Topics in Fourier analysis - Lecture 2.

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1 Infinite Fourier series.

In this section we develop the basic theory of Fourier series of periodic functions of one variable, but only to the extent that is needed for the main applications discussed below. These will include linear and non-linear partial differential equations of interest, such as the non-linear Schrodinger equation (GNLSE), and Ergodic theorems.

1.1 Basics of Forier series.

Let $f : \mathbf{R} \to \mathbf{C}$ be a continuous function, we say it is periodic of period T if $f(x+T) = f(x)$ for all x. If $I = [a, a+T]$ is an interval of length T and f is continuous on I such that $f(a) = f(a+T)$ then f has a uniquely extends to a periodic continuous function on $\mathbf{R}$. Also by a simple scaling: $\tilde{f}(x) = f(x\frac{T}{\pi})$, $\tilde{f}$ becomes a function of period π.

Now let $C(2\pi)$ denote the space of continuous functions of period π, identified with those of $f : [-\pi, \pi] :\to \mathbf{C}$ such that $f(-\pi) = f(\pi)$.

Of special importance are the complex exponentials:

$$e_n \in C(2\pi),\ \ e_n(x) = e^{inx} = \cos(nx) + i\sin(nx) \tag{1}$$

Similarly to the discrete case one has

$$(e_n, e_m) = \delta(n-m) = \begin{cases} 1 & \text{if } \ m = n \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where (f, g) denotes the inner product of the functions f and g

$$(f, g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx$$

Indeed if $n \neq m$ then

$$(e_n, e_m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e_n(x)\overline{e_m(x)}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x}\, dx = \frac{e^{i(n-m)x}}{n-m}|_{-\pi}^{\pi} = 0$$

Note that $i) - i3)$ of Proposition 1.3.1 is valid for the inner product (f, g) of functions in $C(2\pi)$, and so does the Cauchy-Schwarz inequality (Proposition 1.3.2), since it follows only from properties $i) - i3)$ of the inner product. Again analogously to the discrete case we define the L^2- norm by

Definition 1.1.1 *Let $f \in C(2\pi)$, then the L^2- norm of f is defined by*

$$\|f\|_2 = (f, f)^{1/2} = (\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx)^{1/2} \tag{3}$$

This norm satisfy $i) - ii)$ of Proposition 1.3.3, that is the triangle inequality and is homogeneous. The analogy with the discrete case is taken further to define

Definition 1.1.2 *Let $f \in C(2\pi)$, then the for $k \in \mathbf{Z}$ the k-th Fourier coefficient of f is given by*

$$\hat{f}(k) = (f, e_k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\, dx \tag{4}$$

and the function $\hat{f} : \mathbf{Z} \to \mathbf{C}$ is called the Fourier transform of f.

The usefulness of the Fourier transform again is because it has many nice algebraic properties (p.e. it diagonalizes differential operators, see below), and thus certain problems become "simpler in Fourier space", involving $\hat{f}, \hat{g}, \ldots$ instead of $f, g, \ldots$. But to use this one has to have a way of expressing a function in terms of its Fourier transform. To see how such a formula looks like we introduce the spaces of trigonometric polynomials

Definition 1.1.3 *A function $s(x) \in C(2\pi)$ of the form*

$$s(x) = \sum_{|k|\leq N} c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(kx) + b_k \sin(kx) \tag{5}$$

is called a trigonometric polynomial of degree (at most) N. The space of trigonometric polynomials of degree N is denoted by $\mathcal{M}_N$.

Note that by the above definition $\mathcal{M}_N$ is the linear subspace of $C(\pi)$ spanned by the exponentials $e_k(x)$ $-N \leq k \leq N$. These are orthogonal and hence are linearly independent, thus they form an ortho-normal basis (o.n.b) of $\mathcal{M}_N$.

