Math 3100 **Fall 2005**

Practice Final Exam

Part I

Answer all questions

1. Prove by induction that, for $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} k(k!) = (n+1)! - 1.$$

2. (a) Carefully state the definition of the convergence of a sequence $\{a_n\}$ to a real number L.

   (b) Use the definition of convergence to prove that $\lim_{n\to\infty} \frac{4n}{3n-2} = \frac{4}{3}$.

3. (a) State the property of completeness.

   (b) Prove that the sequence $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$ has a limit.

4. (a) Carefully state the definition of uniform convergence of a sequence of functions $\{f_n\}$ to a function f on a set A.

   (b) Prove that the sequence given by $f_n(x) = nx^3e^{-nx^2}$ converges to zero uniformly on the interval $[0, 1]$.

5. Evaluate the following limits (state any special limits that you use)

   (a) $\lim_{n\to\infty} \frac{n \ln n}{n^2+1}$ (b) $\lim_{n\to\infty} \frac{n^3 - 3^{3n}}{(-1)^n 2^{2n} + \sqrt[n]{6}}$ (c) $\lim_{n\to\infty} \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{n!}$

6. Determine which of the following series converge absolutely, converge conditionally, or diverge. Explain your answers.

   (a) $\sum_{n=0}^{\infty} (-1)^n \frac{n}{2^n}$ (b) $\sum_{n=0}^{\infty} (-1)^n \frac{n^3}{2n^3+1}$ (c) $\sum_{n=0}^{\infty} (-1)^n \frac{n^2}{2n^3+1}$

7. For what values of p does $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^p}$ converge?

8. Find the domain of convergence for the following power series.

   (a) $\sum_{n=0}^{\infty} \frac{(2x)^n}{2n+1}$ (b) $\sum_{n=1}^{\infty} \frac{2x+1)^n}{\sqrt{n}}$

9. Evaluate these sums

$$\text{(a)} \quad \sum_{n=0}^{\infty} 2^{-n} \qquad \text{(b)} \quad \sum_{n=3}^{\infty} \frac{4^{n-1}}{2n-1} \qquad \text{(c)} \quad \sum_{n=1}^{\infty} n^2 3^{-n}$$

10. (a) Carefully state Taylor's Theorem.
    (b) Prove that the $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$.
    (c) Find a polynomial which approximates e^x to within 0.1 for all $|x| \leq 2$.

11. Find power series expansions of the form $\sum_{n=1}^{\infty} a_n x^n$ for the functions

$$\text{(a)} \quad f(x) = \frac{x}{3-x^2} \qquad \text{(b)} \quad g(x) = (2+x^2)e^x$$

12. Find the 6th order Maclaurin polynomial for $\dfrac{xe^{-x}}{x^2+1}$.

Part II

Answer <u>three</u> of the following five questions

1. (a) Carefully state the definition of a Cauchy sequence.
   (b) Prove that every convergent sequence is a Cauchy sequence.

2. Prove that if $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

3. State and prove the Weierstrass M-test.

4. Approximate

$$\int_0^1 \sin(x^2)\, dx$$

   to within 0.01.

5. Prove that if $f_n \to f$ uniformly on $[a, b]$, then

$$\lim_{n\to\infty} \int_a^x f_n(t)\, dt = \int_a^x f(t)\, dt$$

   uniformly on $[a, b]$.