Math 3100 **Fall 2005**

Exam 2

1. (20 points) For each of the following series state whether it converges absolutely, converges conditionally, or diverges. Justify your answers.

   (a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{n^2-n}}$ (b) $\displaystyle\sum_{n=1}^{\infty} n^{-2}\sin(n)$

   (c) $\displaystyle\sum_{n=1}^{\infty} n\sin(n^{-2})$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\ln n}$

2. (15 points) Prove that if the sequence $\{|a_n|\}$ is summable, then $\{a_n\}$ is also summable. Give an example illustating that the converse is false.

3. (15 points) For what values of p does $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ converge? Justify your answer.

4. (20 points) For which values of x do the following series converge?

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(3x)^n}{n^2}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n4^n}$

5. (10 points)

   (a) Evaluate the series $\sum_{n=2}^{\infty} 4^{-n}$

   (b) Find a sequence $\{a_n\}$ so that $\sum_{n=0}^{\infty} a_n x^n = \frac{1}{4-x}$ for all $x \in (-4,4)$.

6. (20 points)

   (a) <u>Prove</u> that if $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, then

   $$\sum_{n=1}^{\infty} a_n \text{ convergent} \iff \sum_{n=1}^{\infty} b_n \text{ convergent.}$$

   (b) Use part (a) to show that if $a_n > 0$ and $\lim_{n\to\infty} na_n = c \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ necessarily diverges.

7. **(Bonus Question)**

   (a) Prove that if θ not equal to an integer multiple of 2π, then

   $$\sum_{k=0}^{n} \sin(k\theta) \leq \frac{1}{\sin(\frac{1}{2}\theta)}.$$

   Hint: Use the trigonometric identity

   $$2\sin x \sin y = \cos(x-y) - \cos(x+y).$$

   (b) Use part (a) to show that for any $p > 0$ the series

   $$\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^p}$$

   converges for all $\theta \in \mathbb{R}$. Carefully state any tests that you use.