Math 3100 **Fall 2009**

Exam 1

No calculators. Show your work. Give full explanations. Good luck!

1. (20 points)
   (a) Carefully state the definition of the convergence of a sequence $\{a_n\}$ to a real number L.
   (b) Use the definition of convergence to prove that
   $$\lim_{n\to\infty} \frac{3n+5}{n+3} = 3.$$

2. (20 points) Let $\{a_n\}$ be the sequence given by $a_1 = 2$ and
   $$a_{n+1} = \frac{1+a_n}{2}.$$
   (a) Prove by induction that $a_n - 1 > 0$ for all $n \in \mathbb{N}$.
   (b) i. Carefully state the definition of a sequence $\{a_n\}$ being strictly decreasing.
   ii. Prove that $\{a_n\}$ is a strictly decreasing sequence.

3. (15 points)
   (a) Carefully state the definition of a sequence $\{a_n\}$ being bounded above.
   (b) Use this definition to prove that the sequence $a_n = \sqrt{n}$ is <u>not</u> bounded above.

4. (15 points) Evaluate the following limits or explain why it is divergent.
   (a) $\lim_{n\to\infty} \left(\frac{2n+1}{3-n}\right)^3$
   (b) $\lim_{n\to\infty} \left((-1)^n + \frac{1}{n}\right)$
   (c) $\lim_{n\to\infty} \frac{\cos(n)}{n^2}$

5. (15 points) Let $\{a_n\}$ be a sequence with $\lim_{n\to\infty} a_n = 0$, and let $\{b_n\}$ be a bounded sequence. Use the definition of convergence to prove that $\lim_{n\to\infty} a_n b_n = 0$.

6. (15 points) Prove that if $\lim_{n\to\infty} a_n = L$ and $a_n \leq M$ for all $n \in \mathbb{N}$, then $L \leq M$.