Math 8100 Exam 2

Tuesday 18th of November 2008

1. (a) State and prove Hölder's inequality.
   [*Hint: Recall that* $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$ *if* $a, b \geq 0$, $1 < p < \infty$, *and* $\frac{1}{p} + \frac{1}{p'} = 1$]
   (b) Let $E = [0, 1]$.
      i. Prove that $L^p(E) \subseteq L^1(E)$ and $\|f\|_1 \leq \|f\|_p$ for all $1 < p \leq \infty$.
      ii. Give an example of a function that is in $L^1(E)$, but not in $L^p(E)$ for any $1 < p \leq \infty$.

2. (a) Prove the following version of Tchebyshev's inequality: If $f \in L^2(\mathbb{R}^n)$, then
   $$m(\{x \in \mathbb{R}^n : |f(x)| > \alpha\}) \leq \alpha^{-2}\|f\|_2^2$$
   (b) Prove that $L^2(\mathbb{R}^n)$ is complete.
   [*Hint: Recall that if* $\{f_k\}$ *is Cauchy in measure, then the exists a subsequence* $\{f_{k_j}\}$ *and a function* f *such that* $f_{k_j} \to f$ *almost everywhere.*]

3. (a) State a version of the Lebesgue differentiation theorem.
   (b) Prove that for every $\varepsilon > 0$ there is no set $E \subseteq \mathcal{M}(\mathbb{R})$ with the property that
   $$\varepsilon\, m(I) < m(E \cap I) < (1 - \varepsilon)\, m(I)$$
   for all intervals I.

4. (a) Let $f \in L^p(\mathbb{R}^n)$, with $1 \leq p < \infty$. Prove that
   $$\lim_{|y| \to 0} \|f_y - f\|_p = 0$$
   where $f_y(x) = f(x - y)$.
   [*Hint: Use the fact that continuous functions with compact support are dense in* L^p]
   (b) Give an example showing that the result above fails to hold for $p = \infty$.

5. Let f and g be bounded integrable functions on $\mathbb{R}$.
   (a) Give the definition of the function $f * g$, namely the convolution of f and g.
   (b) Prove that
   $$\lim_{|x| \to \infty} f * g(x) = 0$$
   [*Hint: Recall that* $|x| \leq |x - y| + |y|$]

6. Let $f : [0, 1] \times [0, 1] \to [0, 1]$ be a measurable function, such that for each $x \in [0, 1]$
   $$\int_0^1 f(x, y)\, dy \geq \frac{1}{2}$$
   (a) Show, with justification, that
   $$\int_0^1 \left(\int_0^1 f(x, y)\, dx \right) dy \geq \frac{1}{2}$$
   (b) Prove that
   $$m\left(\left\{ y \in [0, 1] : \int_0^1 f(x, y)\, dx \geq \frac{1}{6} \right\} \right) \geq \frac{1}{3}$$