

Math 8100 Assignment 1
Lebesgue measure and outer measure I

Due date: Thursday 4th of September 2008

1. (a) Prove that the Cantor set \mathcal{C} constructed in the text (and in lecture) is uncountable, totally disconnected, and perfect.
(b) Every real number in $[0, 1]$ has a ternary (or triadic) expansion

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}$$

where $a_k = 0, 1,$ or 2 . Note that this expansion is not unique since, for example, $1/3 = \sum_{k=2}^{\infty} 2/3^k$. Prove that the Cantor set consists of all $x \in [0, 1]$ with a ternary expansion in which the a_k are never equal to 1.

- (c) Find a real number x belonging to the Cantor set which is not the endpoint of one of the intervals used in its construction.
2. The **outer Jordan content** $J_*(E)$ of a set E in \mathbb{R} is defined by

$$J_*(E) = \inf \sum_{j=1}^N |I_j|,$$

where the infimum is taken over every *finite* covering $E \subseteq \cup_{j=1}^N I_j$, by intervals I_j .

- (a) Prove that $J_*(E) = J_*(\bar{E})$ for every set E (here \bar{E} denotes the closure of E).
- (b) Exhibit a countable subset $E \subseteq [0, 1]$ such that $J_*(E) = 1$ while $m_*(E) = 0$.
3. Prove that if E_1 and E_2 are measurable subsets of \mathbb{R}^n , then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

4. **The Borel-Cantelli Lemma.** Suppose $\{E_j\}_{j=1}^{\infty}$ is a countable family of measurable subsets of \mathbb{R}^n and that

$$\sum_{j=1}^{\infty} m(E_j) < \infty.$$

Let

$$E = \limsup_{j \rightarrow \infty} E_j := \{x \in \mathbb{R}^n : x \in E_j, \text{ for infinitely many } j\}.$$

- (a) Show that E is measurable.
- (b) Prove that $m(E) = 0$.

[Hint: Write $E = \cap_{k=1}^{\infty} \cup_{j \geq k} E_j$.]

Challenge Problem I

Hand this in to me at some point in the semester

Given any irrational x one can show (using the pigeonhole principle, for example) that there exists infinitely many fractions a/q , with a and q relatively prime integers, such that

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{q^2}.$$

However, show that the set of those $x \in \mathbb{R}$ such that there exists infinitely many fractions a/q , with a and q relatively prime integers, such that

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{q^3}$$

is a set of measure zero. [Hint: Use the Borel-Cantelli lemma.]