Math 8100 Assignment 3
Lebesgue measurable functions I

Due date: Thursday 18th of September 2008

1. The Cantor set $\mathcal{C}$ is the set of all $x \in [0,1]$ that have a ternary expansion $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ with $a_k \neq 1$ for all k. One can show that $\mathcal{C}$ is uncountable by considering the function $f(x) = \sum_{k=1}^{\infty} b_k 2^{-k}$ where $b_k = a_k/2$, since the series defining $f(x)$ is the binary expansion of a number in $[0,1]$ (and every number in $[0,1]$ can be obtained in this way) it follows that f maps $\mathcal{C}$ onto $[0,1]$.
   (a) Show that f is well defined and continuous on $\mathcal{C}$, and moreover $f(0) = 0$ as well as $f(1) = 1$.
   (b) Prove that there exists a continuous function that maps a measurable set to a non-measurable set.

2. Let $\chi_{[0,1]}$ be the characteristic function of $[0,1]$. Show that there is no everywhere continuous function f on $\mathbb{R}$ such that $f = \chi_{[0,1]}$ almost everywhere.

3. (a) Let Z be a measurable subset of $\mathbb{R}^n$ with $m(Z) = 0$. Show that if $E \cup Z$ is a measurable subset of $\mathbb{R}^n$, then E is also a measurable subset of $\mathbb{R}^n$ with $m(E \cup Z) = m(E)$.
   (b) Suppose that f is measurable and $f = g$ almost everywhere. Show that g is also measurable and
   $$m(\{f > a\}) = m(\{g > a\})$$
   for all $a \in \mathbb{R}$.
   (c) Let g be measurable. Prove that if $g \neq 0$ almost everywhere, then $1/g$ is measurable.

4. Let f and f_k, $k = 1, 2, \ldots$, be measurable functions that are assumed to take finite values at almost every point in a measurable set E. We proved in class that if $f_k \to f$ a.e. on E and $m(E) < \infty$, then f_k also converges in measure on E to f.
   (a) Give an example showing (with full justification) that if $m(E) = \infty$ then the conclusion above may not hold.
   (b) Convergence in measure does not however imply pointwise convergence a.e. (even for sets of finite measure). Give an example (again with full justification) demonstrating this fact.
   (c) Let g and h be measurable functions. Prove that that if f_k converges in measure on E to both g and h, then g must equal h almost everywhere.

Challenge Problem IV

Hand this in to me at some point in the semester

Let us examine the map f defined in Question 1 more closely. One readily sees that if $x, y \in \mathcal{C}$ and $x < y$, then $f(x) < f(y)$ unless x and y are the two endpoints of one of the intervals removed from $[0,1]$ to obtain $\mathcal{C}$. In this case $f(x) = \ell 2^m$ for some integers ℓ and m, and $f(x)$ and $f(y)$ are the two binary expansions of this number. We can therefore extend f to a map $F : [0,1] \to [0,1]$ by declaring it to be constant on each interval missing from $\mathcal{C}$. F is called the **Cantor-Lebesgue function**.

(a) Prove that F is non-decreasing and continuous.
(b) Let $G(x) = F(x) + x$. Show that G is a bijection from $[0,1]$ to $[0,2]$.
(c) i. Show that $m(G(\mathcal{C})) = 1$.
   ii. By considering rational translates of $\mathcal{N}$ (the non-measurable subset of $[0,1]$ that we constructed in class), prove that $G(\mathcal{C})$ necessarily contains a (Lebesgue) non-measurable set $\mathcal{N}'$.
   iii. Let $E = G^{-1}(\mathcal{N}')$. Show that E is Lebesgue measurable, but not Borel.
(d) Give an example of a measurable function φ such that $\varphi \circ G^{-1}$ is not measurable.
[*Hint: Let φ be the characteristic function of a set of measure zero whose image under G is not measurable.*]