TWO-WEIGHT ESTIMATES FOR SINGULAR AND STRONGLY SINGULAR INTEGRAL OPERATORS

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Abstract. In this article we consider conditional *two-weight* estimates for singular and strongly singular integral operators. The conditions governing two-weight estimates shall be simultaneously necessary and sufficient for a quite large class of singular integrals.

1. Introduction

In the sequel we shall assume that K is a distributional kernel that satisfies the estimate

$$|K(x,y)| \le \frac{A}{|x-y|^{n+\alpha}}, \tag{1}$$

whenever $x \neq y$ for some $\alpha \geq 0$. Moreover, we assume that the operator

$$Tf(x) = \int_{\mathbf{R}^n} K(x,y) f(y)\, dy \ , \quad x \notin \operatorname{supp} f, \tag{2}$$

which is initially defined for function $f \in \mathcal{S}(\mathbf{R}^n)$, extends to a bounded operator on $L^2(\mathbf{R}^n)$.

We shall assume that a weight ρ is an almost everywhere positive function on $\mathbf{R}^n$ and denote by $L^p_\rho(\mathbf{R}^n)$, for $1 \le p < \infty$, the space of all measurable functions $f : \mathbf{R}^n \to \mathbf{R}$ for which

$$\|f\|_{L^p_\rho(\mathbf{R}^n)} := \left(\int_{\mathbf{R}^n} |f(x)|^p \rho(x)\, dx \right)^{1/p} < \infty.$$

We denote by $L^{p,\infty}_\rho(\mathbf{R}^n)$, for $1 \le p < \infty$, the space of all measurable functions $f : \mathbf{R}^n \to \mathbf{R}$ for which

$$\|f\|_{L^{p,\infty}_\rho(\mathbf{R}^n)} := \sup_{\lambda > 0} \ \lambda \left(\int_{\{x : |f(x)| > \lambda\}} \rho(x)\, dx \right)^{1/p} < \infty.$$

For convenience we shall often abbreviate $L^p_\rho(\mathbf{R}^n)$ and $L^{p,\infty}_\rho(\mathbf{R}^n)$ by L^p_ρ and $L^{p,\infty}_\rho$ respectively.

We shall say that an operator is of *two-weight* strong-type (p,p) or *two-weight* weak-type (p,p), for $1 \le p < \infty$, if it is bounded from $L^p_{\rho_1}$ to $L^p_{\rho_2}$ or from $L^p_{\rho_1}$ to $L^{p,\infty}_{\rho_2}$ respectively.

In this article we will be concerned with conditional *two-weight* estimates for operators T defined by (2) with kernel satisfying (1). In our arguments we use known boundedness properties of appropriate singular integrals and two-weight criteria for the Hardy transforms. We also establish necessary conditions for such estimates to hold in the case where $\alpha = 0$.

For *one-weight* estimates it is a well known result of Stein [33] that operators given by (2) with kernels satisfying condition (1) for $\alpha = 0$ that are bounded on L^p for $1 < p < \infty$ will also be bounded on $L^p_\rho(\mathbf{R}^n)$, with $\rho(x) = |x|^\lambda$ and $-n < \lambda < n(p-1)$. For related topics when $p = 1$ see Hoffman [18]. The results of [33] were later extended by Soria and Weiss [32] to the case of general

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A_p weights and to certain maximal singular integrals. *One-weight* estimates have been obtained in the case where $\alpha > 0$ by Chanillo [3].

For convenience we recall that ρ is an A_p weight for $1 < p < \infty$, or more succinctly $\rho \in A_p$, if

$$\sup_{B \subset \mathbf{R}^n} \left(\frac{1}{|B|} \int_B \rho(x)\, dx \right)^{1/p} \left(\frac{1}{|B|} \int_B \rho^{-p'/p}(x)\, dx \right)^{1/p'} < \infty,$$

where $p' = \frac{p}{p-1}$ and the supremum is taken over all balls in $\mathbf{R}^n$. Passing to the limit in the definition above we obtain the following characterization of the class A_1, namely that $\rho \in A_1$ if

$$\sup_{B \subset \mathbf{R}^n} \left(\frac{1}{|B|} \int_B \rho(x)\, dx \right) \left\| \rho^{-1} \right\|_{L^\infty(B)} < \infty.$$

Recall also that if $\rho \in A_p$, then $\rho^{-p'/p} \in A_{p'}$, where again $p' = \frac{p}{p-1}$.

2. Main Results

2.1. Positive results in the case where $\alpha \geq 0$. Our first result establishes a sufficient condition for our operators T to be of *two-weight* strong-type (p,p) when $1 < p < \infty$.

Theorem 1. *Let $1 < p < \infty$ and T be an operator defined by (2) with kernel satisfying (1) with $\alpha \geq 0$ that is bounded on $L^p(\mathbf{R}^n)$. If v_0 and w_0 are positive monotonic functions on $(0,\infty)$ such that the weights $v(x) = v_0(|x|)$ and $w(x) = w_0(|x|)$ satisfy the condition*

$$B_{\alpha,d}(v,w) := \sup_{t>0} \left(\int\limits_{t \leq |x|} v(x)(|x|^{-\alpha}+1)^p |x|^{-np} dx \right)^{1/p} \left(\int\limits_{|x| \leq t/d} w^{-p'/p}(x)\, dx \right)^{1/p'} < \infty$$

if v_0 and w_0 are increasing or

$$B'_{\alpha,d}(v,w) := \sup_{t>0} \left(\int\limits_{|x| \leq t/d} v(x)\, dx \right)^{1/p} \left(\int\limits_{t \leq |x|} w^{-p'/p}(x)(|x|^{-\alpha}+1)^{p'} |x|^{-np'} dx \right)^{1/p'} < \infty$$

if v_0 and w_0 are decreasing, for some $d > 1$, then T is bounded from L^p_w to L^p_v. Moreover

$$\|Tf\|_{L^p_v} \leq C_1 B_{\alpha,d}(v,w) \ \left[\text{or } B'_{\alpha,d}(v,w)\right] \|f\|_{L^p_w},$$

where $C_1 = C_1(\|T\|_{L^p \to L^p}, A, p, n, \alpha, d)$.

Remark 1. If w satisfies the doubling condition:

$$\int_{|x| \leq 2t} w(x)\, dx \leq c' \int_{|x| \leq t} w(x)\, dx,$$

then so does $w^{-p'/p}$, and as a consequence $B_{\alpha,d}(v,w) \leq A_\alpha(v,w)$, where

$$A_\alpha(v,w) := \sup_{t>0} t^{-n}(t^{-\alpha}+1) \left(\int_{|x| \leq t} v(x)\, dx \right)^{1/p} \left(\int_{|x| \leq t} w^{-p'/p}(x)\, dx \right)^{1/p'}.$$

We include the proof of this statement as an appendix.

Our second result establishes a sufficient condition for our operators T to be of *two-weight* weak-type (p,p) when $1 \leq p < \infty$.