Proposition 1.1.1 *Let* $f(x) = \mathcal{M}_N$*, then one has*

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx} \tag{6}$$

Proof. Let $f(x) == \sum_{|k|\leq N} c_k e^{ikx}$ then by orthoganility:

$$\hat{f}(l) = \sum_{|k|\leq N} c_k (e_k, e_l) = c_l$$

for $|l| \leq N$ and equal to 0 otherwise, and this is the content of the proposition. □

The left side of (41) is the so-called Fourier series of the function $f(x)$, and the question: to what extent formula (41) remains true for all continuous functions is of basic importance in harmonic analysis. To be more precise we define

Definition 1.1.4 *Let Let*$f \in C(2\pi)$*, its* N*-th partial Fourier series are defined by:*

$$S_N f(x) = \sum_{|k|\leq N} \hat{f}(k) e^{ikx} \tag{7}$$

It is known for a long time that there are functions $f \in C(2\pi)$ such that $S_N f(x)$ diverges at an infinite number of points x, however it is only relatively recent that the set of these exceptional points - where $S_N f(x) \to f(x)$ as $N \to \infty$ is NOT true - is of measure zero, i.e. it can be covered by a countably many intervals whose total length is arbitrary small. This was proved by L. Carleson in 1967, and its subsequent proofs (by C. Fefferman, Lacey-Thiele) had a major influence on the development of Fourier analysis.

We will discuss some (much easier) results in this direction, namely that $\|S_N f - f\|_2 \to 0$ as $N \to \infty$ and that $f(x)$ can be recovered by an easy way from the sequence $S_N f(x)$. Moreover assuming that $f(x)$ has a continuous derivative, the pointwise convergence: $S_N f(x) \to f(x)$, is true at every point.

We start by some simple consequences of orthogonality.

Proposition 1.1.2 *Let $f \in C(2\pi)$ and $s \in \mathcal{M}_N$. Then*

$$\|f - s\|_2 \leq \|f - S_N f\|_2 \tag{8}$$

Proof. Write $f - s = f - S_N f + S_N f - s$. Note that by definition

$$(S_N f, e_k) = \hat{f}(k) = (f, e_k)$$

for all $|k| \leq N$ and thus

$$(f - S_N f, e_k) = 0$$

for all $|k| \leq N$, which then remains true for any trigonometric polynomial in place of e_k. In particular

$$(f - S_N f, S_N f - s) = 0$$

and thus by the "Pythagorean theorem"

$$\|f - s\|_2^2 = \|f - S_N f\|_2^2 + \|S_N f - s\|_2^2$$

which proves the proposition. □.

The geometric meaning of the above is that $S_N f$ is the orthogonal projection of f to the linear subspace $\mathcal{M}_N$. As a corollary we get

Theorem 1.1.1 *(Bessel's inequality) One has for every N*

$$\|f\|_2^2 \geq \sum_{|k| \leq N} |\hat{f}(k)|^2 \tag{9}$$

Proof. Taking $s = 0$ in the above proposition, one gets

$$\|f\|_2^2 \geq \|S_N f\|_2^2 = (S_N f, S_N f) =$$

$$= \sum_{|k| \leq N,\ |l| \leq N} \hat{f}(k)\overline{\hat{f}(l)}\,(e_k, e_l) = \sum_{|k| \leq N} |\hat{f}(k)|^2$$

by orthogonality. □

Corollary 1.1.1 *(Riemann-Lebesgue lemma) Let $f \in C(2\pi)$, then $\hat{f}(k) \to 0$ as $|k| \to \infty$.*

Indeed this is immediate from the fact the the series: $\sum_{k\in\mathbf{Z}}|\hat{f}(k)|^2$ is convergent. Note that the series $\sum_{k\in\mathbf{Z}}|\hat{f}(k)|$ may very well be divergent, however this is not the case if make some "smoothness assumptions" on f.

