# Math 8100 Challenge Problems

*Hand these in to me at some point in the semester*

I. Given any irrational $x$ one can show (using the pigeonhole principle, for example) that there exists infinitely many fractions $a/q$, with $a$ and $q$ relatively prime integers, such that

$$\left|x - \frac{a}{q}\right| \leq \frac{1}{q^2}.$$

However, show that the set of those $x \in \mathbb{R}$ such that there exists infinitely many fractions $a/q$, with $a$ and $q$ relatively prime integers, such that

$$\left|x - \frac{a}{q}\right| \leq \frac{1}{q^3}$$

is a set of measure zero. [*Hint: Use the Borel-Cantelli lemma.*]

II. Let $A$ and $B$ be two measurable subsets of $\mathbb{R}$. Prove that if $m(A) > 0$ and $m(B) > 0$, then the *sumset*

$$A + B = \{a + b \,|\, a \in A,\, b \in B\}$$

contains an interval.

[*Hint: Modify the proof of the special case when $B = -A$ that was presented in class.*]

III. Let $\mathcal{C}$ denote the usual (middle-third) Cantor set. Prove that $\mathcal{C} + \mathcal{C} = [0, 2]$.

[*Hint: Consider the intersection of the set $\mathcal{C} \times \mathcal{C} \subset \mathbb{R}^2$ and the family of lines $\{x + y = c \,|\, c \in [0, 2]\}$ and use the property of nested compact sets.*]

IV. Let us examine the map $f$ defined in Question 1 of Assignment 3 more closely. One readily sees that if $x, y \in \mathcal{C}$ and $x < y$, then $f(x) < f(y)$ unless $x$ and $y$ are the two endpoints of one of the intervals removed from $[0, 1]$ to obtain $\mathcal{C}$. In this case $f(x) = \ell 2^m$ for some integers $\ell$ and $m$, and $f(x)$ and $f(y)$ are the two binary expansions of this number. We can therefore extend $f$ to a map $F : [0, 1] \to [0, 1]$ by declaring it to be constant on each interval missing from $\mathcal{C}$. $F$ is called the **Cantor-Lebesgue function**.

(a) Prove that $F$ is non-decreasing and continuous.

(b) Let $G(x) = F(x) + x$. Show that $G$ is a bijection from $[0, 1]$ to $[0, 2]$.

(c) i. Show that $m(G(\mathcal{C})) = 1$.

ii. By considering rational translates of $\mathcal{N}$ (the non-measurable subset of $[0, 1]$ that we constructed in class), prove that $G(\mathcal{C})$ necessarily contains a (Lebesgue) non-measurable set $\mathcal{N}'$.

iii. Let $E = G^{-1}(\mathcal{N}')$. Show that $E$ is Lebesgue measurable, but not Borel.

(d) Give an example of a measurable function $\varphi$ such that $\varphi \circ G^{-1}$ is not measurable.

[*Hint: Let $\varphi$ be the characteristic function of a set of measure zero whose image under $G$ is not measurable.*]

V. Let $\{f_k\}$ be a sequence of measurable functions on $[0, 1]$ with $|f_k(x)| < \infty$ for a.e. $x$. Show that there exists a sequence of positive real numbers $\{a_k\}$ such that $a_k f_k \to 0$ a.e.

[*Hint: Pick $a_k$ such that $m(\{x : a_k|f_k(x)| > 1/k\}) < 2^{-k}$, and apply the Borel-Cantelli lemma.*]

VI. Let $f$ be a measurable function on $[0, 1]$ with $|f(x)| < \infty$ for a.e. $x$. Prove that there exists a sequence of continuous functions $\{g_k\}$ on $[0, 1]$ such that $g_k \to f$ for a.e. $x \in [0, 1]$.

VII. Let $E \in \mathcal{M}(\mathbb{R}^n)$ and $f : E \times [a,b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, be such that for each $t \in [a,b]$, $f(x,t)$ is an integrable function of $x$. Let $F(t) = \int f(x,t)\,dx$.

(a) Suppose that there exists an integrable function $g$ such that $|f(x,t)| \leq g(x)$ for all $x$ and $t$. Prove that if $\lim_{t\to t_0} f(x,t) = f(x,t_0)$ for every $x$, then $\lim_{t\to t_0} F(t) = F(t_0)$. In particular, if $f$ is continuous in $t$ for each fixed $x$, then $F$ is continuous.

(b) Suppose that $\partial f(x,t)/\partial t$ exists and there exists an integrable function $g$ such that $|\partial f(x,t)/\partial t| \leq g(x)$ for all $x$ and $t$. Prove that $F$ is differentiable and

$$F'(t) = \frac{d}{dt}\int f(x,t)\,dx = \int \frac{\partial f(x,t)}{\partial t}\,dx.$$

[*Hint: Use the dominated convergence theorem with any sequence* $\{t_k\}$ *in* $[a,b]$ *converging to* $t_0$.]

VIII. Complete the following outline (also given in class) to prove that a bounded function on an interval $[a,b]$ is Riemann integrable if and only if its set of discontinuities has measure zero.

Let $f$ be a bounded function on a compact interval $[a,b]$ and

$$\operatorname{osc}(f,c) = \lim_{\delta\to 0} \sup_{x,y\in B_\delta(c)\cap[a,b]} |f(x) - f(y)|$$

define the oscillation of $f$ at $c$. Clearly, $f$ is continuous at $c \in [a,b]$ if and only if $\operatorname{osc}(f,c) = 0$.

