Topics in Fourier analysis - 4

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1 The Fourier transform on $\mathbf{R}^n$

We assume familiarity with basic notions of Lebesgue integration. As for the notation, $\mathbf{R}^n$ denotes the n dimensional Euclidean space, endowed with the norm: $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$ where $x = (x_1, \ldots, x_n)$ and the inner product: $x \cdot y = x_1 y_1 + \ldots + x_n y_n$. The space of continuous (complex valued) functions are denoted by $\mathcal{C}(\mathbf{R}^n)$ (or by $\mathcal{C}$) while if moreover $\lim_{|x|\to\infty} f(x) = 0$ then we say $f \in \mathcal{C}_0(\mathbf{R}^n)$.

The space of Lebesgue integrable functions is $L^1(\mathbf{R}^n)$ (or by L^1), which means that:

$$\|f\|_1 = \int_{\mathbf{R}^n} |f(x)|\, dx < \infty \tag{1}$$

Note that $L^1(\mathbf{R}^n)$ with the norm $\|\ \|_1$ is a Banach space. One way to think about Lebesgue integrable functions is to consider them as all possible limits in the L^1 norm of continuous functions with compact support.

Definition 1 *Let $f \in L^1$, and let $\xi \in \mathbf{R}^n$. Then the Fourier transform of f at ξ is defined by*

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-ix\cdot\xi}\, dx \tag{2}$$

Theorem 1 *Let $f \in L^1$, then $\hat{f} \in \mathcal{C}_0$ that is $\hat{f}$ is uniformly continuous function vanishing at infinity, moreover*

$$\sup_\xi |\hat{f}(\xi)| = \|\hat{f}\|_\infty \leq \|f\|_1 \tag{3}$$

The fact that $\hat{f}(x) \to 0$ as $|x| \to \infty$ is called the Riemann-Lebesgue lemma. However not all functions in $\mathcal{C}_0$ appear as Fourier transforms of L^1 functions, in fact there is simple characterization known of those. Another fact we'll need is proved using the same approximation argument

Proposition 1 *Let $f \in L^1$, then*

$$\int_{\mathbf{R}^n} |f(x-y) - f(x)|\, dx \to 0 \quad as \quad y \to 0$$

Proof. This is immediate if f is the characteristic function of a box, and hence if it is simple. For general $f \in L^1$ and $\varepsilon > 0$, write: $f = g + h$ where g is simple and $\|h\|_1 \leq \varepsilon$. Then

$$\int_{\mathbf{R}^n} |f(x-y) - f(x)|\, dx \leq_{\mathbf{R}^n} |g(x-y) - g(x)|\, dx + 2\varepsilon \leq 3\varepsilon$$

if $|y|$ is small enough. $\square$

Proof. Notice that (3) is obvious. If $x_n \to x$ then $\hat{f}(x_n) \to \hat{f}(x)$ follows from the fact that the integral in (2) is absolutely and uniformly convergent (or alternatively from the Lebesgue dominated convergence theorem). Finally notice that by a simple computation: $\hat{f} \in \mathcal{C}_0$ if f is the characteristic function of a box, thus also if it is a finite linear combination of such functions i.e. for a simple function. The result for general $f \in L^1$ follows easily by approximating f in the L^1 norm by: $f = g + (f - g)$ where g is simple and $\hat{f} - \hat{g}$ is uniformly small. $\square$

As before the Fourier transform has nice algebraic properties.

