Math 8100 Assignment 9
Differentiation II & L^p Spaces

Due date: Thursday 13th of November 2008

1. (a) Let $f(x) = x^2 \sin(x^{-2})$ if $x \neq 0$, $f(0) = 0$. Show that f' exists everywhere, but $f' \notin L^1([0,1])$.
   (b) Show directly that the Cantor-Lebesgue function is not absolutely continuous.

2. Prove that if F and G are absolutely continuous on $[a,b]$, then so is FG, and

$$\int_a^b (FG' + GF')(x)\,dx = F(b)G(b) - F(a)G(a)$$

3. Let $F_\delta(x) = \delta^{-1}F(x/\delta)$ where

$$F(x) = \frac{1}{\pi}\Big(\frac{\sin x}{x}\Big)^2.$$

Prove that if $1 \leq p < \infty$ and $f \in L^p(\mathbb{R})$, then

$$\lim_{\delta \to 0} \|f * F_\delta - f\|_p = 0$$

4. Let $a = \{a_j\}_{j=-\infty}^{\infty}$ be a sequence of complex numbers, and let

$$\|a\|_p = \Big(\sum_{j=-\infty}^{\infty} |a_j|^p\Big)^{1/p} \quad \text{if } 0 < p < \infty \qquad \text{and} \qquad \|a\|_\infty = \sup_j |a_j| \quad \text{if } p = \infty.$$

Then, for $0 < p \leq \infty$ we define

$$\ell^p(\mathbb{Z}) = \{a = \{a_j\}_{j\in\mathbb{Z}} \,:\, \|a\|_p < \infty\}.$$

   (a) Let p and p' be conjugate exponents with $1 \leq p \leq \infty$, $a = \{a_j\}$, $b = \{b_j\}$, and $ab = \{a_j b_j\}$. Prove Hölder's inequality: $\|ab\|_1 \leq \|a\|_p \|b\|_{p'}$.
   (b) Suppose $1 \leq p \leq \infty$, $a = \{a_j\}$, $b = \{b_j\}$, and $a + b = \{a_j + b_j\}$. Prove Minkowski's inequality: $\|a + b\|_p \leq \|a\|_p + \|b\|_p$.
   (c) Prove that if $0 < p < q \leq \infty$, then $\ell^p(\mathbb{Z}) \subseteq \ell^q(\mathbb{Z})$ and $\|a\|_q \leq \|a\|_p$. [*Hint: Consider $q = \infty$ first.*]
   (d) Prove that $\ell^p(\mathbb{Z})$ is a Banach space whenever $1 \leq p < \infty$.

5. (Hilbert's Inequality) Let

$$Tf(x) = \int_0^\infty \frac{f(y)}{x+y}\,dy$$

   (a) Show that Tf satisfies the norm inequality

$$\Big(\int_0^\infty |Tf(x)|^p dx\Big)^{1/p} \leq C_p \Big(\int_0^\infty |f(x)|^p dx\Big)^{1/p}$$

   for $1 < p < \infty$, with

$$C_p = \int_0^\infty \frac{1}{x^{1/p}(x+1)}\,dx.$$

   (b) Show, without using complex analysis, that

$$C_p \leq \frac{p^2}{p-1}.$$

   Remark: It is a standard exercise in contour integration to show that in fact $C_p = \pi/\sin(\pi/p)$.

Challenge Problem XI, XII & XIII

Hand this in to me at some point in the semester

XI. Suppose that $0 < p < q < \infty$. Prove that if $L^q(E) \subseteq L^p(E)$, then $m(E) < \infty$.

XII. Let $g(x)$ be absolutely continuous and increasing on $[a, b]$. Show that if f is an integrable function on $[g(a), g(b)]$, then

$$\int_{g(a)}^{g(b)} f(y)\,dy = \int_a^b f(g(x))g'(x)\,dx$$

XIII. (**Young's Inequality**) Suppose $1 \leq p, q, r \leq \infty$ with $p^{-1} + q^{-1} = r^{-1} + 1$. Prove that if $f \in L^p$ and $g \in L^q$, then $f * g \in L^r$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q$$

[*Hint: For* $f, g \geq 0$ *and* $p, q, r < \infty$, *write*

$$f * g(x) = \int f(y)^{p/r} g(x-y)^{q/r} \cdot f(y)^{p(1/p-1/r)} \cdot g(x-y)^{q(1/q-1/r)}\,dy$$

and apply Hölder for <u>*three*</u> *functions with exponents* r, p_1, p_2 *where* $1/p_1 = 1/p - 1/r$ *and* $1/p_2 = 1/q - 1/r$.]