Math 3100 **Fall 2005**

Handout 3 – Sequences and Series of Functions

1. Uniform Convergence of Sequences of Functions

Definition (Pointwise convergence)**.** A sequence $\{f_n\}$ of functions *converges pointwise on* A to a function f if <u>for any given $x \in A$</u> and $\varepsilon > 0$ there exists a $N = N(x, \varepsilon) \in \mathbb{N}$ such that

$$n > N \Longrightarrow |f_n(x) - f(x)| < \varepsilon.$$

We saw in class that pointwise limits do not interact well with the concepts of continuity, integrability and differentability from calculus. For this reason we introduce the following *stronger* notion of convergence.

Definition (Uniform convergence)**.** A sequence $\{f_n\}$ of functions *converges uniformly on* A to a function f if for any given $\varepsilon > 0$ there exists a $N = N(\varepsilon) \in \mathbb{N}$ such that

$$n > N \Longrightarrow |f_n(x) - f(x)| < \varepsilon, \ \underline{\text{for every } x \in A}.$$

The following Theorems illustrate that in contrast to pointwise convergence, *uniform convergence* interacts well with the concepts of calculus.

Theorem 1.1 (Uniform convergence preserves continuity)**.** *If $f_n \longrightarrow f$ uniformly on A and each function f_n is continuous at a point $a \in A$, then the limit function f is also continuous at a.*

Note: Symbolically we can express this result as

$$\lim_{x\to a}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to a} f_n(x).$$

Theorem 1.2 (Integration preserves uniform convergence)**.** *Suppose that $f_n \longrightarrow f$ uniformly on $A = [a, b]$ and each function f_n is continuous on A. If for a fixed $x_0 \in A$ we define*

$$F_n(x) = \int_{x_0}^{x} f_n(t)\,dt \quad \text{and} \quad F(x) = \int_{x_0}^{x} f(t)\,dt$$

for all $x \in A$, then $F_n \longrightarrow F$ uniformly on A.

Note: Symbolically we can express this result as

$$\lim_{n\to\infty}\int_{x_0}^{x} f_n(t)\,dt = \int_{x_0}^{x}\lim_{n\to\infty} f_n(t)\,dt.$$

Note that it is only as a consequence of Theorem 1.1 that the definition of the function F above makes sense.

Unfortunately, differentiation does not preserve uniform convergence; recall that the sequence of functions $f_n(x) = \frac{1}{n}\sin(n^2 x)$ converges to 0 uniformly, but the derived sequence $\{f_n'\}$ does not even converge pointwise! The following result follows from Theorem 1.2 (and the Fundamental Theorem of Calculus).

Theorem 1.3 (Differentiation)**.** *Let $\{f_n\}$ be a sequence of functions on $A = [a, b]$ such that each f_n is differentiable on A with continuous derivative. If $\lim_{n\to\infty} f_n(x_0)$ exists for some $x_0 \in A$ and f_n' converges uniformly on A to some function g, then f_n converges uniformly on A to a <u>differentiable</u> function f with $f' = g$.*

Note: Symbolically we can express this result as

$$\frac{d}{dx}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{d}{dx} f_n(x).$$

2. Uniform Convergence of Infinite Series of Functions

We can apply this notion of convergence (and hence the above results) to series of functions.

Definition (Uniform convergence of series)**.** A series $\sum_{n=0}^{\infty} f_n$ of functions *converges uniformly on* A to a function s if the sequence of partial sums $\{s_n\}$ of the series *converges uniformly on* A to s.

The following result gives a sufficient condition for uniform convergence.

Weierstrass M-test. *If $|f_n(x)| \le M_n$ for each $x \in A$ and the sequence $\{M_n\}$ is summable, then the series $\sum_{n=0}^{\infty} f_n$ converges uniformly on A.*

The following results follow by applying Theorems 1.1, 1.2 and 1.3 respectively to the sequence of partial sums $s_n(x) = \sum_{k=0}^{n} f_k(x)$.

Theorem 2.1. *If $\sum_{n=0}^{\infty} f_n$ converges uniformly on A and each term f_n is continuous at a point $a \in A$, then the sum function s is also continuous at a.*

Note: Symbolically we can express this result as

$$\lim_{x\to a} \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \lim_{x\to a} f_n(x).$$

Theorem 2.2. *Suppose that $\sum_{n=0}^{\infty} f_n$ converges uniformly on $A = [a, b]$ to a sum function s where each term f_n is continuous on A. If for a fixed $x_0 \in A$ we define*

$$S_n(x) = \int_{x_0}^{x} s_n(t)\,dt \left(= \sum_{k=0}^{n} \int_{x_0}^{x} f_n(t)\,dt \right) \quad \text{and} \quad S(x) = \int_{x_0}^{x} s(t)\,dt$$

for all $x \in A$, then $S_n \longrightarrow S$ uniformly on A.

