

Final Exam

Math 4100 students: Answer questions 1–8

Math 6100 students: Answer all of the questions below

1. (15 points) Give examples of the following. No proofs are required.
 - (a) A function that is continuous on a bounded set but fails to be uniformly continuous.
 - (b) A convergent series $\sum_{n=1}^{\infty} a_n$ with $\limsup |a_{n+1}/a_n| > 1$.
 - (c) An everywhere differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ whose derivative is discontinuous at 0.
 - (d) A sequence of differentiable functions that converge uniformly to a function that is not differentiable.
 - (e) A bounded subset of a metric space (X, d) that is complete but not compact.
2. (10 points) Suppose $x_1 = 5$ and $x_{n+1} = \frac{x_n}{2} + \frac{2}{x_n}$.
 - (a) Prove that $x_n > 2$ for all $n \in \mathbb{N}$.
 - (b) Prove that (x_n) is a convergent sequence and find its limit.
3. (10 points) Let A be an uncountable subset of \mathbb{R} .
 - (a) Show that there exists an $n \in \mathbb{N}$ such that $[-n, n]$ contains uncountably many points of A .
 - (b) Prove that A must have a limit point in \mathbb{R} .
4. (10 points) Let (X, d) be a metric space. Prove that a sequence (x_n) converges if and only if it is Cauchy and has a convergent subsequence.
5. (10 points) Suppose f is a bounded integrable function on $[a, b]$. Prove that there exists some $c \in [a, b]$ so that

$$\int_a^c f(t) dt = \int_c^b f(t) dt.$$

6. (5 points) If g is continuous on $[a, b]$, show that there exists some $c \in [a, b]$ so that

$$g(c) = \frac{1}{b-a} \int_a^b g(t) dt.$$

7. (10 points) Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable everywhere and f' is bounded. Prove that f is uniformly continuous.
8. (10 points) For each $n \in \mathbb{N}$ let $f_n(x) = x^n / (x^n + n)$ for $x \geq 0$.
- (a) Find the pointwise limit of (f_n) and deduce that the convergence is *not* uniform with relative to $[0, a]$, when $a > 1$.
 - (b) Show that the convergence is uniform relative to $[0, 1]$.
- 9.* (10 points) Let A be a compact set and $\mathcal{F} = \{U_\alpha\}$ a family of open sets which cover A . Prove that there exists a fixed $\varepsilon > 0$ such that for every $a \in A$ the ball $V_\varepsilon(a)$ is contained in one of the open sets of \mathcal{F} .
- 10.* (10 points) Let (p_n) be a sequence of polynomials defined inductively by

$$p_1(x) = 0 \quad \text{and} \quad p_{n+1}(x) = p_n(x) + \frac{x^2 - p_n(x)^2}{2}$$

for each $n \in \mathbb{N}$, $x \in \mathbb{R}$.

- (a) Prove that $0 \leq p_n(x) \leq x \leq p_n(x) + 2/n$ for each $n \in \mathbb{N}$, and $0 \leq x \leq 1$. [For the third inequality, consider $x \leq 2/(n+1)$ and $x > 2/(n+1)$ separately.]
- (b) Noting that each p_n is even, deduce that $p_n \rightarrow |x|$ uniformly on $[-1, 1]$.