Theorem 2. *Let* $1 \le p < \infty$ *and* T *be an operator defined by (2) with kernel satisfying (1) with* $\alpha \ge 0$ *that is bounded from* $L^p(\mathbf{R}^n)$ *to* $L^{p,\infty}(\mathbf{R}^n)$*. If* v_0 *and* w_0 *are positive increasing functions on* $(0,\infty)$ *such that the weights* $v(x) = v_0(|x|)$ *and* $w(x) = w_0(|x|)$ *satisfy the condition*

$$B^{(p)}_{\alpha,d}(v,w) := \sup_{0<t<\tau} \frac{\tau^{-\alpha}+1}{\tau^n} \left(\int\limits_{t\le|x|\le\tau} v(x)\,dx \right)^{1/p} \left(\int\limits_{|x|\le t/d} w^{-p'/p}(x)\,dx \right)^{1/p'} < \infty$$

if $1 < p < \infty$*, and*

$$B^{(1)}_{\alpha,d}(v,w) := \sup_{0<t<\tau} \frac{\tau^{-\alpha}+1}{\tau^n} \left(\int\limits_{t\le|x|\le\tau} v(x)\,dx \right) \left\| w^{-1} \right\|_{L^\infty(\{|\cdot|<t/d\})} < \infty$$

if $p = 1$*, for some* $d > 1$*, then* T *is bounded from* L^p_w *to* $L^{p,\infty}_v$*. Moreover*

$$\|Tf\|_{L^{p,\infty}_v} \le C_2 B^{(p)}_{\alpha,d}(v,w) \|f\|_{L^p_w},$$

where $C_2 = C_2(\|T\|_{L^p\to L^{p,\infty}}, A, p, n, \alpha, d)$*.*

2.1.1. **Examples**. Let $1 < p < \infty$ and recall that $|x|^\gamma \in A_p(\mathbf{R}^n)$ if and only if $-n < \gamma < n(p-1)$.

For simplicity we shall restrict our examples to the case where $n = 1$. It is known (see [8]) that if

$$v(x) = \begin{cases} |x|^{p-1} & \text{if } 0 < |x| \le 1 \\ |x|^\gamma & \text{if } |x| > 1 \end{cases}$$

$$w(x) = \begin{cases} |x|^{p-1}(1-\log|x|)^p & \text{if } 0 < |x| \le 1 \\ |x|^\gamma & \text{if } |x| > 1 \end{cases}$$

with $0 < \gamma < p-1$, then the Hilbert transform is bounded from L^p_w to L^p_v. Furthermore, if

$$v(x) = \begin{cases} |x|^{p-1}(1-\log|x|)^p & \text{if } 0 < |x| \le 1 \\ |x|^\gamma & \text{if } |x| > 1 \end{cases}$$

$$w(x) = \begin{cases} |x|^{p-1}(1-\log|x|) & \text{if } 0 < |x| \le 1 \\ |x|^\gamma & \text{if } |x| > 1 \end{cases}$$

with $0 < \gamma < p-1$, then the Hilbert transform is bounded from L^p_w to $L^{p,\infty}_v$, but is not bounded from L^p_w to L^p_v. See [10] page 557.

The following two examples are an immediate consequence of Theorem 1.

Example 1. Suppose that T is an operator defined by (2) with kernel satisfying (1) with $\alpha \ge 0$ that is bounded on $L^p(\mathbf{R})$. If we set

$$v(x) = \begin{cases} |x|^{\gamma+\alpha p} & \text{if } 0 < |x| \le 1 \\ |x|^\gamma & \text{if } |x| > 1 \end{cases}$$

$$w(x) = |x|^\gamma \quad \text{if } |x| > 0$$

with $0 < \gamma < p-1$, then T is bounded from L^p_w to L^p_v.

Example 2. Suppose that T is an operator defined by (2) with kernel satisfying (1) with $\alpha \geq 0$ that is bounded on $L^p(\mathbf{R})$. If we set

$$v(x) = \begin{cases} |x|^{p-1+\alpha p} & \text{if } 0 < |x| \leq 1 \\ |x|^\gamma & \text{if } |x| > 1 \end{cases}$$

$$w(x) = \begin{cases} |x|^{p-1}(1-\log|x|)^p & \text{if } 0 < |x| \leq 1 \\ |x|^\gamma & \text{if } |x| > 1 \end{cases}$$

with $0 < \gamma < p-1$, then T is bounded from L^p_w to L^p_v.

The following is an immediate consequence of Theorem 2.

Example 3. Suppose that T is an operator defined by (2) with kernel satisfying (1) with $\alpha \geq 0$ that is bounded from $L^p(\mathbf{R})$ to $L^{p,\infty}(\mathbf{R})$. If we set

$$v(x) = \begin{cases} |x|^{p-1+\alpha p}(1-\log|x|)^p & \text{if } 0 < |x| \leq 1 \\ |x|^\gamma & \text{if } |x| > 1 \end{cases}$$

$$w(x) = \begin{cases} |x|^{p-1}(1-\log|x|) & \text{if } 0 < |x| \leq 1 \\ |x|^\gamma & \text{if } |x| > 1 \end{cases}$$

with $0 < \gamma < p-1$, then T is bounded from L^p_w to $L^{p,\infty}_v$.

2.1.2. **Local Properties in the case where** $\alpha \geq 0$. Our third and final result in the generality of $\alpha \geq 0$ concerns the local properties of our operator T.

We make the assumption here that our operators T are local; that the boundedness of T on L^p is equivalent to the following estimate holding uniformly in x_0,

$$\int_{|x-x_0|\leq 1} |Tf(x)|^p dx \leq C_0 \int_{|x-x_0|\leq 10} |f(x)|^p dx. \tag{3}$$

Theorem 3. *Let* $1 < p < \infty$ *and* T *be an operator defined by (2) with kernel satisfying (1) with* $\alpha \geq 0$ *that satisfies (3). If* v_0 *and* w_0 *are positive monotonic functions on* $(0,10)$ *such that the weights* $v(x) = v_0(|x|)$ *and* $w(x) = w_0(|x|)$ *satisfy the condition*

$$B^{\mathrm{loc}}_{\alpha,d}(v,w) := \sup_{0<t<1} \left(\int\limits_{t\leq|x|\leq 1} v(x)|x|^{-(n+\alpha)p} dx \right)^{1/p} \left(\int\limits_{|x|\leq t/d} w^{-p'/p}(x)\, dx \right)^{1/p'} < \infty$$

if v_0 *and* w_0 *are increasing or*

$$B'^{\mathrm{loc}}_{\alpha,d}(v,w) := \sup_{0<t<1} \left(\int\limits_{|x|\leq t/d} v(x)\, dx \right)^{1/p} \left(\int\limits_{t\leq|x|\leq 1} w^{-p'/p}(x)|x|^{-(n+\alpha)p'} dx \right)^{1/p'} < \infty$$

if v_0 *and* w_0 *are decreasing, for some* $d > 1$, *then*

$$\int\limits_{|x-x_0|\leq 1} |Tf(x)|^p\, v(x-x_0)\, dx \leq C_3 B^{\mathrm{loc}}_{\alpha,d}(v,w)\ \left[\text{or } B'^{\mathrm{loc}}_{\alpha,d}(v,w)\right] \int\limits_{|x-x_0|\leq 10} |f(x)|^p\, w(x-x_0)\, dx,$$

where $C_3 = C_3(C_0, A, p, n, \alpha, d)$ *is independent of* x_0 *and* f.

2.2. **Positive results in the case where $\alpha = 0$.** The restriction to $\alpha = 0$ in (1) enables us to formulate more general statements.