Note: Symbolically we can express this result as

$$\sum_{n=0}^{\infty} \int_{x_0}^{x} f_n(t)\,dt = \int_{x_0}^{x} \sum_{n=0}^{\infty} f_n(t)\,dt.$$

Again it is only as a consequence of Theorem 2.1 that the definition of the function S above makes sense.

Unfortunately, "term-by-term" differentiation may destroy convergence, even if the original series converges uniformly; the series $\sum_{n=0}^{\infty} (\sin nx)/n^2$ converges uniformly for all x by the M-test, but the series obtained by differentiating "term-by-term" is $\sum_{n=0}^{\infty} (\cos nx)/n$, which converges if and only if x is not an integer multiple of 2π.

Theorem 2.3. *Let $\{f_n\}$ be a sequence of functions on $A = [a, b]$ such that each f_n is differentiable on A with continuous derivative. If $\sum_{n=0}^{\infty} f_n(x_0)$ converges for some $x_0 \in A$ and the series of derivatives $\sum_{n=0}^{\infty} f_n'$ converges uniformly on A to a sum function g, then $\sum_{n=0}^{\infty} f_n$ converges uniformly on A to a differentiable sum function s with $s' = g$.*

Note: Symbolically we can express this result as

$$\frac{d}{dx} \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{d}{dx} f_n(x).$$

Note that the problem of interchanging the operations of differntiation and summation is, in general, more serious than in the case of integration. Manipulations that we are familiar with carrying out on finite sums do not alway carry over to infinite series, even if the series involved are uniformly convergent. We turn next to special series of functions, known as power series, which can be manipulated in many respects as though ther were finite sums.

3. Power Series

Definition. A *power series (centered at a)* is a series of the form $\sum_{n=0}^{\infty} a_n(x-a)^n$ where a is some fixed real number, its *domain of convergence* is given by $\{x \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n(x-a)^n \text{ converges}\}$.

Proposition 3.1. *If the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges for $x-a=c$, then it converges absolutely for all x such that $|x-a|<|c|$.*

Corollary 3.2. *The domain of convergence of any power series centered at a is an interval centered at a, it may, however, include neither, one, or both of the endpoints of the interval.*

Definition. If R is a nonnegative real number such that the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges for all real x with $|x-a|<R$ and diverges for all real x with $|x-a|>R$ then R is called the *radius of convergence* for the power series. If the power series converges for all real x, then we say that the radius of convergence is infinite.

Proposition 3.3. *If $\lim_{n\to\infty} |a_n/a_{n+1}| = R$ exists as a finite number or is infinite, then R is the radius of convergence for the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$.*

The following proposition is an immediately consequence of the Weierstrass M-test.

Proposition 3.4. *If the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges for all $|x-a|<R$, then for any $0<c<R$ it converges uniformly on the interval $[a-c,a+c]$.*

In summary, it follows from Proposition 3.1 that if the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ has a radius of convergence R, then it converges absolutely on the open interval $(a-R,a+r)$, and from Proposition 3.4 that it converges uniformly on every closed interval $[a-c,a+c]$, where $0<c<R$.

Proposition 3.5 (Properties of power series)**.** *Let $\sum_{n=0}^{\infty} a_n(x-a)^n$ have radius of convergence $R>0$ and suppose $\sum_{n=0}^{\infty} b_n(x-a)^n$ also converges for $x \in A=(a-R,a+R)$. Define $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ by*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n(x-a)^n.$$

1. (Arithmetic) *If $x \in A$, then*

$$\text{(a)} \quad \lambda f(x) = \sum_{n=0}^{\infty} \lambda a_n(x-a)^n \text{ for all } \lambda \in \mathbb{R}$$

$$\text{(b)} \quad f(x)+g(x) = \sum_{n=0}^{\infty} (a_n+b_n)(x-a)^n$$

$$\text{(c)} \quad f(x)g(x) = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k b_{n-k}\Big)(x-a)^n$$

2. (Calculus) *On $A=(a-R,a+R)$, f is differentiable (and hence continuous) and, for $x \in A$,*

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x-a)^{n-1} \quad \text{and} \quad \int_a^x f(t)\,dt = \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1}.$$

   In paricular these series also have radius of convergence R.

3. (Equating Coefficients) *If $f=g$, then $a_n = b_n = \dfrac{f^{(n)}(a)}{n!}$ for all $n \in \mathbb{N}$.*

The second set of results above follow from the corresponding results in section 2, recall that we actually proved the differentiation result above a little more directly.

