

## A SHORT PROOF OF ROTH'S THEOREM

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**Roth's Theorem.** *Let  $\varepsilon > 0$  and  $N \geq \exp \exp(C\varepsilon^{-1})$  for some absolute constant  $C$ . Then any  $A \subset [1, N]$  of size  $|A| = \varepsilon N$  necessarily contains a (non-trivial) arithmetic progression of length three.*

With the aid of the Fourier transform (on  $\mathbf{Z}$ ) we can count the number of 3AP in  $A$ , that is the number of solutions to the equation  $m_1 + m_2 = 2m_3$  with  $m_1, m_2, m_3 \in A$ , namely

$$\mathcal{N} = \sum_{m_1 \in A} \sum_{m_2 \in A} \sum_{m_3 \in A} \int_0^1 e^{2\pi i(m_1+m_2-2m_3)\alpha} d\alpha = \int_0^1 \widehat{1}_A(\alpha)^2 \widehat{1}_A(-2\alpha) d\alpha.$$

**Definition 1.** We say that  $A$  is  $(\delta, L)$  regular if every progression  $P$ , with  $|P| \geq L$ , satisfies

$$|A \cap P| \leq (\varepsilon + \delta)|P|.$$

**Definition 2.** We define the *balanced* function of  $A$  to be

$$f_A(m) = 1_A(m) - \varepsilon 1_{[1, N]}(m).$$

We note that  $\sum f_A(m) = 0$  and now fix  $\delta = \varepsilon^2/10$  and  $L = \delta N^{1/2}$ .

**Proposition.** *If  $A$  is  $(\delta, L)$  regular, then  $|\widehat{f}_A(\alpha)| \leq 8\delta N$  uniformly in  $\alpha$ .*

**Corollary.** *If  $A$  is  $(\delta, L)$  regular, then  $A$  contains at least  $\varepsilon^3 N^2/10$  non-trivial 3AP.*

*Proof of Corollary.* Writing  $1_A = \varepsilon 1_{[1, N]} + f_A$  we obtain

$$\mathcal{N} = \varepsilon \int_0^1 \widehat{1}_A(\alpha)^2 \widehat{1}_{[1, N]}(-2\alpha) d\alpha + \int_0^1 \widehat{1}_A(\alpha)^2 \widehat{f}_A(-2\alpha) d\alpha = \text{Main} + \text{Error}.$$

Now it is easy to see that

$$\text{Main} = \varepsilon \sum_{m_1 \in A} \sum_{m_2 \in A} \sum_{m_3 \in [1, N]} \int_0^1 e^{2\pi i(m_1+m_2-2m_3)\alpha} d\alpha = \varepsilon |A|^2 = \varepsilon^3 N^2,$$

while

$$|\text{Error}| \leq \sup_{\alpha} |\widehat{f}_A(\alpha)| \int_0^1 |\widehat{1}_A(\alpha)|^2 d\alpha \leq 8\delta \varepsilon N^2.$$

This completes the proof of the corollary as there are only  $\varepsilon N$  trivial 3AP in  $A$ . □

*Proof of Proposition.* Let  $\alpha \in [0, 1]$ . By Dirichlet, there exists  $q \leq 2L$  such that  $\langle q\alpha \rangle \leq 1/2L$ . Defining

$$P_0 = \{\ell q : |\ell| \leq L/2\}$$

it then follows that

$$\widehat{1}_{P_0}(\alpha) = \sum_{|\ell| \leq L/2} e^{2\pi i \ell q \alpha} \geq L/2$$

and hence that

$$\sum_m |f_A * 1_{P_0}(m)| \geq |\widehat{f}_A(\alpha)| |\widehat{1}_{P_0}(\alpha)| \geq \frac{L}{2} |\widehat{f}_A(\alpha)|.$$

But,

$$f_A * 1_{P_0}(m) = \sum_{|\ell| \leq L/2} f_A(m - \ell q) = |A \cap P_m| - \varepsilon |P_m \cap [1, N]| \leq \delta L$$

where  $P_m = m + P_0 \subseteq [1, N]$ . Thus, after accounting for the ‘‘tails’’, we can conclude that

$$\sum_m (f_A * 1_{P_0})_+(m) \leq \delta L N + 2L^3 \leq 2\delta L N$$

and consequently, since  $\sum_m f_A * 1_{P_0}(m) = 0$  and  $|g| = 2g_+ - g$ , that

$$\sum_m |f_A * 1_{P_0}(m)| \leq 4\delta L N. \quad \square$$