Again our first result establishes a sufficient condition for our operators T to be of *two-weight* strong-type (p, p) when $1 < p < \infty$. We introduce the following notation,

$$\begin{aligned} B_d(v,w) &:= B_{0,d}(v,w) = \sup_{t>0} \left(\int\limits_{t\le|x|} v(x)|x|^{-np} dx \right)^{1/p} \left(\int\limits_{|x|\le t/d} w^{-p'/p}(x)\, dx \right)^{1/p'} \\ B'_d(v,w) &:= B'_{0,d}(v,w) = \sup_{t>0} \left(\int\limits_{|x|\le t/d} v(x)\, dx \right)^{1/p} \left(\int\limits_{t\le|x|} w^{-p'/p}(x)|x|^{-np'} dx \right)^{1/p'} . \end{aligned}$$

Theorem 4. *Let $1 < p < \infty$, $\rho \in A_p$, and T be an operator defined by (2) with kernel satisfying (1) with $\alpha = 0$ that is bounded on $L^p_\rho(\mathbf{R}^n)$. If v_0 and w_0 are positive monotonic functions on $(0, \infty)$ such that the weights $v(x) = v_0(|x|)\rho(x)$ and $w(x) = w_0(|x|)\rho(x)$ satisfy the condition*

$$B_d(v,w) < \infty$$

if v_0 and w_0 are increasing or

$$B'_d(v,w) < \infty$$

if v_0 and w_0 are decreasing, for some $d > 1$, then T is bounded from L^p_w to L^p_v. Moreover

$$\|Tf\|_{L^p_v} \le C_4 B_d(v,w) \ \left[\text{or } B'_d(v,w)\right] \|f\|_{L^p_w},$$

where $C_4 = C_4(\|T\|_{L^p_\rho \to L^p_\rho}, A, p, n, d)$.

Theorem 4 has been already been proven in the case of Calderón-Zygmund singular integrals; see [8]. The following corollary generalizes results presented in [9], see also [10], p517.

Corollary 5. *Let $1 < p < \infty$ and T be an operator defined by (2) with kernel satisfying (1) with $\alpha = 0$ that is bounded on $L^p_\varrho(\mathbf{R}^n)$ for $\rho \in A_p$. Let $\rho_1 \in A_1$, if v_0 and w_0 are positive monotonic functions on $(0, \infty)$ such that the weights $v(x) = v_0(|x|)$ and $w(x) = w_0(|x|)$ satisfy the condition*

$$B_d(v,w) < \infty$$

if v_0 and w_0 are increasing [or $B'_d(v,w) < \infty$ if v_0 and w_0 are decreasing], for some $d > 1$, then it follows that T is bounded from $L^p_{w\rho_1}$ to $L^p_{v\rho_1}$ [or from $L^p_{w\rho_1^{1-p}}$ to $L^p_{v\rho_1^{1-p}}$]. Moreover

$$\|Tf\|_{L^p_{v\rho_1}} \le C_5 B_d(v,w) \|f\|_{L^p_{w\rho_1}}$$

$$\left[\text{or } \ \|Tf\|_{L^p_{v\rho_1^{1-p}}} \le C_5 B'_d(v,w)\right] \|f\|_{L^p_{w\rho_1^{1-p}}} \Bigg],$$

where $C_5 = C_5(\|T\|_{L^p_\rho \to L^p_\rho}, A, p, n, d)$.

Proof. We shall assume that v_0 and w_0 are increasing. Using the fact that $\rho_1 \in A_1 \subset A_p$ it follows that

$$\begin{aligned}
B_d(v\rho_1, w\rho_1) &= \sum_{k=0}^{\infty} \Bigg(\int\limits_{2^k t \le |x| < 2^{k+1} t} v(x)\rho_1(x)|x|^{-np} dx \Bigg)^{1/p} \Bigg(\int\limits_{|x| \le t/d} w^{-p'/p}(x) \rho_1^{-p'/p}(x)\, dx \Bigg)^{1/p'} \\
&\le [A_1(\rho_1)]^{1/p} \sum_{k=0}^{\infty} v^{1/p}(2^{k+1})(2^k t)^{-n+n/p} \Bigg(\int\limits_{|x| \le t/d} w^{-p'/p}(x)\, dx \Bigg)^{1/p'} \\
&\le [A_1(\rho_1)]^{1/p} \sum_{k=1}^{\infty} \Bigg(\int\limits_{2^k t \le |x| < 2^{k+1} t} v(x)|x|^{-np} dx \Bigg)^{1/p} \Bigg(\int\limits_{|x| \le t/d} w^{-p'/p}(x)\, dx \Bigg)^{1/p'} \\
&\le C B_d(v, w).
\end{aligned}$$

The argument for v_0 and w_0 decreasing is similar, in this case one instead uses the fact that $\rho_1^{-p/p'} \in A_{p'}$ if $\rho_1 \in A_1$. □

Our second main result when $\alpha = 0$ establishes a sufficient condition for our operators T to be of *two-weight* weak-type (p,p) when $1 \le p < \infty$. We introduce the following notation,

$$\begin{aligned}
B_d^{(p)}(v,w) &:= B_{0,d}^{(p)}(v,w) = \sup_{0<t<\tau} \frac{1}{\tau^n} \Bigg(\int\limits_{t \le |x| \le \tau} v(x)\, dx \Bigg)^{1/p} \Bigg(\int\limits_{|x| \le t/d} w^{-p'/p}(x)\, dx \Bigg)^{1/p'} \\
B_d^{(1)}(v,w) &:= B_{0,d}^{(1)}(v,w) = \sup_{0<t<\tau} \frac{1}{\tau^n} \Bigg(\int\limits_{t \le |x| \le \tau} v(x)\, dx \Bigg) \left\| w^{-1} \right\|_{L^\infty(\{|\cdot| < t/d\})}.
\end{aligned}$$

Theorem 6. *Let $1 \le p < \infty$, $\rho \in A_p$, and T be an operator defined by (2) with kernel satisfying (1) with $\alpha = 0$ that is bounded from $L^p_\rho(\mathbf{R}^n)$ to $L^{p,\infty}_\rho(\mathbf{R}^n)$. If v_0 and w_0 are positive increasing functions on $(0,\infty)$ such that the weights*

$$v(x) = v_0(|x|)\rho(x) \quad \text{and} \quad w(x) = w_0(|x|)\rho(x)$$

satisfy the condition

$$B_d^{(p)}(v,w) < \infty$$

if $1 < p < \infty$, and

$$B_d^{(1)}(v,w) < \infty$$

if $p = 1$, for some $d > 1$, then T is bounded from L^p_w to $L^{p,\infty}_v$. Moreover

$$\|Tf\|_{L^{p,\infty}_v} \le C_6 B_d^{(p)}(v,w) \|f\|_{L^p_w},$$

where $C_6 = C_6(\|T\|_{L^p_\rho \to L^{p,\infty}_\rho}, A, p, n, d)$.

Remark 2. If $\rho \in A_p$ with $p > 1$, then *one-weight* weak-type (p,p) estimates for the Riesz transforms are equivalent to *one-weight* strong-type (p,p) estimates. It has however been shown that for $p > 1$ the class of weight pairs guaranteeing *two-weight* weak-type (p,p) estimates for the Hilbert transform is larger than the class that ensures *two-weight* strong-type (p,p) estimates; see [9] and [10], Chapter 8.

2.3. Necessary conditions in the case where $\alpha = 0$.

Our first result establishes a necessary condition for our operators T to be of *two-weight* strong-type (p,p) when $1 < p < \infty$.

Theorem 7. *Let $1 < p < \infty$ and T be an operator defined by (2) with kernel $K(x,y) = \widetilde{K}(x-y)$ satisfying the estimates*

$$|\nabla \widetilde{K}(x)| \leq \frac{A_0}{|x|^{n+1}}, \tag{4a}$$

$$|\widetilde{K}(x)| \geq \frac{A_1}{|x|^{n}}, \tag{4b}$$

whenever $x \neq 0$, in addition to (1) with $\alpha = 0$. If T is bounded from L^p_w to L^p_v then this implies the condition

$$B_{\frac{4A_0 n}{A_1}}(v,w) < \infty.$$

Our second result of this section establishes a necessary condition for our operators T to be of *two-weight* weak-type (p,p) when $1 \leq p < \infty$.