4. Taylor Series

Applying Proposition 3.5 (part 2) to the geometric series we are able to establish the validity of the expressions

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} \quad \text{and} \quad \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

for all $|x| < 1$. The above series actually converge for $x = 1$ and $|x| = 1$ respectively, do they also converge to the functions there?

Another consequence of Proposition 3.5 (part 2) is that the sum function of a power series has derivatives of every order, these are obtained by repeated term by term differentiation of the power series. Hence if a given function f has a power series expansion then it must take the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This is called the *Taylor series generated by f at a*. The nth partial sum

$$P_n^f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is called the *nth order Taylor polynomial of f at a*.

Two questions naturally arise: Given a function f, does its Taylor series converge for any x other than $x = a$? If so, is its sum equal to $f(x)$? The answer to both these questions is, in general, no.

Example. The Maclaurin series of the function

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is identically 0, and hence does not converge to f for any $x \neq 0$.

Necessary and sufficient conditions for answering both questions in the affirmative can be expressed in terms of

$$R_n^f(x) := f(x) - P_n^f(x),$$

this is called the *nth remainder of f about a*.

Proposition 4.1. $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \iff P_n^f(x) \longrightarrow f(x) \iff R_n^f(x) \longrightarrow 0.$

Theorem 4.2 (Taylor's Theorem)**.** *Suppose that $f, f', \ldots, f^{(n)}$ are continuous on $[\alpha, \beta]$ and $f^{(n)}$ is differentiable on (α, β), then for any x and a in $[\alpha, \beta]$ there exists a number c between x and a so that*

$$R_n^f(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

We finish this section with a result which answers in the affirmative the first question we posed.

Theorem 4.3 (Abel's Theorem)**.** *Suppose that $\sum_{n=0}^{\infty} a_n$ converges and $f(x) = \sum_{n=0}^{\infty} a_n x^n$, for $|x| < 1$, then $\lim_{x \to 1^-} f(x) = \sum_{n=0}^{\infty} a_n$.*

Standard Maclaurin Series (Taylor series centered at 0)

1. $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n \quad (-1 < x < 1)$
2. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (x \in \mathbb{R})$
3. $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \quad (-1 < x \leq 1)$
4. $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad (-1 \leq x \leq 1)$
5. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad (x \in \mathbb{R})$
6. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad (x \in \mathbb{R})$
7. Binomial Series: For $r \in \mathbb{R}$

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n \quad (x \in A)$$

where $\binom{r}{0} = 1$ and, for $n \in \mathbb{N}$,

$$\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}.$$

The set A depends on r. When $r \in \mathbb{N}$, then $A = \mathbb{R}$. In all cases, $(-1,1) \subseteq A$.

5. Operations on Taylor Polynomials and Series (Informal)

Throughout this section f is assumed to have derivatives of all orders on the open interval $(-R, R)$. We recall that by definition, the nth order Maclaurin polynomial of f is given by

$$P_n^f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

and is related to the nth Maclaurin remainder of f by

$$f(x) - P_n^f(x) = R_n^f(x).$$

The following results are useful when performing calculations involving Taylor series. The first result allows us to reduce our discussions to simply Maclaurin series.

Proposition 5.1 (Translation). *Let $a \in \mathbb{R}$, and define a new function g by $g(x) = f(x-a)$. Then the nth order Taylor polynomial of g about a is $P_n^f(x-a)$ and nth order remainder of g is $R_n^f(x-a)$.*

The following results concern the calculus of Maclaurin polynomials.

Proposition 5.2 (Differentiation). *The nth order Maclaurin polynomial of f' is the derivative of $P_{n+1}^f(x)$ and nth order remainder of f' is the derivative of $R_{n+1}^f(x)$.*

Proposition 5.3 (Integration). *If $F(x) = \int_0^x f(t)\,dt$, for $|x| < R$, then the nth order Maclaurin polynomial of F is $P_n^F(x) = \int_0^x P_{n-1}^f(t)\,dt$ and nth order remainder of F is $R_n^F(x) = \int_0^x R_{n-1}^f(t)\,dt$.*

The result below is useful if you have to find Maclaurin polynomials or remainders of functions like $g(x) = x\sin(x^2)$, for example.

Proposition 5.4 (Monomial multiplication and composition). *Suppose that k and ℓ are positive integers and λ is a real number. If $g(x) = x^k f(\lambda x^\ell)$, then the $(n\ell+k)$th order Maclaurin polynomial of g is $P_{n\ell+k}^g(x) = x^k P_n^f(\lambda x^\ell)$ and $(n\ell+k)$th order remainder of g is $R_{n\ell+k}^g(x) = x^k R_n^f(\lambda x^\ell)$.*