ii. the subsequence $\{b_{2n-1}\}$ is increasing

   (c) Conclude that $\{b_n\}$ converges and $\lim\limits_{n\to\infty} b_n = \dfrac{1+\sqrt{5}}{2}$