Theorem 8. *Let $1 \leq p < \infty$ and T be an operator defined by (2) with kernel $K(x,y) = \widetilde{K}(x-y)$ satisfying estimates (4) and (1) with $\alpha = 0$. If T is bounded from L^p_w to $L^{p,\infty}_v$ then it follows that*

$$B^{(p)}_{\frac{4nA_0}{A_1}}(v,w) < \infty,$$

if $1 < p < \infty$, and

$$B^{(1)}_{\frac{4nA_0}{A_1}}(v,w) < \infty,$$

if $p = 1$.

Corollary 9. *Let $K(x,y) = \widetilde{K}(x-y)$ satisfy conditions (1) and (4) and P be a real polynomial on $\mathbf{R}^n \times \mathbf{R}^n$. If we, in the sense of (2), define*

$$T_P f(x) = \int K(x,y) e^{iP(x,y)} f(y)\, dy$$

then in order for T_P to be bounded from L^p_w to L^p_v with bounds independent of the coefficients of P it is necessary that

$$B_{\frac{4A_0 n}{A_1}}(v,w) < \infty.$$

Proof. For $\varepsilon > 0$ we denote $P_\varepsilon(x,y) := \varepsilon P(x,y)$. Let f be a non-negative belonging to $L^p_w$with support in $\chi_{B(0,t)}$, $t > 0$. It is then easy to see, using the Lebesgue dominated convergence theorem, that if $|x| > \frac{4nA_0}{A_1} t$, then

$$\lim_{\varepsilon \to 0} \int_{|y|<t} e^{iP_\varepsilon(x,y)} K(x,y) f(y) dy = \int_{|y|<t} K(x,y) f(y) dy. \tag{5}$$

□

3. Background

3.1. Model Operators.

We now list some model operators that are of the form that we are considering, namely of the form (2) with kernels satisfying (1). Recall that we are also making the *a priori* assumption that our operators are bounded on $L^2(\mathbf{R}^n)$.

3.1.1. *Calderón-Zygmund Singular Integrals.* These operators, which we shall denote by S, are defined as in (2) with integral kernel that in addition to satisfying (1) for $\alpha = 0$ also satisfy the differential inequality

$$|\partial_x^\nu \partial_y^\mu K(x,y)| \leq A|x-y|^{-n-|\nu|-|\mu|}.$$

Key examples are the following;

(1) The key example when $n = 1$ is the *Hilbert transform* H, this is a convolution operator (defined as a principle value) with distributional kernel

$$K(x) = x^{-1}.$$

(2) The analogues of the Hilbert transform in higher dimensions are the *Reisz transforms* $R_1, \ldots, R_n$, where each operator R_j is given by convolution with

$$K_j(x) = x_j|x|^{-n-1}.$$

(3) Another important example are the convolution kernels of the form

$$K(x) = |x|^{-n-it}, \text{ with } t \neq 0.$$

The following result is of course well known; see for example [34].

Theorem A. *If* $1 < p < \infty$ *then* S *extends to a bounded operator on* $L^p(\mathbf{R}^n)$ *and if* $p = 1$ *then* S *extends to an operator which is of weak-type* $(1,1)$.

3.1.2. *Oscillatory Singular Integrals.* Let K be a Calderón-Zygmund kernel as described above and P be a real polynomial on $\mathbf{R}^n \times \mathbf{R}^n$. If we, in the sense of (2), define

$$T_P f(x) = \int K(x,y) e^{iP(x,y)} f(y)\, dy$$

then the following is known, see [30] and [4].

Theorem B. *If* $1 < p < \infty$ *then* T_P *extends to a bounded operator on* $L^p(\mathbf{R}^n)$ *and if* $p = 1$ *then* T_P *extends to an operator which is of weak-type* $(1,1)$. *In both instances the bounds on* T_P *can be taken independent of the coefficients of* P.

For extensions of Theorem B to more general phase functions see [29], see also [34].

3.1.3. *Strongly Singular Integrals.* These are operators, which we shall denote by T_α, whose integral kernels take the form

$$K_\alpha(x,y) = a(x,y)e^{i\varphi(x,y)},$$

where the amplitude[1] and phase satisfy the differential inequalities

$$|\partial_x^\mu \partial_y^\nu a(x,y)| \leq C_{\mu,\nu}|x-y|^{-d-\alpha-|\mu|-|\nu|}$$
$$|\partial_x^\mu \partial_y^\nu \varphi(x,y)| \leq C_{\mu,\nu}|x-y|^{-\beta-|\mu|-|\nu|},$$

that φ is real-valued and furthermore that

$$|\nabla_x \varphi(x,y)|,\ |\nabla_y \varphi(x,y)| \geq C|x-y|^{-\beta-1} \tag{6}$$

with $\beta > 0$ and $0 \leq \alpha \leq n\beta/2$. In additions to these two assumptions one also makes the non-degeneracy assumption that

$$\left|\det\left(\frac{\partial^2 \varphi_\lambda(x,y)}{\partial x_i \partial y_j}\right)\right| \geq C > 0$$

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[1] In the case where $\alpha = 0$ we must make the further assumption that our amplitude a is compactly supported in a neighborhood of the diagonal $x = y$, this is of course also the only region of any interest when $\alpha > 0$.

uniformly in λ. Note that this non-degeneracy assumption ensures that T_α extends to a bounded operator on $L^2(\mathbf{R}^n)$.

The key example of such a kernels are $K_\alpha(x,y) = \widetilde{K}_\alpha(x-y)$ where $\widetilde{K}_\alpha$ is a distribution on $\mathbf{R}^d$ that away from the origin agrees with the function

$$\widetilde{K}_\alpha(x) = |x|^{-d-\alpha} e^{i|x|^{-\beta}} \chi(|x|),$$

with χ smooth and compactly supported in a small neighborhood of the origin. Operators of this type where first studied in one dimension by Hirschman [17] and then in higher dimensions by Wainger [36]. For the following result see [25].

Theorem C. *If* $1 < p < \infty$*, then* T_α *extends to a bounded operator on* $L^p(\mathbf{R}^n)$ *if and only if*

$$\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{1}{2} - \frac{\alpha}{n\beta}.$$

The prototype non-convoltion operator of this strongly singular type are the pseudo-differential operators with symbols in the class $S^m_{\rho,\delta}$ that where introduced by Hörmander. In the special case where $K_\alpha(x,y) = \widetilde{K}_\alpha(x-y)$ and $\alpha = 0$ it was shown by C. Fefferman [11] that T_α extends to an operator which is of weak-type $(1,1)$. In fact this result, and the one above, can be extended to the general class introduced by Hörmander, see [12].

We now recall some well known *one-weight* and *two-weight* estimates for these singular integrals.

3.2. **One-weight estimates for singular integrals.** For completeness we choose to state here some results which pre-date those in [32].

3.2.1. *Calderón-Zygmund singular integrals.*

Theorem D. *If* $1 < p < \infty$ *and* $\rho \in A_p$ *then* S *is bounded on* L^p_ρ*, while if* $\rho \in A_1$ *then* S *is bounded from* L^1_ρ *to* $L^{1,\infty}_\rho$ *(*S *is of* one-weight *weak-type* $(1,1)$*).*

In the case of the Hilbert transform H *having* $\rho \in A_p$ *for* $1 < p < \infty$ *and* $\rho \in A_1$ *is also necessary for* H *to be of* one-weight *strong-type* (p,p) *and* one-weight *weak-type* $(1,1)$ *respectively.*

Theorem D was proved for the Hilbert transform in [21] and for general Calderón-Zygmund intgrals in [7], see also [13].

Moreover, in [13] (page 417) it is shown that if the Reisz transforms R_j are of *one-weight* weak-type (p,p) for $1 \leq p < \infty$ then one must necessarily have $\rho \in A_p$.

