Handout 1 – Sequences

1. Basic Fact

Theorem 1 (Binomial Theorem)**.** *For all $n \in \mathbb{N}$ and $x \in \mathbb{R}$*

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots + nx^{n-1} + x^n$$

where the coefficient of x^k is given by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

2. Special Limits

1. $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$
2. $\lim_{n\to\infty} \frac{1}{n^p} = 0$ if $p > 0$
3. $\lim_{n\to\infty} \sqrt[n]{p} = 1$ if $p > 0$
4. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
5. $\lim_{n\to\infty} \frac{\ln(n)}{n^p} = 0$ if $p > 0$
6. $\lim_{n\to\infty} \frac{n^p}{r^n} = 0$ if $p > 0$ and $|r| > 1$
7. $\lim_{n\to\infty} \frac{r^n}{n!} = 0$ if $r \in \mathbb{R}$
8. $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

3. Two Convergence Tests

Proposition 2 (Ratio test for sequences)**.** *If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\lim_{n\to\infty} a_n = 0$.*

Proposition 3 (Root test for sequences)**.** *If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\lim_{n\to\infty} a_n = 0$.*

4. Useful Theorems

The following theorems were covered in class but do not appear in the notes, they may be used in conjuction with the special limits above to determine the behaviour of more general sequences.

Theorem 4 (Squeeze Theorem)**.** *If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences that satisfy the inequality*

$$a_n \le b_n \le c_n$$

for all sufficiently large $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then

$$\lim_{n\to\infty} b_n = L.$$

Theorem 5 (Reciprocal Theorem)**.** *If $a_n \neq 0$ for all $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} a_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \frac{1}{|a_n|} = \infty$$