Proposition 2

$$(i) \quad \widehat{\tau_h f}(\xi) = e^{-h \cdot \xi} \hat{f}(\xi) \quad where \quad \tau_h f(x) = f(x-h)$$

$$(ii) \quad \widehat{e^{ix \cdot h} f}(x)(\xi) = (\tau_h \hat{f})(\xi)$$

Next our main objective will be to establish the Fourier inversion formula

$$f(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} \hat{f}(\xi) e^{ix \cdot \xi}\, d\xi \tag{4}$$

in the appropriate sense for functions $f(x) \in L^1$. There are some difficulties as first, the right side of (4) may not be absolute convergent, and also it is not clear how the orthogonality of the exponentials: $e_\xi(x) = e^{ix \cot \xi}$ should make sense. The idea is to use appropriate weight functions in the integrals involved, also called Abel summation. First we recall

Proposition 3 *Let $\xi, \eta \in \mathbf{R}^n$ and let $\varepsilon > 0$. Then*

$$\int_{\mathbf{R}^n} e^{ix\cdot\xi} e^{-ix\cdot\eta} e^{-\varepsilon|x|^2}\, dx = \pi^{n/2} \varepsilon^{-n/2} e^{-\frac{|\xi-\eta|^2}{4\varepsilon}} \tag{5}$$

Proof. This follows immediately from formula (12) for the heat kernel, by noticing that (5) is the n-th power of the one-dimensional integral in (12), when $t = \varepsilon$, $x = \xi - \eta$ and $y = x$. $\square$

Proposition 4 *The family of functions: $K_\varepsilon(y) = 2^{-n}\pi^{n/2}\varepsilon^{-n/2}e^{-\frac{|y|^2}{4\varepsilon}}$ is an approximation of unity, that is one has*

$$(I)\quad K_\varepsilon(y) > 0, \quad \int_{\mathbf{R}^n} K_\varepsilon(y)\, dy = 1 \quad \forall \varepsilon > 0$$

$$(II)\quad \forall \delta > 0 \quad \int_{|y|\geq} K_\varepsilon(y)\, dy \to 0 \quad \text{as} \quad \varepsilon \to 0$$

Proof. Let $K(y) = K_1(y)$ and note that $K_\varepsilon(y) = \varepsilon^{-n/2} K(y/\sqrt{\varepsilon})$ and hence

$$\int_{\mathbf{R}^n} K_\varepsilon(y)\, dy = \int_{\mathbf{R}^n} K(y)\, dy = \left(\frac{1}{2\sqrt{\pi}} \int_{\mathbf{R}} e^{-\frac{y^2}{4}\, dy} \right)^n = 1$$

By scaling again: $y = y/\sqrt{\varepsilon}$ one has

$$\int_{|y|\geq\delta} K_\varepsilon(y)\, dy = \int_{|y|\geq\delta/\sqrt{\varepsilon}} K(y)\, dy$$

Since as $\varepsilon \to 0$ one has $\delta/\sqrt{\varepsilon} \to \infty$ the integral tend to 0 by the Lebesgue monotone convergence theorem, or alternatively it is easy to estimate it directly using the exponential decay. $\square$

We recall the convolution of L^1 functions and some immediate facts about it

Proposition 5 *Let $f, g \in L^1(\mathbf{R}^n)$ and let K_ε be as defined above. Let $f * g(x) = \int_{\mathbf{R}^n} f(x-y)g(y)\, dy$ be the convolution of the functions f and g. Then*

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1 \tag{6}$$

$$\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi) \tag{7}$$

$$\lim_{\varepsilon\to 0} \|f - f * K_\varepsilon\|_1 \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{8}$$

Proof. It is enough to show (6) when $f \geq 0$ and $g \geq 0$. The by Fubini's theorem

$$\int \int f(x-y)g(y)\,dy\,dx = \int g(y) \int f(x-y)\,dx\,dy = \|f\|_1 \|g\|_1$$

The same way

$$\int \int f(x-y)g(y)e^{-x\cdot\xi}\,dy\,dx = \int g(y)e^{-y\cdot\xi} \int f(x-y)e^{-(x-y)\cdot\xi}\,dx\,dy$$

$$= \hat{f}(\xi)\hat{g}(\xi)$$

To show (8) we proceed as in Fejer's theorem. Let $\alpha > 0$ and using $\int_{\mathbf{R}^n} K_\varepsilon(y)\,dy = 1$ one has for $\delta > 0$

$$\|f - f * K_\varepsilon\|_1 \leq \int_x \int_{|y|\leq\delta} |f(x) - f(x-y)| K_\varepsilon(y)\,dy\,dx + \tag{9}$$

$$\int_{|y|>\delta} \int_x |f(x) + |f(x-y)|\, K_\varepsilon(y)\,dx\,dy = I_1 + I_1$$