Definition 1.1.5 *Let $f \in C(2\pi)$, we say f is n-times continuously differentiable if the n-th derivative $f^{(n)} \in C(2\pi)$ as well. The space of such functions will be denoted by $C^{(n)}(2\pi)$.*

The Fourier coefficients of functions in $C^{(n)}(2\pi)$ have characteristic decay, based on the following important property, which is also the starting point of the applicability of Fourier analysis to differential equations;

Lemma 1.1.1 *Let $f \in C^{(n)}(2\pi)$. Then one has*

$$\widehat{f^{(n)}}(k) = (ik)^n \hat{f}(k) \tag{10}$$

Proof. We proceed by induction. For $k = 1$ using integration by parts, one has

$$\widehat{f'}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-ikx}\,dx = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(e^{-ikx})'\,dx =$$

$$= ik\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx = ik\hat{f}(k)$$

then one applies the same argument to $f^{(k)}(x)$. □

Now, we discuss our first convergence result, due to Chernoff

Theorem 1.1.2 *Let $f \in C^{(1)}(2\pi)$, then there exists a function $S(x) \in C(2\pi)$ such that for every x The Fourier series*

$$S(x) = \sum_{k\in\mathbf{Z}} \hat{f}(k)e^{ikx} \tag{11}$$

is absolutely convergent.

Proof. Using that $|e^{ikx}| = 1$ to prove the uniform convergence of the series in (46) it is enough to show that

$$\sum_{k\in\mathbf{Z}} |\hat{f}(k)| < \infty$$

For $k \neq 0$: $|\hat{f}(k)| = |\hat{f}'(k)|/|k|$ and thus by Cauchy-Schwartz

$$\sum_{k\in\mathbf{Z}} |\hat{f}(k)| \leq |\hat{f}(0)| + \sum_{k\neq 0} |\hat{f}'(k)|^2 \sum_{k\neq 0} |k|^{-2} \sum_{<} \leq |\hat{f}(0)| + C\|f'\|_2^2$$

This means that he partial sums $S_N(x) = \sum_{k=N}^{N} \hat{f}(k)e^{ikx}$ uniformly converge to the function $S(x)$ which is then continuous (and has period 2π). $\square$

Note that if $|k| \leq N$ then: $\hat{S_N}(k) = \hat{f}(k)$ and thus by continuity: $\hat{S}(k) = \hat{f}(k)$ for all k's. Thus we expect that $S(x) = f(x)$ for all x. This would follow from the fact that the Fourier coefficients uniquely determine the function f. This follows from Fejer's theorem, which in fact shows much more.

Theorem 1.1.3 *Let $f \in C(2/pi)$. Define the averages of the partial sums S_N by:*

$$T_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} S_N(x) \tag{12}$$

Then $T_N(x) \to f(x)$ uniformly as $N \to \infty$.

The proof consists of two main parts, first to rewrite $T_N(x)$ as the convolution of f with a specific trigonometric polynomial $K_N(x)$ - called the Fejer-kernel - and then to use the properties of K_N to show the convergence.

First we review convolution in this context

Definition 1.1.6 *Let $f, g \in C(2\pi)$. Then their convolution is defined by*

$$f * g = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)g(t)\, dt \tag{13}$$

Similarly as in the discrete case it has the basic properties

Proposition 1.1.3 *Let $f, g, h \in C(2\pi)$. Then*

(i) $f * g \in C(\pi)$

(ii) $f * g = g * f$ *and* $f * (\alpha g + \beta h) = \alpha f * g + \beta f * h$

(i3) $f \hat{*} g(k) = \hat{f}(k)\hat{g}(k)$

Proof. (i) follows from the uniform continuity of f. We remark that the integral of a function in $C(\pi)$ over any interval of length 2π is the same. Then a substitution $s = x - t$ proves (ii). One shows (i3) the same way as in the discrete case. $\square$.

One can rewrite the partial sums $S_N(x)$ and their averages $T_N(x)$ as follows

$$S_N(x) = \sum_{k=-N}^{N} \hat{f}(k)e^{ikx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-N}^{N} e^{ik(x-t)} f(t)\, dt$$

in other words

$$S_N(x) = f * D_N(x) \quad \text{where} \quad D_N(t) = \sum_{k=-N}^{N} e^{ikt} \tag{14}$$

is the so-called Dirichlet kernel. It can be explicitly evaluated

$$(e^{it/2} - e^{-it/2})D_N(t) = e^{i(N+\frac{1}{2})t} - e^{-i(N+\frac{1}{2})t}$$

