Math 4100/6100 Addition Problems 6

Metric Spaces

1. Let $\mathcal{D}$ denote the set of all *dyadic sequences*, i.e., sequences $(x_n) = x_0, x_1, x_2, \ldots$ in $\{0, 1\}$.
   (a) Prove that $\rho((x_n), (y_n)) = \sum_{n=0}^{\infty} 2^{-n}|x_n - y_n|$ defines a metric on $\mathcal{D}$.
   (b) Prove that the set of all sequences in $\mathcal{D}$ which begin $0, 1$ (in that order) is both *open* and *closed*.

2. Prove that $\{f \in C[0, 1] \,|\, f(0) > 0\}$ is open with respect to the uniform metric d_∞, but *not* open with respect to the "taxicab metric" d_1.

3. Let (X, d) be a metric space, $f : X \to \mathbb{R}$ be continuous, and $a \in \mathbb{R}$. Prove that $\{x \in X \,|\, f(x) > a\}$ is open, while the sets $\{x \in X \,|\, f(x) \geq a\}$ and $\{x \in X \,|\, f(x) = a\}$ are both closed.

4. Show that the function $T : C[0, 1] \to C[0, 1]$ defined by $T(f) = f^2$ is continuous, but not uniformly continuous, with respect to the uniform metric. [Hint: Use the sequential characterization of continuity/uniform continuity.]

Termwise integration and differentiation of sequences/series

1. Let $f_n(x) = n^2 x^n (1 - x)$ for each $n \in \mathbb{N}$ and $x \in [0, 1]$. Show that (f_n) converges pointwise to f, say, on $[0, 1]$ and uniformly on $[0, a]$ with $0 < a < 1$ but that $\int_0^1 f_n \nrightarrow \int_0^1 f$. What is the significance of this example in relation to the theorem we learned in class concerning termwise integration of sequences?

2. Let $a > 1$ and $f_n(x) = n^{-1}(1 + x^n)^{-1}$ for each $n \in \mathbb{N}$ and $x \geq 1$. Show that $|f_n'(x)| \leq x^{-n}$ for such n and x. Hence show that $\sum f_n'$ converges uniformly on $[a, \infty)$. Deduce then that $\sum f_n$ converges with C^1 (continuously differentiable) sum, uniformly on $[a, \infty)$.

3. In each case below f_n is defined on $[0, 1]$ for each $n \in \mathbb{N}$.
   (a) Let $f_n(x) = 1/x$ if $1/n \leq x \leq 1$, 0 otherwise.Show that (f_n) has a non-integrable pointwise limit.
   (b) Let $f_n(x) = n^2$ if $0 < x \leq 1/n$, 0 otherwise. Show that (f_n) tends pointwise to 0, but that $\left(\int_0^x f_n\right)$ does not converge pointwise.
   (c) Let $f_n(x) = n$ if $0 < x \leq 1/n$, 0 otherwise. Show that (f_n) tends pointwise to 0, uniformly on $[a, 1]$ for any $a > 0$, but that $\left(\int_0^x f_n\right)$ tends pointwise to 1 on $(0, 1]$.

   What is the significance of these example in relation to the theorem we learned in class concerning termwise integration of sequences?

4. (a) Let $f_n(x) = (x^2 + 1/n)^{1/2}$ for each $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Show that $f_n(x) \to |x|$ uniformly.
   (b) Let $f_n(x) = n^{-1} \sin nx$ for each $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Show that (f_n) tends uniformly to 0, but that (f_n') does not converge pontwise.
   (c) Let $f_n(x) = x(1 + nx^2)^{-1}$ for each $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Show that (f_n) tends uniformly to 0, that (f_n') converges pontwise but not to 0.

   What is the significance of these example in relation to the theorem we learned in class concerning termwise differentiation of sequences?