Math 4900/6900 **Spring 2007**

Final Exam

Due in class on Monday April 30

Math 6900 students should attempt all of the questions below, Math 4900 students need only answer the unstarred questions. You are permitted to use your own class notes and the course textbook ONLY, please do not discuss the solution to these problems with anyone (except me). Please be careful to state any results from these sources that you use in your arguments.

Have fun and good luck!

Part I: Fourier analysis on $\mathbb{T}$ and $\mathbb{R}$

1. (a) Let f be a continuously differentiable function on the circle. Use the Cauchy-Schwarz inequality and Plancherel's identity for f' to prove that the Fourier series of f converges absolutely (and hence uniformly to f).

   (b)* Prove Bernstein's theorem: If f is 2π-periodic and Hölder continuous[1] of order $\alpha > 1/2$, then the Fourier series of f converges absolutely.
   Hint: Modify the argument in the case $\alpha = 1$ which is outlined in exercise 3.16.

2. For $t > 0$ we define

$$f_t(x) := e^{-2\pi t|x|} \quad \text{and} \quad g_t(x) := \frac{t}{\pi(t^2 + x^2)}.$$

   (a) Show that

$$\widehat{f_t}(\xi) = \frac{t}{\pi(t^2 + \xi^2)}.$$

   (b) What is the Fourier transform of the function g_t? Be sure to explain your answer.

   (c) Show that for all $t > 0$,

$$\frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{t}{t^2 + n^2} = \frac{1 + e^{-2\pi t}}{1 - e^{-2\pi t}}.$$

   (d) Show that the k-fold convolution $g_1 * \cdots * g_1(x) = g_k(x)$.

3. (a) Show that if $B = [-b, b]$, then

$$\widehat{1_B}(\xi) = \frac{\sin 2b\pi\xi}{\pi\xi},$$

   where 1_B is the characteristic function (indicator function) of the set B.

   (b) Let $f \in \mathcal{M}(\mathbb{R})$ such that its Fourier transform $\widehat{f}$ is supported in $B = [-b, b]$. Show that

$$f(t) = \sum_{n \in \mathbb{Z}} f(n/2b) \frac{\sin \pi(2bt - n)}{\pi(2bt - n)}.$$

   Hint: Write $\widehat{f}$ as a Fourier series, then use Fourier inversion (on $\mathbb{R}$), see exercise 5.20.

   This is the *sampling formula for band-limited functions*, also employed by your CD-player. It says that if you have an acoustic signal f such that $f(t) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{2\pi i t\xi} d\xi$ (in some sense), t time, with a limited range of frequencies ξ, i.e. $\text{supp}(\widehat{f}) \subseteq [-b, b]$ for some $b > 0$ (*band-limited*), then the signal f can be reconstructed from the *samples* $f(n/2b)$, $n \in \mathbb{Z}$.

[1] Recall that a function f is said to be *Hölder continuous* of order $\alpha > 0$ if there exists a constant $C > 0$ such that $|f(x + h) - f(x)| \leq C|h|^\alpha$ for all x and h.

Part II: Fourier analysis on $\mathbb{Z}$

4. Prove that if $f : \mathbb{Z} \to \mathbb{C}$ satisfies $\sum_{n\in\mathbb{Z}} |n||f(n)| < \infty$, then the Fourier transform of f on $\mathbb{Z}$, defined by
$$\widehat{f}(\alpha) = \sum_{n\in\mathbb{Z}} f(n)e^{-2\pi i n\alpha}$$
is a continuously differentiable periodic function on $\mathbb{R}$, with period 1.

5. Let B be an interval of M consecutive integers, that is $B = \mu + \{1, \dots, M\}$, for some $\mu \in \mathbb{Z}$.
   (a) Prove that if $\alpha \notin \mathbb{Z}$, then
   $$\widehat{1_B}(\alpha) = \frac{\sin \pi M\alpha}{\sin \pi\alpha}\, e^{-\pi i(2\mu+M+1)\alpha}$$
   and hence that
   $$|\widehat{1_B}(\alpha)| \le \min\left\{M, \frac{1}{2\|\alpha\|}\right\}$$
   for all $\alpha \in \mathbb{R}$ where $\|\alpha\|$ denotes the distance from α to the nearest integer.
   (b) Verify that if $1 < q < \infty$, then
   $$\int_0^1 |\widehat{1_B}(\alpha)|^q \, d\alpha \le C_q M^{q-1},$$
   where C_q is a constant which depends only on q.