3.2.2. *Oscillatory singular integrals.* In [31] it was shown that if $K(x,y) = \widetilde{K}(x-y)$ satisfies the conditions[2]

$$|\widetilde{K}(x)| \leq A|x|^{-n} \quad \text{and} \quad |\nabla \widetilde{K}(x)| \leq A_0 |x|^{-n-1}$$

and

$$\int_{\epsilon<|x|<N} \widetilde{K}(x)\, dx = 0,$$

for all $0 < \epsilon < N < \infty$, and $\rho \in A_1$, then T_P is *one-weight* weak-type $(1,1)$. See also [5].

Let $1 < p < \infty$, $P(x,y) = Q(x-y)$ and $\widetilde{K}(x,y) = (x-y)^{-1}$, then T_P is bounded on $L^p_\rho(\mathbf{R})$ if and only if $\rho \in A_p$, see [19]. See also [20].

---

[2] These are precisely the necessary and sufficient conditions in order for the Calderón-Zygmund singular integrals with this convolution kernel $\widetilde{K}$ to extend top a bounded operator on $L^2(\mathbf{R}^n)$.

3.2.3. *Strongly singular integrals.* The following results have been established in the in the 'model' convolution case.

Theorem E. *Let* $K_\alpha(x,y) = \widetilde{K}_\alpha(x-y)$.

(i) *If* $\alpha = 0$, $1 < p < \infty$ *and* $\rho \in A_p$ *then* T_α *is bounded on* L^p_ρ, *while if* $\rho \in A_1$ *then* T_α *is bounded from* L^1_ρ *to* $L^{1,\infty}_\rho$.

(ii) *If* $0 < \alpha \le \alpha_p := n\beta\left(\frac{1}{2} - \left|\frac{1}{p} - \frac{1}{2}\right|\right)$, $\gamma = (\alpha - \alpha_p)/\alpha_p$ *and* $\rho \in A_p$ *then* T_α *is bounded on* $L^p_{\rho^\gamma}$.

These two results were established in [3], in the same paper it was also shown that if $1 < p < \infty$ and $\rho(x) = |x|^\lambda$, where $\lambda \le -n$ or $\lambda \ge n(p-1)$, then T_α is not bounded on L^p_ρ. For extensions to the prototype non-translation invariant setting discussed in §3.1.3 see [6].

3.3. **Two-weight estimates for Calderón-Zygmund singular integrals.** Two-weight inequalities for Calderón-Zygmund singular integrals have been studied in [28] and [8] (see also [16], [15], [23], [24], [10] Chapter 8, and [14]).

Theorem F. *Let* $1 < p < \infty$ *and* K *be a Calderón-Zygmund kernel. We put* $v(x) = v_0(|x|)\rho(x)$ *and* $w(x) = w_0(|x|)\rho(x)$, *where* v_0 *and* w_0 *are positive monotonic functions on* $(0,\infty)$ *and* $\rho \in A_p$. *If* v_0 *and* w_0 *are increasing and*

$$B_2(v,w) < \infty$$

or if v_0 *and* w_0 *are decreasing and*

$$B_2'(v,w) < \infty$$

then S *is bounded from* L^p_w *to* L^p_v.

Conversely, if the Hilbert transform H *is to be bounded from* L^p_w *to* L^p_v *then the weights* v *and* w *must satisfy conditions* $B_2(v,w) < \infty$ *and* $B_2'(v,w) < \infty$.

For the *two-weight* weak-type inequality we have the following, see [9] and [10].

Theorem G. *Let* $1 \le p < \infty$ *and* K *be a Calderón-Zygmund kernel. We put* $v(x) = v_0(x)\rho(x)$ *and* $w(x) = w_0(x)\rho(x)$, *where* v_0 *and* w_0 *are positive increasing functions on* $(0,\infty)$ *and* $\rho \in A_1$. *Now if the weights* v *and* w *satisfy*

$$B_2^{(p)}(v,w) < \infty$$

if $1 < p < \infty$, *and*

$$B_2^{(1)}(v,w) < \infty$$

if $p = 1$, *then* T *is bounded from* L^p_w *to* $L^{p,\infty}_v$.

Conversely, if the Hilbert transform H *is to be bounded from* L^p_w *to* $L^{p,\infty}_v$ *then the weights* v *and* w *must satisfy conditions* $B_2^{(p)}(v,w) < \infty$ *and* $B_2^{(1)}(v,w) < \infty$.

3.4. **Hardy operators.** Before presenting the proofs of the main results, we formulate some well known statements concerning two-weight norm estimates for Hardy-type transforms defined on $\mathbf{R}^n$. The two-weight problem for the classical Hardy operator

$$\mathcal{H}f(x) = \int_0^x f(y)\,dy$$

has been solve in [27], [2], [22], and [26].

Let

$$\mathcal{H}_{\alpha,d}f(x) := \frac{1}{|x|^{n+\alpha}} \int_{|y| \le |x|/d} f(y)\,dy$$

and

$$\mathcal{H}'_{\alpha,d}f(x) := \int_{|y|\geq d|x|} \frac{f(y)}{|y|^{n+\alpha}}\, dy$$

for measurable $f : \mathbf{R}^n \to \mathbf{R}$, where $d > 1$.

For the following *two-weight* strong-type and weak-type (p,p) estimates see [10], Chapter 1.

Theorem H. *Let $1 < p < \infty$ and $\alpha \geq 0$.*

(i) *$\mathcal{H}_{\alpha,d}$ is bounded from L^p_w to L^p_v if and only if*

$$D_{\alpha,d}(v,w) := \sup_{t>0} \left(\int\limits_{t\leq|x|} v(x)|x|^{-(n+\alpha)p} dx \right)^{1/p} \left(\int\limits_{|x|\leq t/d} w^{-p'/p}(x)\, dx \right)^{1/p'} < \infty.$$

Moreover, there exists constants c_1 and c_2 such that

$$c_1 D_{\alpha,d}(v,w) \leq \|\mathcal{H}_{\alpha,d}\|_{L^p_w \to L^p_v} \leq c_2 D_{\alpha,d}(v,w).$$

(ii) *$\mathcal{H}'_{\alpha,d}$ is bounded from L^p_w to L^p_v if and only if*

$$D'_{\alpha,d}(v,w) := \sup_{t>0} \left(\int\limits_{|x|\leq t/d} v(x)\, dx \right)^{1/p} \left(\int\limits_{t\leq|x|} w^{-p'/p}(x)|x|^{-(n+\alpha)p'} dx \right)^{1/p'} < \infty.$$

Moreover, there exists constants c'_1 and c'_2 such that

$$c'_1 D'_{\alpha,d}(v,w) \leq \|\mathcal{H}'_d\|_{L^p_w \to L^p_v} \leq c'_2 D'_{\alpha,d}(v,w).$$

Remark 3. It is easy to see that for all $\alpha \geq 0$ one has

$$D_{\alpha,d}(v,w) \leq B_{\alpha,d}(v,w) \ \text{ and } \ D'_{\alpha,d}(v,w) \leq B'_{\alpha,d}(v,w).$$

From Theorem H is easy to establish the following local result.

