Math 4100/6100 **Fall 2006**

Exam 2

Math 4100 students: Answer three *out of the following four questions*
Math 6100 students: Answer all *of the following four questions*

1. (a) Define what it means to say that a function $f : A \to \mathbb{R}$ is *uniformly continous* on A.
   (b) Give an example of a function that is continuous on a set A but fails to be uniformly continuous on A (no proofs required).
   (c) Prove that a function that is continuous on a compact set K is uniformly continuous on K.

2. Let f be a differentiable function on $[a, b]$. We say that f is *uniformly differentiable* on $[a, b]$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \varepsilon$$

   whenever $|x - y| < \delta$ with $x, y \in [a, b]$.

   (a) Prove that f is uniformly differentiable on $[a, b]$ if and only if f' is continuous on $[a, b]$.
   (b) Give an example of a function that is differentiable on $[a, b]$ but fails to be uniformly differentiable on $[a, b]$ (no proofs required).

3. (a) Give a definition of what it means for a bounded function f to be *(Riemann) integrable* on $[a, b]$.
   (b) Give an example of a function f that is not integrable on $[0, 1]$ such that f^2 is integrable on $[0, 1]$ (no proofs required).
   (c) Suppose $f : [0, 1] \to \mathbb{R}$ is integrable and $g : [0, 1] \to \mathbb{R}$ satisfies

$$|g(x) - g(y)| \leq |f(x) - f(y)|$$

   for all $x, y \in [0, 1]$. Use your definition from (a) to prove that g is also integrable.

4. (a) Define what it means for a sequence of functions f_n to *converge uniformly* on A to a function f and what it means to say that the series $\sum_{n=1}^{\infty} f_n$ *converge uniformly* on A to a sum function s.
   (b) State the Weierstrass M-test for the uniform convergence of $\sum_{n=1}^{\infty} f_n$.
   (c) Let $b > 1$ and $f_n(x) = \dfrac{x}{1 + x^n}$ for each $n \in \mathbb{N}$, $x \geq 1$.
      i. Show that $\sum_{n=1}^{\infty} f_n$ converges uniformly to a continuously diffentiable[1] sum function s on $[b, \infty)$ and that $s'(x) = \sum_{n=1}^{\infty} f_n'(x)$ for all $x \in [b, \infty)$.
      ii. Show that f_n converges uniformly to 0 on $[b, \infty)$, but does not converge uniformly to 0 on $(1, \infty)$.

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[1] Recall that a function s is said to be continuously differentiable at a point c if it is differentiable at c and its derivative s' is continuous at c.