(a) Let $A_\varepsilon = \{c \in [a,b] : \operatorname{osc}(f,c) \geq \varepsilon\}$. Prove that for every $\varepsilon > 0$, the set $A_\varepsilon$ is compact.

(b) Prove that if the set of discontinuities of $f$ has measure zero, then $f$ is Riemann integrable.

[*Hint: Let* $\varepsilon > 0$. *Cover* $A_\varepsilon$ *by a finite number of open intervals whose total length is* $\leq \varepsilon$. *Select and appropriate partition of* $[a,b]$ *and estimate the difference between the upper and lower sums of* $f$ *over this partition.*]

(c) Prove that if $f$ is Riemann integrable on $[a,b]$, then its set of discontinuities has measure zero.

[*Hint: The set of discontinuities of* $f$ *is contained in* $\bigcup_n A_{1/n}$. *Given* $\varepsilon > 0$, *choose a partition* $P$ *such that* $U(f,P) - L(f,P) < \varepsilon/n$. *Show that the total length of the intervals in* $P$ *whose interiors intersect* $A_{1/n}$ *is* $\leq \varepsilon$. ]

IX. (a) Prove that if $A, B \in \mathcal{M}(\mathbb{R})$, then $A \times B \in \mathcal{M}(\mathbb{R}^2)$ with $m(A \times B) = m(A)m(B)$.

(b) i. The *continuum hypothesis* asserts that whenever $S$ is an infinite subset of $\mathbb{R}$, then either $S$ is countable, or $S$ has the cardinality of $\mathbb{R}$. Accepting the validity of the continuum hypothesis show that there exists an ordering $\prec$ of $\mathbb{R}$ with the property that for each $y \in \mathbb{R}$ the set $\{x \in \mathbb{R} : x \prec y\}$ is at most countable.

ii. Given the ordering $\prec$ from part (i) we define

$$E = \{(x,y) \in [0,1] \times [0,1] : x \prec y\}.$$

Show that $E$ is <u>not</u> measurable, even though the slices

$$E_x = \{y \in \mathbb{R} : (x,y) \in E\} \quad \text{and} \quad E^y = \{x \in \mathbb{R} : (x,y) \in E\}$$

are both measurable with $m(E_x) = 1$ and $m(E^y) = 0$ for each $x, y \in [0,1]$.

[*Hint for part (i): Let* $\prec$ *denote a well-ordering of* $\mathbb{R}$, *and define*

$$X = \{y \in \mathbb{R} : \textit{the set } \{x : x \prec y\} \textit{ is not countable}\}.$$

*If* $X$ *is empty we are done. Otherwise, consider the smallest element* $y'$ *in* $X$, *and use the continuum hypothesis.*]

X. (a) Let $f : \mathbb{R}^n \to \mathbb{C}$ be measurable. For each $t > 0$, let $\lambda_f(t) = m(\{x : |f(x)| > t\})$. Prove that

$$\sup_{t>0} t^p \lambda_f(t) \leq \int_{\mathbb{R}^n} |f(x)|^p \, dx = p \int_0^\infty t^{p-1} \lambda_f(t) \, dt$$

whenever $1 \leq p < \infty$. Actually the first inequality (which generalizes Tchebychev's inequality) holds for all $0 < p < \infty$.

(b) Prove that if $1 < p < \infty$, then

$$\int_{\mathbb{R}^n} |f^*(x)|^p \, dx \leq \frac{3^n 2^p p}{p-1} \int_{\mathbb{R}^n} |f(y)|^p \, dy.$$

[*Hint: Let* $g(x) = f(x)$ *if* $|f(x)| > t/2$, *and 0 otherwise. Show that* $f^*(x) \leq g^*(x) + t/2$.]

XI. Suppose that $0 < p < q < \infty$. Prove that if $L^q(E) \subseteq L^p(E)$, then $m(E) < \infty$.

XII. Let $g(x)$ be absolutely continuous and increasing on $[a, b]$. Show that if $f$ is an integrable function on $[g(a), g(b)]$, then

$$\int_{g(a)}^{g(b)} f(y) \, dy = \int_a^b f(g(x)) g'(x) \, dx$$

XIII. (**Young's Inequality**) Suppose $1 \leq p, q, r \leq \infty$ with $p^{-1} + q^{-1} = r^{-1} + 1$. Prove that if $f \in L^p$ and $g \in L^q$, then $f * g \in L^r$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q$$

[*Hint: For* $f, g \geq 0$ *and* $p, q, r < \infty$, *write*

$$f * g(x) = \int f(y)^{p/r} g(x-y)^{q/r} \cdot f(y)^{p(1/p-1/r)} \cdot g(x-y)^{q(1/q-1/r)} \, dy$$

*and apply Hölder for* <u>*three*</u> *functions with exponents* $r, p_1, p_2$ *where* $1/p_1 = 1/p - 1/r$ *and* $1/p_2 = 1/q - 1/r$.]

XIV. Suppose that $0 < p_0 < p_1 \leq \infty$. Find examples of functions $f$ on $(0, \infty)$, such that $f \in L^p$ iff

(a) $p_0 < p < p_1$

(b) $p_0 \leq p \leq p_1$

(c) $p = p_0$

[*Hint: Consider functions of the form* $f(x) = x^{-a} |\log x|^b$]