Corollary I. *Let $1 < p < \infty$, then the* two-weight *inequality*

$$\int_{|x-x_0|\leq 1} \left|\mathcal{H}_{\alpha,d}|f_{x_0}|(x)\right|^p v(x-x_0)\, dx \leq C_0 \int_{|x-x_0|\leq 10} |f(x)|^p\, w(x-x_0)\, dx, \tag{7}$$

where $f_{x_0}(y) = f(y+x_0)$ holds if and only if $B^{\text{loc}}_{\alpha,d}(v,w) < \infty$. Moreover, there exists constants c_1 and c_2 such that if C_0 is the best possible constant in (7) then

$$c_1[B^{\text{loc}}_{\alpha,d}(v,w)]^p \leq C_0 \leq c_2[B^{\text{loc}}_{\alpha,d}(v,w)]^p.$$

Theorem J. *Let $1 \leq p < \infty$ and $\alpha > -n$. $\mathcal{H}_{\alpha,d}$ is bounded from L^p_w to $L^{p,\infty}_v$ if and only if*

$$D^{(p)}_{\alpha,d}(v,w) := \sup_{0<t<\tau} \tau^{-n-\alpha} \left(\int\limits_{t\leq|x|\leq\tau} v(x)\, dx \right)^{1/p} \left(\int\limits_{|x|\leq t/d} w^{-p'/p}(x)\, dx \right)^{1/p'} < \infty$$

when $1 < p < \infty$, and

$$D^{(1)}_{\alpha,d}(v,w) := \sup_{0<t<\tau} \tau^{-n-\alpha} \left(\int\limits_{t\leq|x|\leq\tau} v(x)\, dx \right) \left\| w^{-1} \right\|_{L^\infty(\{|\cdot|<t/d\})} < \infty$$

when $p = 1$ for some $d > 1$. Moreover, there exists constants c_1 and c_2 depending only on α and p such that

$$c_1 D_{\alpha,d}(v,w) \leq \|\mathcal{H}_{\alpha,d}\|_{L^p_w \to L^{p,\infty}_v} \leq c_2 D_{\alpha,d}(v,w).$$

This statement was proved in more generality in [10]. For *two-weight* weak-type estimates for the one-dimensional Hardy operator see [1].

Remark 4. For all $1 \le p < \infty$ and $\alpha \ge 0$ one has that

$$D^{(p)}_{\alpha,d}(v,w) \le B^{(p)}_{\alpha,d}(v,w).$$

4. PROOF OF MAIN RESULTS

We shall need the following lemmata, they are easily established so we omit their proofs.

Lemma 1. *Let $1 < p < \infty$ and $\alpha \ge 0$. Suppose that $v(x) = v_0(|x|)$ and $w(x) = w_0(|x|)$, where v_0 and w_0 are positive monotonic functions on $(0,\infty)$.*

(i) *If v_0 and w_0 are increasing and*

$$B_{\alpha,d}(v,w) < \infty$$

for some $d > 1$, then there exists a positive constant C depending only on p, n, α, and d such that

$$v_0(dt) \le C[B_{\alpha,d}(v,w)]^p \; w_0(t)$$

for all $t > 0$.

(ii) *If v_0 and w_0 are decreasing and*

$$B'_{\alpha,d}(v,w) < \infty$$

for some $d > 1$, then there exists a positive constant C depending only on p, n, α, and d such that

$$v_0(t/d) \le C[B'_{\alpha,d}(v,w)]^p w_0(t)$$

for all $t > 0$.

Lemma 2. *Let $1 \le p < \infty$ and $\alpha \ge 0$. If $v(x) = v_0(|x|)$ and $w(x) = w_0(|x|)$, where v_0 and w_0 are positive increasing functions on $(0,\infty)$ satisfy, for some constant $d > 1$, the condition*

$$B^{(p)}_{\alpha,d}(v,w) < \infty$$

if $1 < p < \infty$, and

$$B^{(1)}_{\alpha,d}(v,w) < \infty$$

if $p = 1$, then there exists a positive constant C depending only on p, n, α, and d such that

$$v_0(dt) \le C[B^{(p)}_{\alpha,d}(v,w)]^p w_0(t)$$

for all $t > 0$.

When $\alpha = 0$ we have the following two lemmata, see [8], [9], and [10].

Lemma 3. *Let $1 < p < \infty$ and $\alpha = 0$. Suppose that $v(x) = v_0(|x|)\rho(x)$ and $w(x) = w_0(|x|)\rho(x)$, where v_0 and w_0 are positive monotonic functions on $(0,\infty)$ and $\rho \in A_p$.*

(i) *If v_0 and w_0 are increasing and*

$$B_d(v,w) < \infty$$

for some $d > 1$, then there exists a positive constant C depending only on p, n, and d such that

$$v_0(dt) \le C[B_d(v,w)]^p \; w_0(t)$$

for all $t > 0$.

(ii) *If v_0 and w_0 are decreasing and*

$$B'_d(v,w) < \infty$$

for some $d > 1$, then there exists a positive constant C depending only on p, n, and d such that

$$v_0(t/d) \le C[B'_d(v,w)]^p w_0(t)$$

for all $t > 0$.

Lemma 4. *Let $1 \le p < \infty$ and $\alpha = 0$. If $v(x) = v_0(|x|)\rho(x)$ and $w(x) = w_0(|x|)\rho(x)$, where v_0 and w_0 are positive increasing functions on $(0,\infty)$ and $\rho \in A_p$ satisfy, for some constant $d > 1$, the condition*

$$B_d^{(p)}(v,w) < \infty$$

if $1 < p < \infty$, and

$$B_d^{(1)}(v,w) < \infty$$

if $p = 1$, then there exists a positive constant C depending only on p, n, and d such that

$$v_0(dt) \le C[B_d^{(p)}(v,w)]^p w_0(t)$$

for all $t > 0$.

Proof of Theorem 1. We shall assume that v_0 and w_0 are increasing. Without loss of generality we can assume that the weight $v(x) = v_0(|x|)$ has the form

$$v(x) = v(0) + \int_0^{|x|} \varphi(t)\,dt, \tag{8}$$

where $\varphi \ge 0$ and $v(0) := \lim\limits_{|x|\to 0} v(x)$. In fact there exists a sequence of absolutely continuous functions v_k such that

$$v_k(x) \le v(x) \quad \text{and} \quad \lim_{k\to\infty} v_k(x) = v(x),$$

that are given by

$$v_k(x) = v(0) + k\int_0^{|x|} [v_0(t) - v_0(t-\tfrac{1}{k})]\,dt.$$

Now using representation (8) we have

$$\int |Tf(x)|^p v(x)\,dx = \int |Tf(x)|^p v(0)\,dx + \int |Tf(x)|^p \Big(\int_0^{|x|} \varphi(t)\,dt\Big)\,dx =: I_1 + I_2.$$

Now if $v(0) = 0$ then $I_1 = 0$, while if $v(0) \ne 0$ it follows from the L^p boundedness of T and Lemma 1 (part (i)) that

$$I_1 \le v(0)\|T\|^p_{L^p\to L^p} \int |f(x)|^p dx \le C[B_{\alpha,d}(v,w)]^p \|T\|^p_{L^p\to L^p} \int |f(x)|^p w(x)\,dx.$$

For I_2 we have that

$$\begin{aligned} I_2 &= \int_0^\infty \varphi(t)\Big(\int_{|x|\ge t} |Tf(x)|^p dx\Big)\,dt \\ &\le 2^{p-1}\left[\int_0^\infty \varphi(t)\Big(\int_{|x|\ge t} |Tf_{1,t}(x)|^p dx\Big)\,dt + \int_0^\infty \varphi(t)\Big(\int_{|x|\ge t} |Tf_{2,t}(x)|^p dx\Big)\,dt\right] \\ &= I_{2,1} + I_{2,2}, \end{aligned}$$

where

$$f_{1,t}(x) = f(x)\chi_{\{|x|\geq t/d\}}(x) \ \text{ and } \ f_{2,t}(x) = f(x) - f_{1,t}(x).$$

Using again the L^p boundedness of T and Lemma 1 (part (i)) it follows that

$$\begin{aligned} I_{2,1} &\leq \|T\|^p_{L^p\to L^p}\int_0^\infty \varphi(t)\Big(\int_{|x|\geq t/d}|f(x)|^p dx\Big)\,dt \\ &= \|T\|^p_{L^p\to L^p}\int |f(x)|^p\Big(\int_0^{d|x|}\varphi(t)\,dt\Big)\,dx \\ &\leq C[B_{\alpha,d}(v,w)]^p\|T\|^p_{L^p\to L^p}\int |f(x)|^p w(x)\,dx. \end{aligned}$$

