

Exam 1 - Fourier Series

Instructions

4900 students: Answer any three of the following four problems.

6900 students: Answer all four of the following four problems.

Carefully state any results from class which you use in your arguments.

1. We say that a function f is *Hölder continuous* of order $\alpha > 0$ if

$$|f(x+h) - f(x)| \leq C|h|^\alpha$$

for some constant $C > 0$ and all x and h . Let f be a 2π -periodic Riemann integrable function.

- (a) Show that if f is Hölder continuous of order $\alpha > 0$ as defined above, then

$$|\widehat{f}(n)| \leq \left(\frac{C\pi^\alpha}{2}\right)|n|^{-\alpha}$$

for all $n \neq 0$.

Hint: Show that

$$\widehat{f}(n) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + \pi/n) e^{-inx} dx.$$

- (b) Show that every continuously differentiable 2π -periodic function is Hölder continuous of order $\alpha = 1$. What can you say about a function f which is Hölder continuous of order $\alpha > 1$?

Functions which are Hölder continuous of order $\alpha = 1$ are sometimes called Lipschitz continuous. Thus all continuously differentiable functions are Lipschitz continuous.

- (c) Show that if f is continuously differentiable, then in fact $\widehat{f}(n) = o(|n|^{-1})$, that is

$$n\widehat{f}(n) \rightarrow 0 \quad \text{as} \quad |n| \rightarrow \infty.$$

Hint: Use the identity in problem 3(a) below, namely that

$$\widehat{f'}(n) = in\widehat{f}(n)$$

for all $n \in \mathbb{Z}$.

(4900 students should prove this identity if not attempting problem 3).

2. A function f on the circle is said to be *Wiener* if it is continuous and its Fourier series converges absolutely.

- (a) Give examples that show that this is a stronger notion than continuity, but weaker than the notion of being continuously differentiable.
- (b) Prove that if f is a Wiener function, then the partial sums of the Fourier series of f in fact converge uniformly to f .

You may assume the uniqueness of Fourier series: If f is continuous on the circle and $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$.

- (c) Prove that if g and h are Riemann integrable functions on the circle, then $f = g * h$ is actually a Wiener function.

You may assume (as you proved it in your homework) that f is continuous.

3. Let f be a continuously differentiable function which is 2π -periodic.

- (a) Prove that

$$\widehat{f'}(n) = in\widehat{f}(n)$$

for all $n \in \mathbb{Z}$. Be careful to treat both $n \neq 0$ and $n = 0$.

- (b) *Wirtinger's inequality*: Show that if $\int_0^{2\pi} f(x) dx = 0$, then

$$\int_0^{2\pi} |f(x)|^2 dx \leq \int_0^{2\pi} |f'(x)|^2 dx$$

with equality if and only if $f(x) = A \cos x + B \sin x$.

Wirtinger's inequality is in fact equivalent to a stronger version of the isoperimetric inequality where Γ not necessarily assumed to be simple, see Exercise 4.4.

4. (a) Show that the N th Cesàro mean of the Fourier series of f can be expressed as

$$\sigma_N(f)(x) = \sum_{|n| \leq N} \left(1 - \frac{|n|}{N}\right) \widehat{f}(n) e^{inx}.$$

- (b) State Fejér's theorem and the periodic analogue of the Weierstrass approximation theorem (a corollary of Fejér's theorem of course).
- (c) *Hausdorff's moment problem*: Let $[a, b]$ be an interval and let $f, g : [a, b] \rightarrow \mathbb{C}$ be continuous. Prove that if

$$\int_a^b x^k f(x) dx = \int_a^b x^k g(x) dx$$

for all integers $k \geq 0$, then $f(x) = g(x)$ for all $x \in [a, b]$.

Hint: Set $h = f - g$ and use the Weierstrass approximation theorem to approximate $\overline{h(x)}$ by a sequence of polynomials.