Math 3100 **Fall 2009**

Exam 3 - Practice Questions

1. (a) Carefully state the definition of uniform convergence of a sequence of functions $\{f_n\}$ to a function f on a set A.

   (b) Prove that the sequence given by $f_n(x) = nx^3e^{-nx^2}$ converges to zero uniformly on the interval $[0, 1]$.

2. Evaluate these sums

   (a) $\displaystyle\sum_{n=0}^{\infty} 2^{-n}$ (b) $\displaystyle\sum_{n=3}^{\infty} \frac{4^{n-1}}{2n-1}$ (c) $\displaystyle\sum_{n=1}^{\infty} n^2 3^{-n}$

3. (a) State and prove the Weierstrass M-test.

   (b) Prove that if the power series $\sum a_n x^n$ converges for all $|x| < R$, then for any $0 < c < R$ it converges uniformly on the interval $[-c, c]$.

4. Prove that if $f_n \to f$ uniformly on $[a, b]$, then
$$\lim_{n\to\infty} \int_a^x f_n(t)\,dt = \int_a^x f(t)\,dt$$
   uniformly on $[a, b]$.

5. Let $f(x) = \frac{1}{1+3x^2}$. Without differentiating, find $f^{(8)}(0)$. Show your work.

6. Find the Taylor Polynomial of order n generated by f at a.

   (a) $f(x) = \ln x, \quad a = 1, \quad n = 3$

   (b) $f(x) = \sqrt{x+4}, \quad a = 0, \quad n = 2$

7. Find the 6th order Maclaurin polynomial for $\dfrac{xe^{-x}}{x^2+1}$.

8. Find the Taylor Series at $x = 0$ (the Maclaurin Series) of the following functions.

   (a) $x^2 \sin x$

   (b) $\sin^2 x$ *Hint:* $\sin^2 x = (1 - \cos 2x)/2$.

9. Find the Taylor series generated by f at $x = a$.

   (a) $f(x) = x^4 + x^2 + 1, \quad a = -2$

   (b) $f(x) = x^{-2}, \quad a = 1$

10. (a) State Taylor's Theorem and prove that the $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$.

    (b) Find a polynomial which approximates e^x to within 0.1 for all $|x| \le 2$.

11. For what values of x does $1 - x^2/2$ approximate $\cos x$ to within 0.1?

12. (a) How accurately does $1 - x^2 + x^4/2$ approximate e^{-x^2} for $-1 \le x \le 1$?

    (b) Can you find a polynomial that approximates e^{-x^2} to within 0.01 on this interval?

13. Find a polynomial that will approximate
$$F(x) = \int_0^x t^2 e^{-t^2}\,dt$$
    for all x in the interval $[0, 1]$ with an error of magnitude less than 10^{-3}.