Math 8100 Assignment 10
Hilbert Spaces

Due date: Thursday 4th of December 2008

1. For each $k \in \mathbb{Z}$, define $\varphi_k \in \ell^2(\mathbb{Z})$ by $\varphi_k(j) = 1$ if $j = k$, $\varphi_k(j) = 0$ otherwise. Verify that the set $\{\varphi_k\}_{k\in\mathbb{Z}}$ forms a complete orthonormal system in $\ell^2(\mathbb{Z})$.

2. In $L^2(0,1)$ let $e_0(x) = 1$, $e_1(x) = \sqrt{3}(2x-1)$ for all $x \in (0,1)$.

   (a) Show that e_0, e_1 is an orthonormal system in $L^2(0,1)$.

   (b) Show that the polynomial of degree 1 which is closest with respect to the norm of $L^2(0,1)$ to the function $f(x) = x^2$ is given by $g(x) = x - 1/6$. What is $\|f-g\|_2$?

3. Let E be a subset of a Hilbert space H.

   (a) Show that $E^\perp$ is a closed subspace of H.

   (b) Show that $(E^\perp)^\perp$ is the smallest closed subspace of H that contains E.

4. (a) The first three Legendre polynomials are

   $$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = (3x^2-1)/2.$$

   Show that the orthonormal system in $L^2(-1,1)$ obtained by applying the Gram-Schmidt process to $1, x, x^2$ are scalar multiples of these.

   (b) Compute

   $$\min_{a,b,c} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 \, dx$$

   (c) Find

   $$\max \int_{-1}^{1} x^3 g(x) \, dx$$

   where g is subject to the restrictions

   $$\int_{-1}^{1} g(x)\,dx = \int_{-1}^{1} xg(x)\,dx = \int_{-1}^{1} x^2 g(x)\,dx = 0\,; \quad \int_{-1}^{1} |g(x)|^2\,dx = 1.$$

5. (a) Verify that the following systems are orthogonal in $L^2(E)$:

   i. $\{1/2, \cos x, \sin x, \dots, \cos kx, \sin kx, \dots\}$, when E is any interval of length 2π.

   ii. $\{e^{2\pi i k x/(b-a)}\}_{k=-\infty}^{\infty}$, when $E = (a,b)$.

   (b) Let $f \in L^1(0, 2\pi)$.

   i. Show that for any $\epsilon > 0$ we can write $f = g + h$, where $g \in L^2$ and $\|h\|_1 < \epsilon$.

   ii. Use this decomposition of f to prove the Riemann-Lebesgue lemma:

   $$\lim_{k\to\infty} \int_0^{2\pi} f(x) \cos kx \, dx = \lim_{k\to\infty} \int_0^{2\pi} f(x) \sin kx \, dx = 0$$

Challenge Problem XIV

Hand this in to me at some point in the semester

XIV. Suppose that $0 < p_0 < p_1 \le \infty$. Find examples of functions f on $(0, \infty)$, such that $f \in L^p$ iff

(a) $p_0 < p < p_1$

(b) $p_0 \le p \le p_1$

(c) $p = p_0$
[*Hint: Consider functions of the form* $f(x) = x^{-a} |\log x|^b$]