Math 8100 Assignment 4
Lebesgue measurable functions II

Due date: Thursday 25th of September 2008

1. Let $f_n(x) = n^2 x^n (1-x)$ on the interval $0 \le x \le 1$.
   (a) Does $\lim_{n\to\infty} f_n(x)$ exist for every $x \in [0,1]$?
   (b) On what closed subsets $F \subseteq [0,1]$ does the sequence $\{f_n\}$ converge uniformly?
   (c) Does $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$ exist?
   (d) Let $\alpha > 0$. Does $\lim_{n\to\infty} m(\{x \in [0,1] : f_n(x) \ge \alpha\})$ exist?

2. Let $\{f_k\}$ be a sequence of measurable functions on a measurable set E.
   (a) We sometimes (in particular now and in Question 3b below) say that $\{f_k\}$ *converges almost uniformly* to f on E if for every $\varepsilon > 0$ there exists a closed set $F \subseteq E$ with $m(E \setminus F) < \varepsilon$ such that $f_k \to f$ uniformly on F.
   Prove that if $f_k \to f$ almost uniformly, then $f_k \to f$ almost everywhere and $f_k \to f$ in measure.
   (b) Suppose that $m(E) < \infty$ and $|f_k(x)| \le M_x$ for all $k \in \mathbb{N}$ for every $x \in E$.
      i. Prove that f_k is *almost uniformly bounded* on E: For every $\varepsilon > 0$ there exists a $M > 0$ and a closed set $F \subseteq E$ with $m(E \setminus F) < \varepsilon$ such that $|f_k(x)| \le M$ for all $x \in F$ and $k \in \mathbb{N}$.
      ii. Does the same conclusion hold if $m(E) = \infty$?
   [*Hint: For part (i) consider the sets* $E_{k,N} = \{x \in E : |f_k(x)| \le N\}$.]

3. Let f be defined and continuous on the square $\{(x,y) \,|\, 0 \le x \le 1,\ 0 < y \le 1\}$. Suppose that $f(x) = \lim_{y\to 0} f(x,y)$ exists for every $x \in [0,1]$.
   (a) Prove that if $0 < \varepsilon < 1$ and $0 < \delta < 1$, the set
   $$E_{\varepsilon,\delta} = \{x \in [0,1] : |f(x,y) - f(x)| \le \varepsilon \text{ for all } 0 < y \le \delta\}$$
   is measurable.
   [*Hint: If* $\{y_k\}_{k=1}^{\infty}$ *is a dense subset of* $(0,\delta]$, *show that* $E_{\varepsilon,\delta} = \bigcap_k \{x \in [0,1] : |f(x,y_k) - f(x)| \le \varepsilon\}$.]
   (b) Prove that $f(\cdot, y)$ converges almost uniformly (see Question 2a above) to f on $[0,1]$ as $y \to 0$.
   [*Hint: Follow the proof of Egorov's theorem, using the sets* $E_{\varepsilon,1/j}$ *defined above for the sets* E_j.]

4. Let $Q = \{(x,y) \in \mathbb{R}^2 : |x| \le 1 \text{ and } |y| \le 1\}$.
   (a) Construct a function f defined on Q such that f is continuous in each variable separately, such that $|f(x,y)| \le 1$ on Q, but such that f is not continuous at $(0,0)$.
   (b) Construct a function g defined on Q such that g is continuous in each variable separately, such that $|g(x,y)| \le 1$ on Q, but such that g is not continuous at a dense set of points of Q.
   (c) Let h be any function defined on Q which is continuous in each variable separately. Show that h is measurable.
   [*Hint: For each* k *divide the* xy*-plane into vertical strips of width* $1/k$. *Set* $h_k(x,y) = h(x,y)$ *on each of the lines* $x = j/k$, *and interpolate linearly (holding* y *fixed and letting* x *vary) in between.*]

Challenge Problems V & VI

Hand these in to me at some point in the semester

V. Let $\{f_k\}$ be a sequence of measurable functions on $[0,1]$ with $|f_k(x)| < \infty$ for a.e. x. Show that there exists a sequence of positive real numbers $\{a_k\}$ such that $a_k f_k \to 0$ a.e.
[*Hint: Pick* a_k *such that* $m(\{x : a_k|f_k(x)| > 1/k\}) < 2^{-k}$, *and apply the Borel-Cantelli lemma.*]

VI. Let f be a measurable function on $[0,1]$ with $|f(x)| < \infty$ for a.e. x. Prove that there exists a sequence of continuous functions $\{g_k\}$ on $[0,1]$ such that $g_k \to f$ for a.e. $x \in [0,1]$.