Proposition 1.6.5 Every bounded sequence $\{a_n\}$ of real numbers has a convergent subsequence.

Proof: For each $k \in \mathbb{N}$ we let $S_k = \{a_n : n > k\}$. We describe how to use these sets S_k to pick an appropriate subsequence from $\{a_n\}$. There are two separate cases to consider.

Case 1: For each $k \in \mathbb{N}$ the set S_k has a maximum element

In this case we define a subsequence of $\{a_n\}$ as follows: $b_1 = a_{n_1}$ is the maximum element of S_1, $b_2 = a_{n_2}$ is the maximum element of S_{n_1}, $b_3 = a_{n_3}$ is the maximum element of S_{n_2}, and so on.

It is easy to see that $\{b_n\}$ is a decreasing subsequence of $\{a_n\}$. Since $\{a_n\}$ is bounded, so too is $\{b_n\}$. Thus we can apply the Property of Completeness to conclude that the sequence $\{b_n\}$ converges to some real number.

Case 2: There exists $K \in \mathbb{N}$ such that the set S_K does not have a maximum element

It follows that for any $k > K$ there exists $n > k$ such that $a_n > a_k$ (otherwise the largest of $a_{K+1}, \ldots, a_k$ would be the maximum of S_K).

In this case we define a subsequence of $\{a_n\}$ as follows: $c_1 = a_{K+1}$ and let c_2 be the first term of $\{a_n\}$ following c_1 for which $c_2 > c_1$. Now let c_3 be the first term of $\{a_n\}$ following c_2 for which $c_3 > c_2$, and so on.

It is easy to see that $\{c_n\}$ is an increasing subsequence of $\{a_n\}$. Since $\{a_n\}$ is bounded, so too is $\{c_n\}$. Thus we can apply the Property of Completeness to conclude that the sequence $\{c_n\}$ converges to some real number. $\square$

Exercise: Prove Proposition 1.6.4 using Proposition 1.6.5.