Using the fact that if $|x|\geq t$ and $|y|\leq t/d$ then $(d-1)|x|/d\leq |x-y|$ and Theorem H (part (i)) we see that

$$\begin{aligned} I_{2,2} &\leq CA^p\int_0^\infty \varphi(t)\left(\int_{|x|\geq t}|x|^{-(n+\alpha)p}\Big(\int_{|y|\leq t/d}|f(y)|\,dy\Big)^p dx\right)\,dt \\ &\leq CA^p\int_0^\infty \varphi(t)\left(\int_{|x|\geq t}|x|^{-(n+\alpha)p}\Big(\int_{|y|\leq |x|/d}|f(y)|\,dy\Big)^p dx\right)\,dt \\ &\leq CA^p\int |x|^{-(n+\alpha)p}\Big(\int_{|y|\leq|x|/d}|f(y)|\,dy\Big)^p\Big(\int_0^{|x|}\varphi(t)\,dt\Big)\,dx \\ &\leq CA^p\int v(x)|x|^{-(n+\alpha)p}\big|\mathcal{H}_{\alpha,d}|f|(x)\big|^p dx \\ &\leq C[B_{\alpha,d}(v,w)]^pA^p\int |f(x)|^p w(x)\,dx. \end{aligned}$$

This completes the proof in the case when v_0 and w_0 are increasing. The proof in the decreasing case follows in exactly then same manner using the representation

$$v(x) = v(\infty) + \int_{|x|}^\infty \varphi(t)\,dt,\ \ \varphi\geq 0,\ \ v(\infty) := \lim_{|x|\to\infty} v(x), \tag{9}$$

and part (ii) of both Theorem H and Lemma 1. □

Proof of Theorem 2. Using representation (8) we have

$$\int_{\{|Tf(x)|>\lambda\}} v(x)\,dx = v(0)|\{x : |Tf(x)|>\lambda\}| + \int_{\{|Tf(x)|>\lambda\}}\Big(\int_0^{|x|}\varphi(t)\,dt\Big)\,dx =: I_1 + I_2.$$

Now if $v(0)=0$ then $I_1=0$, while if $v(0)\neq 0$ it follows from the assumption that T is of weak-type (p,p) and Lemma 2 that

$$I_1 \leq v(0)\|T\|^p_{L^p\to L^{p,\infty}}\frac{1}{\lambda^p}\int |f(x)|^p dx \leq C[B^{(p)}_{\alpha,d}(v,w)]^p\|T\|^p_{L^p\to L^{p,\infty}}\frac{1}{\lambda^p}\int |f(x)|^p w(x)\,dx.$$

To estimate I_2 we introduce the following notation:

$$\begin{aligned} J_t(\lambda) &= \{x : |Tf(x)|>\lambda\}\cap\{x : |x|\geq t\} \\ J_{1,t}(\lambda) &= \{x : |Tf_{1,t}(x)|>\lambda/d\}\cap\{x : |x|\geq t\} \\ J_{2,t}(\lambda) &= \{x : |Tf_{2,t}(x)|>\lambda/d\}\cap\{x : |x|\geq t\}, \end{aligned}$$

where again

$$f_{1,t}(x) = f(x)\chi_{\{|x|\geq t/d\}}(x) \ \text{ and } \ f_{2,t}(x) = f(x) - f_{1,t}(x).$$

Now it is easy to see that

$$I_2 = \int_0^\infty \varphi(t)\,|J_t(\lambda)|\,dt \le \int_0^\infty \varphi(t)\,|J_{1,t}(\lambda)|\,dt + \int_0^\infty \varphi(t)\,|J_{2,t}(\lambda)|\,dt = I_{2,1} + I_{2,2}.$$

Using again that T is of weak-type (p,p) and Lemma 2 it follows that

$$\begin{aligned} I_{2,1} &\le \|T\|^p_{L^p\to L^{p,\infty}} \frac{1}{\lambda^p} \int_0^\infty \varphi(t) \Big(\int_{|x|\ge t/d} |f(x)|^p dx \Big)\, dt \\ &= \|T\|^p_{L^p\to L^{p,\infty}} \int |f(x)|^p \Big(\int_0^{d|x|} \varphi(t)\,dt \Big)\, dx \\ &\le C[B^{(p)}_{\alpha,d}(v,w)]^p \|T\|^p_{L^p\to L^{p,\infty}} \int |f(x)|^p w(x)\,dx. \end{aligned}$$

Using, as in the proof of Theorem 1, the fact that $|x|\ge t$ and $|y|\le t/d$ ensures $(d-1)|x|/d \le |x-y|$ and Theorem J we see that

$$\begin{aligned} I_{2,2} &\le \int_0^\infty \varphi(t)\,\big|\{x : |x|\ge t\}\cap\{x : \mathcal{H}_{\alpha,d}|f|(x) > \lambda/d'\}\big|\,dt \\ &= \int_{\{\mathcal{H}_{\alpha,d}|f|(x)>\lambda/d'\}} \Big(\int_0^{|x|} \varphi(t)\,dt \Big)\, dx \\ &\le \int_{\{\mathcal{H}_{\alpha,d}|f|(x)>\lambda/d'\}} v(x)\,dx \\ &\le C[B^{(p)}_{\alpha,d}(v,w)]^p \frac{1}{\lambda^p} \int |f(x)|^p w(x)\,dx, \end{aligned}$$

where $d' = d\left(\frac{d}{d-1}\right)^{n+\alpha}$. ☐

The proofs of Theorems 4 and 6 are similar to those for Theorems 1 and 2 above, one simply instead uses the *one-weight* strong-type and weak-type (p,p) assumptions respectively together with Lemmata 3 and 4.

Arguing as in the proof of Theorem 1 and using Corollary I one can easily obtain Theorem 3.

Before proving Theorems 7 and 8, we present the following Lemma.

Lemma 5. *If* $|x| \ge \frac{4nA_0}{A_1}t$, *then*

$$|Tf(x)| \ge \frac{A_1}{4}|x|^{-n} \int_{|y|\le t} f(y)\,dy \tag{10}$$

for all non-negative f *supported in* $B(0,t)$.

Proof. It follows from (4a) and (4b) that

$$\Big|\widetilde{K}(x-y) - \widetilde{K}(x)\Big| \le \frac{A_1}{4}|x|^{-n} \tag{11}$$

whenever $|x| \ge \frac{4nA_0}{A_1}|y|$ and that either

$$\Big|\operatorname{Re}\widetilde{K}(x)\Big| \ge \frac{A_1}{2}|x|^{-n} \quad\text{or}\quad \Big|\operatorname{Im}\widetilde{K}(x)\Big| \ge \frac{A_1}{2}|x|^{-n}.$$

Lets assume that $\big|\operatorname{Re}\widetilde{K}(x)\big| \geq \dfrac{A_1}{2}|x|^{-n}$, then it follows from (11) that

$$\Big|\big|\operatorname{Re}\widetilde{K}(x-y)\big| - \big|\operatorname{Re}\widetilde{K}(x)\big|\Big| \leq \Big|\widetilde{K}(x-y)-\widetilde{K}(x)\Big| \leq \frac{1}{2}\big|\widetilde{K}(x)\big|,$$

whenever $|x| \geq \frac{4nA_0}{A_1}|y|$ and thus that

$$\frac{1}{2}\big|\operatorname{Re}\widetilde{K}(x)\big| \leq \big|\operatorname{Re}\widetilde{K}(x-y)\big| \leq \frac{3}{2}\big|\operatorname{Re}\widetilde{K}(x)\big| \tag{12}$$

It is then immediate from the continuity of $\widetilde{K}$ on $\mathbf{R}^n \setminus \{0\}$ that $\operatorname{Re}\widetilde{K}(x-y)$ does not change sign for $|y| \leq \dfrac{A_1}{4nA_0}|x|$.