thus

$$D_N(t) = \frac{\sin\left((N+\frac{1}{2})t\right)}{sin\,\frac{t}{2}} \tag{15}$$

The Fejer kernels can be evaluated in a similar way:

$$K_N(t) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(t) = \frac{1}{(N+1)\sin(t/2)} Im\left(\sum_{n=0}^{N} e^{i(n+1/2)t}\right) =$$

$$= \frac{1}{(N+1)\sin(t/2)}\, Im\left(e^{i(N+1)t/2}\, \frac{e^{i(N+1)t/2} - e^{-i(N+1)t/2}}{e^{it/2} - e^{-it/2}} \right)$$

where Im denotes the imaginary part. Thus

$$K_N(t) = \frac{1}{N+1}\frac{\sin^2(\frac{N+1}{2}t)}{\sin^2(\frac{t}{2})} \tag{16}$$

if $t \neq 0$ and - it is easy to see that - $K_N(0) = N+1$.

The family of functions $K_N(t)$ has some properties of basic importance, which are summarized below

Definition 1.1.7 *A family of functions $K_N(t) \in C(2\pi)$ is called an approximation of unity or δ- sequence, if it has*

(i) $K_N(t) \geq 0$

(ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)\,dt = 1$ *for all* $N = 0, 1, 2, \ldots$

(i3) For every $\delta > 0$ the integrals:

$$I(\delta) = \int_{\delta \leq |t| \leq \pi} |K_N(t)|\,dt \to 0$$

converge to 0 as $N \to \infty$.

Proposition 1.1.4 *The Fejer kernels $K_N(t)$ form an approximation of unity.*

Proof. (i) is clear from (51). Also (ii) is true for the Dirichlet kernels $D_n(t)$, (why?) and then for their averages $K_N(t)$. To show (i3) notice that when $\delta \leq |t| \leq \pi$ then $|\sin(t/2)| \geq |\sin(\delta/2)| > 0$ and thus:

$$|K_N(t)| \leq \frac{2}{(N+1)\sin^2(\delta/2)}$$

uniformly, which tends to 0 as N tends to infinity. □

The second part of Fejer's theorem is a based on the following

Theorem 1.1.4 *Let $f \in C(2\pi)$ and K_N be an approximation of unity. Then the convolutions*

$$f * K_N \to f \tag{17}$$

as $N \to \infty$.

Proof. The idea is that, for large N, the functions $K_N(t)$ are roughly 0 for $|t| > \delta$. However for $|t| \leq \delta$ the function f is roughly constant by continuity, hence

$$f * K_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\,dt \approx \frac{1}{2\pi}\int_{|t|\leq\delta} f(x-t)K_N(t)\,dt$$

$$\approx f(x)\frac{1}{2\pi}\int_{|t|\leq\delta} K_N(t)\,dt \approx f(x)$$

More precisely, let $\epsilon > 0$ be fixed, and let $\delta > 0$ be chosen such that $|f(x) - f(x-t)| \leq \epsilon$ holds uniformly in x when $|t| \leq \delta$. Let $M = \sup_x |f(x)|$. Then by (ii)

$$|f * K_N(x) - f(x)| = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (f(x-t) - f(x))\, K_N(t)\, dt \right|$$

$$\leq \int_{|t| \leq \delta} |f(x-t) - f(x)| K_N(t)\, dt + \int_{\delta \leq |t| \leq \pi} |f(x-t) - f(x)| K_N(t)\, dt$$

The first integral in the above expression is estimated by:

$$I_1 \leq \epsilon \int_{|t| \leq \delta} K_N(t)\, dt \leq \epsilon$$

Now we choose N_0 such that for $N > N_0$ one has $I(\delta) = \int_{\delta \leq |t| \leq \pi} K_N(t)\, dt \leq \epsilon$ according to (i3). Then the second integral is estimated by

$$I_2 \leq 2M \int_{\delta \leq |t| \leq \pi} K_N(t)\, dt \leq 2M\epsilon$$

Thus we have for $N > N_0$:

$$|f(x) - f * K_N(x)| \leq (2M + 1)\epsilon$$

uniformly in x and this proves the Theorem. □