6.* Let $A \subseteq \{1, \dots, N\}$ with $|A| = \delta N$ be an ε-uniform set for some $\varepsilon > 0$, that is
$$\sup_{\alpha\in[0,1]} |\widehat{f_A}(\alpha)| \le \varepsilon N$$
where $f_A = 1_A - \delta 1_{\{1,\dots,N\}}$ is the so-called balanced function of A.
   (a) Show that if $2 < p < \infty$, then
   $$\int_0^1 |\widehat{f_A}(\alpha)|^p \, d\alpha \le (\varepsilon N)^{p-2}|A| = \delta\varepsilon^{p-2}N^{p-1}.$$
   (b) Show that if $B \subseteq \{1, \dots, N\}$ is an arbitrary subset with $|B| = \eta N$, then
   $$\big||A \cap B| - \delta\eta N\big| \le \int_0^1 |\widehat{f_A}(\alpha)||\widehat{1_B}(\alpha)| \, d\alpha.$$
   (c) Conclude that if $B \subseteq \{1, \dots, N\}$ is in fact an interval of ηN consecutive integers, then
   $$\big||A \cap B| - \delta\eta N\big| \le C\varepsilon^{1/2}(\delta\eta)^{1/4}N.$$
   Hint: Set $p = 4$ and apply Hölder's inequality: If f, g are integrable function on $[0, 1]$ and $1 < p < \infty$ with $1 = 1/p + 1/q$, then
   $$\int_0^1 |f(x)||g(x)| \, dx \le \left(\int_0^1 |f(x)|^p dx\right)^{1/p} \left(\int_0^1 |g(x)|^q dx\right)^{1/q}.$$
   (d) Carefully explain why part (c) illustrates that ε-uniform sets are indeed "uniform" in a natural sense.

Part III: Fourier analysis on $\mathbb{Z}_N$

Let $f : \mathbb{Z}_N \to \mathbb{C}$ be a function on the cyclic group $\mathbb{Z}_N$ (which we think of as the additive group of integer modulo N), and let $\widehat{f} : \mathbb{Z}_N \to \mathbb{C}$ be its Fourier transform defined by

$$\widehat{f}(r) = \frac{1}{N} \sum_{k \in \mathbb{Z}_N} f(k) e^{-2\pi i k r/N}.$$

7. Derive a formula connecting the Fourier transform of g with the Fourier transform of f when
   (a) $g(k) := f(k-h)$ with $h \in \mathbb{Z}_N$,
   (b) $g(k) := f(k) e^{2\pi i k h/N}$ with $h \in \mathbb{Z}_N$,
   (c) $g(k) := f(h^{-1}k)$ with $h \in \mathbb{Z}_N^*$.

8. Let $f : \mathbb{Z}_N \to \mathbb{C}$ be a function such that $f(k)$ is non-zero for exactly A different values of k, and $\widehat{f}(r)$ is non-zero for exactly B different values of r.
   (a) Verify the validity of the following inequalities
   $$\sum_{r=0}^{N-1} |\widehat{f}(r)|^2 \le B\Big(\max_{r \in \mathbb{Z}_N} |\widehat{f}(r)|\Big)^2 \quad \text{and} \quad \max_{r \in \mathbb{Z}_N} |\widehat{f}(r)| \le \frac{1}{N} \sum_{k=0}^{N-1} |f(k)|.$$
   (b) Use the Cauchy-Schwarz inequality and Plancherel's identity to prove that $AB \ge N$.

   This is the *uncertainty principle for* $\mathbb{Z}_N$, it says that a function and its Fourier transform cannot simultaneously have "narrow support".

9. If $f : \mathbb{Z}_N \to \mathbb{C}$ and $g : \mathbb{Z}_N \to \mathbb{C}$ are two functions, we define their convolution $f * g : \mathbb{Z}_N \to \mathbb{C}$ by the formula
   $$f * g(k) := \frac{1}{N} \sum_{m=0}^{N-1} f(m) g(k-m).$$
   (a) Prove that $\widehat{f * g}(r) = \widehat{f}(r)\widehat{g}(r)$.
   (b) What is the relationship between $\widehat{fg}(r)$ and $\widehat{f} * \widehat{g}(r)$?
   (c) Does there exist a function $\delta : \mathbb{Z}_N \to \mathbb{C}$ such that $f * \delta = \delta * f = f$ for all functions $f : \mathbb{Z}_N \to \mathbb{C}$? If so, what is this function δ? What is the Fourier transform of δ?

10. Suppose that $N = q_1 q_2$ with $q_1, q_2 \in \mathbb{N}$ and define
    $$f(k) = \begin{cases} 1 & \text{if } \quad q_1 | k \\ 0 & \text{otherwise} \end{cases}.$$
    (a) Show that the Fourier transform
    $$\widehat{f}(r) = \begin{cases} 1/q_1 & \text{if } \quad q_2 | r \\ 0 & \text{otherwise} \end{cases}.$$
    (b) For any function $g : \mathbb{Z}_N \to \mathbb{C}$ prove the identity
    $$\frac{1}{q_2} \sum_{n=1}^{q_2} g(nq_1) = \sum_{n=1}^{q_1} \widehat{g}(nq_2).$$

    This is an analogue of the *Poisson summation formula for* $\mathbb{Z}_N$.

    Hint: Apply Parseval's identity:
    $$\frac{1}{N} \sum_{k=0}^{N-1} f(k)\overline{g(k)} = \sum_{r=0}^{N-1} \widehat{f}(r)\overline{\widehat{g}(r)}.$$