If we now let $0 < t \leq \dfrac{A_1}{4nA_0}|x|$ and

$$f_t(y) = f(y)\chi_{\{|y|\leq t\}},$$

from (12) it then follows that

$$|Tf_t(x)| \geq \int_{|y|\leq t} f(y)\big|\operatorname{Re}\widetilde{K}(x-y)\big|\,dy \geq \frac{A_1}{4}|x|^{-n}\int_{|y|\leq t} f(y)\,dy.$$

Arguing in a similar manner for the case where $|\operatorname{Im}\widetilde{K}(x)| \geq \dfrac{A_1}{2}|x|^{-n}$ we obtain the same conclusion.

□

Proof of Theorems 7 and 8. Let us first prove Theorem 8. We consider the case $p > 1$, the case $p = 1$ is similar. We claim that if the operator T is bounded from L^p_w to $L^{p,\infty}_v$, then

$$I(r) := \int_{|x|<r} w^{-p'/p}(x)dx < \infty \tag{13}$$

for all $r > 0$.

Indeed, first observe that $I(r) = \|w^{-1/p}\chi_{|\cdot|<r}\|^{p'}_{L^p}$. If $I(r) = \infty$ for some $r > 0$, then by the duality properties there exists non-negative $g \in L^p$ supported in $B(0,r)$ such that $\int_{|\cdot|<r} gw^{-1/p} = \infty$.

Let us take the function $f_r(y) = g(y)w^{-1/p}(y)\chi_{\{|y|<r\}}$. Then by Lemma 5 we have

$$|T_r f(x)| \geq \frac{A_1}{4}|x|^{-n}\int_{|y|\leq r} g(y)w^{-1/p}(y)dy = \infty,$$

whenever $|x| > \frac{4nA_0}{A_1}r$.

Due to *two-weight* weak-type inequality and the latter estimate we have

$$\int_{|x|>\frac{4nA_0}{A_1}r} v(x)dx \leq \int_{\{x:|Tf_r(x)|>\lambda\}} v(x)dx \leq \frac{c}{\lambda^p}\int_{|y|<r} g(y)dy < \infty$$

for all positive λ. Consequently, passing λ to ∞ we find that the left-hand side of the latter inequality is equal to 0 which contradicts the assumption that the weight v is positive almost everywhere.

Now let us derive the condition $B^{(p)}_{\frac{4nA_0}{A_1}}(v,w) < \infty$.

Applying Lemma 5 we conclude that

$$|Tf(x)| \geq \frac{A_1}{4}|x|^{-n}\int_{|y|<\frac{A_1}{4nA_0}t} w^{-p'/p}(y)dy \geq \frac{A_1}{4}\tau^{-n} I\big(\frac{A_1}{4A_0 n}t\big) \tag{14}$$

whenever $0 < t \le |x| < \tau$ and $f(y) = w^{-p'/p}(y)\chi_{\{|y|<\frac{A_1}{4nA_0}t\}}(y)$.

The *two-weight* weak-type inequality for T leads to the estimates

$$\int_{t<|x|<\tau} v(x)dx \le \int_{\{x:|Tf(x)|\ge (A_1\tau^{-n}/4)I\left(\frac{A_1}{4nA_0}\right)\}}$$

$$\le \left(\frac{4\tau^n\|T\|_{L^p_w\to L^{p,\infty}}}{A_1}\right)^p \frac{1}{I^p\left(\frac{A_1}{4nA_0}t\right)} I\left(\frac{A_1}{4nA_0}t\right) < \infty$$

for all t, τ, $0 < t < \tau < \infty$. This completes the proof of Theorem 8.

To prove Theorem 7 we observe that due to (12) which is true also for all $|x| \ge t$ because of Lemma 5, we have

$$\|Tf\|^p_{L^p_v} \ge \int_{|x|>t} |Tf(x)|^p v(x)dx \ge \frac{A_1}{4}\left(\int_{|x|>t} |x|^{-np}v(x)dx\right)\left(\int_{|y|<\frac{A_1}{4A_0n}t} w^{-p'/p}(y)dy\right)^p. \tag{15}$$

On the other hand, by (11) we have

$$\|f\|^p_{L^p_w} = \int_{|x|<\frac{A_1}{4nA_0}} w^{-p'/p}(x)dx < \infty.$$

Finally, from the boundedness of T from L^p_w to L^p_v we conclude that $B_{\frac{4A_0n}{A_1}}(v, w) < \infty$. ☐

APPENDIX

Here we shall verify the statement made in Remark 1. We first note that if the measure

$$w^{-p'/p}(E) = \int_E w^{-p'/p}(x)\,dx$$

is doubling then it also satisfies the reverse doubling condition: that there exists constants η_1, $\eta_2 > 1$ such that for all $t > 0$ the inequality

$$\int_{|x|\le\eta_1 t} w^{-p'/p}(x)\,dx \ge \eta_2 \int_{|x|\le t} w^{-p'/p}(x)\,dx$$

holds, see [35] page 21.

Using this fact we find that

$$\int\limits_{\eta_1^k t\le|x|\le\eta_1^{k+1}t} w^{-p'/p}(x)\,dx = \int\limits_{|x|\le\eta_1^{k+1}t} w^{-p'/p}(x)\,dx \;-\int\limits_{|x|\le\eta_1^k t} w^{-p'/p}(x)\,dx \ge (\eta_2-1)\eta_2^k \int\limits_{|x|\le t} w^{-p'/p}(x)\,dx,$$

and hence

$$\int\limits_{|x|\le t} w^{-p'/p}(x)\,dx \le \frac{1}{(\eta_2-1)\eta_2^k} \int\limits_{\eta_1^k t\le|x|\le\eta_1^{k+1}t} w^{-p'/p}(x)\,dx. \tag{16}$$

Arguing as in the proof of Corollary 5 leads to the following string of inequalities

$$\begin{aligned}
B_{\alpha,d}(v,w) &= \sup_{t>0}\Bigg(\int\limits_{t\le|x|} v(x)(|x|^{-\alpha}+1)^p|x|^{-np}dx\Bigg)^{1/p}\Bigg(\int\limits_{|x|\le t/d} w^{-p'/p}(x)\,dx\Bigg)^{1/p'}\\
&= \sup_{t>0}\sum_{k=0}^{\infty}\Bigg(\int\limits_{\eta_1^k t\le|x|<\eta_1^{k+1}t} v(x)|x|^{-np}dx\Bigg)^{1/p}\Bigg(\int\limits_{|x|\le t/d} w^{-p'/p}(x)\,dx\Bigg)^{1/p'}\\
&\le \sup_{t>0}\sum_{k=0}^{\infty}\frac{(\eta_1^k t)^{-n}[(\eta_1^k t)^{-\alpha}+1]}{[(\eta_2-1)\eta_2^k]^{1/p'}}\Bigg(\int\limits_{\eta_1^k t\le|x|<\eta_1^{k+1}t} v(x)\,dx\Bigg)^{1/p}\Bigg(\int\limits_{\eta_1^k t\le|x|<\eta_1^{k+1}t} w^{-p'/p}(x)\,dx\Bigg)^{1/p'}\\
&\le A_\alpha(v,w)\sum_{k=0}^{\infty}\frac{1}{[(\eta_2-1)\eta_2^k]^{1/p'}}\\
&\le CA_\alpha(v,w).
\end{aligned}$$

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