

Handout 2 – Series

1. IMPORTANT INFINITE SERIES

Geometric series: $\sum_{n=1}^{\infty} x^n$ converges $\iff |x| < 1$. If $|x| < 1$, then $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$.

The p -series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff p > 1$.

[**Telescoping series:** If $\lim_{n \rightarrow \infty} b_n = L$, then $\sum_{n=1}^{\infty} (b_n - b_{n+1}) = b_1 - L$.]

2. SERIES TESTS

Definition. Given a sequence $\{a_n\}$ let $s_n = a_1 + \cdots + a_n$ denote its n th partial sum, then

$$\{a_n\} \text{ summable} \iff \sum_{n=1}^{\infty} a_n \text{ converges} \iff \underline{\{s_n\} \text{ converges}}.$$

Proposition 1 (Cauchy Criterion).

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \text{for every } \varepsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that } \left| \sum_{k=n}^m a_k \right| \leq \varepsilon \text{ if } m \geq n \geq N.$$

Corollary 2. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

2.1 SERIES OF NON-NEGATIVE TERMS

Proposition 3. If $a_n \geq 0$ and $s_n = a_1 + \cdots + a_n$, then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \{s_n\} \text{ bounded}.$$

Proposition 4 (First Comparison Test). If $0 \leq a_n \leq b_n$ for all sufficiently large $n \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges}.$$

Proposition 5 (Second Comparison Test). Suppose $a_n > 0$, $b_n > 0$, and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \neq 0$, then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges}.$$

Proposition 6 (Cauchy Condensation Test). If $\{a_n\}$ is a decreasing sequence of non-negative terms, then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots \text{ converges}.$$

Proposition 7 (Integral Test). Let $f : (0, \infty) \rightarrow (0, \infty)$ be a decreasing function, $a_n = f(n)$ for all $n \in \mathbb{N}$, and $b_n = \int_1^{n+1} f(x) dx$, then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \{b_n\} \text{ converges}.$$

2.2 SERIES OF BOTH NEGATIVE AND NON-NEGATIVE TERMS

Proposition 8 (Ratio Test). Suppose that $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$, then

$$(a) \sum_{n=1}^{\infty} a_n \text{ converges if } L < 1 \quad (b) \sum_{n=1}^{\infty} a_n \text{ diverges if } L > 1.$$

Proposition 9 (Root Test). Suppose that $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$, then

$$(a) \sum_{n=1}^{\infty} a_n \text{ converges if } L < 1 \quad (b) \sum_{n=1}^{\infty} a_n \text{ diverges if } L > 1.$$

Recall that both the Root and Ratio tests are inconclusive if $L = 1$.

Partial Summation: If $s_n = a_1 + \cdots + a_n$, then it is easy to verify that

$$\sum_{k=1}^n a_k b_k = s_n b_{n+1} + \sum_{k=1}^n s_k (b_k - b_{k+1}).$$

If we let $n \rightarrow \infty$ we see that the series $\sum a_k b_k$ converges if both the series $\sum s_k (b_k - b_{k+1})$ and the sequence $\{s_n b_{n+1}\}$ converge, the next two tests give sufficient conditions for this to indeed happen.

Proposition 10 (Dirichlet Test).

$$\{s_n\} \text{ bounded and } \{b_n\} \text{ decreasing with limit } 0 \implies \sum_{n=1}^{\infty} a_n b_n \text{ converges.}$$

Corollary 11 (Leibniz Test). If $\{b_n\}$ is decreasing with limit 0, then $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.

Proposition 12 (Abel Test).

$$\sum_{n=1}^{\infty} a_n \text{ convergent and } \{b_n\} \text{ monotone and bounded} \implies \sum_{n=1}^{\infty} a_n b_n \text{ converges.}$$

3. STRATEGY FOR ANALYZING $\sum_{n=1}^{\infty} a_n$

1. Does $a_n \rightarrow 0$?

If NO, then $\sum_{n=1}^{\infty} a_n$ diverges.

2. Does $\sum_{n=1}^{\infty} |a_n|$ converge?

If YES, then $\sum_{n=1}^{\infty} a_n$ converges absolutely, and hence converges. Try using

- geometric series and p -series
- first and second comparison tests
- ratio and root tests
- Cauchy condensation and integral tests

3. If $\sum_{n=1}^{\infty} |a_n|$ does not converge or you cannot decide, then try

- alternating series test (Leibniz test)
- partial summation (tests of Dirichlet or Abel)

If either of these tests apply, then $\sum_{n=1}^{\infty} a_n$ converges.

Recall that if

$\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, then we say $\sum_{n=1}^{\infty} a_n$ converges conditionally.