Math 3100 **Fall 2005**

Practice Exam 1

1. (10 points) Let $\{a_n\}$ be the sequence which is given recursively by $a_1 = 4$, and $a_{n+1} = a_n + (n+1)(3n+4)$. Use induction to prove that $a_n = n(n+1)^2$ for all $n \in \mathbb{N}$.

2. (20 points) Let $\{a_n\}$ be the sequence given by $a_1 = 2$ and $a_{n+1} = \frac{3a_n+2}{a_n+2}$. Prove that this sequence is convergent and evaluate its limit.
   (Hint: Show that $a_n > 2$ for all $n \in \mathbb{N}$ and that $\{a_n\}$ is decreasing.)

3. (15 points)
   (a) Carefully state the definition of the convergence of a sequence $\{a_n\}$ to a real number L.
   (b) Using the definition of convergence, prove that
   $$\lim_{n\to\infty} \frac{2n+1}{2-n} = -2.$$

4. (15 points)
   (a) Prove the so-called **root test for sequences**, which states that if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\lim_{n\to\infty} a_n = 0$.
   (b) Use this test to determine
      i. $\lim_{n\to\infty} n^2/n^n$
      ii. $\lim_{n\to\infty} n!/n^n$ **(very hard!)**

5. (20 points) Assume that $\lim_{n\to\infty} a_n = \infty$, that is the sequence $\{a_n\}$ diverges to infinity, and that $\lim_{n\to\infty} b_n = L$ with $L > 0$. Prove the following two statements;
   (a) There exists some $N \in \mathbb{N}$ such that if $n > N$, then $b_n > L/2$.
   (b) The sequence $\{a_n b_n\}$ diverges to infinity.

6. (20 points) Evaluate the following limits. Justify your answers and **do not** use l'Hôpital's rule.
   (a) $\lim_{n\to\infty} \sqrt{\frac{n-1}{4n+3}}$
   (b) $\lim_{n\to\infty} \frac{n^2+2^n}{n!+3n^3}$
   (c) $\lim_{n\to\infty} \frac{\sin(n)}{\sqrt{n}}$
   (d) $\lim_{n\to\infty} n! - 2^n$

7. **(Bonus points)** Prove that if $\lim_{n\to\infty} a_n = L$ and $a_n \le M$ for all $n \in \mathbb{N}$, then $L \le M$.