Note that

$$I_2 = 2\|f\|_1 \int_{|y|>\delta} K_\varepsilon(y)\,dy \leq \alpha$$

if δ is small enough by (II). To estimate I_1 one uses Proposition 1.

$$I_1 \leq \int_y \left(\int_x |f(x) - f(x-y)|\,dx \right) K_\varepsilon(y)\,dy \leq \alpha$$

This proves (8) $\square$

We can prove now the Fourier inversion, which will have two formulations, the first uses the notion of "smooth partial sums"

Theorem 2 *(Fourier inversion)*

(i) Let $f \in L^1$ *and* $\varepsilon > 0$ *and define:*

$$S_\varepsilon f(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} e^{-\varepsilon|\xi|^2} \hat{f}(\xi)\,d\xi \tag{10}$$

Then $\|f - S_\varepsilon f\|_1 \to 0$ *as* $\varepsilon \to 0$.

(ii) Assume that both $f \in L^1$ *and* $\hat{f} \in L^1$. *Then for almost every* $x \in \mathbf{R}^n$ *one has*

$$f(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\,d\xi \tag{11}$$

Proof. By plugging (2) into (10) and changing the order of integration

$$S_\varepsilon f(x) = (2\pi)^{-n} \int \int e^{i(x-y)\cdot\xi} e^{-\varepsilon|\xi|^2} f(y)\, d\xi\, dy =$$

$$\int f(y) K_\varepsilon(x-y)\, dy = f * K_\varepsilon(x)$$

and (i) follows from (8). Next, if $\hat{f} \in L^1$ then by the Lebesgue dominated convergence theorem it is clear that for every sequence $\varepsilon_k \to 0$ one has $S_{\varepsilon_k} f(x)$ converges to the function defined in the right side of (11) at every x. However since $S_\varepsilon f(x)$ converges to $f(x)$ in L^1 norm there is a sequence $\varepsilon_k \to 0$ such that $S_{\varepsilon_k} f(x)$ converges to $f(x)$ pointwise. This proves (ii). $\square$

Note that the function defined in (11) is continuous, thus in case f is also continuous we have equality at all points, otherwise this can be achieved by changing f on a set of measure 0. We also proved uniqueness of the Fourier transform

Corollary 1 *If f and g belong to $L^1\mathbf{R}^n$ and $\hat{f}(\xi) = \hat{g}(\xi)$ for $\xi \in \mathbf{R}^n$ then $f(x) = g(x)$ for almost every $x \in \mathbf{R}^n$.*

Proof. Indeed if $h = f - g$ then for every $\varepsilon > 0$ then one has $S_\varepsilon(h) = S_\varepsilon(f) - S_\varepsilon(g) = 0$ hence $h = 0$ $\square$

The Fourier transform becomes a unitary operator when considered on $L^2(\mathbf{R}^n)$ that is on the space of square integrable functions. This is usually called Plancherel's formula and infact is an easy corollary of Theorem 2. We need the following

Proposition 6 *Let $f \in L^2$ and define $\tilde{f}(x) = \bar{f}(-x)$ where $\bar{z}$ denotes the complex conjugate. Then the function: $h(x) = f * \tilde{f}(x)$ is well-defined and continuous.*

Proof. By Cauchy- Schwartz

$$|h(x)|^2 \le \int |f(x-y)|^2\, dy \int |f(-y)|^2\, dy = \|f\|_2^4 < \infty$$

Note that this also shows: $h(0) = \|f\|_2^2$. Continuity follows from that fact that

$$\int |f(x-h) - f(x)|^2\, dx \to 0 \quad \text{as} \quad h \to 0$$

which is proved the same way as Prop. 1. $\square$

Theorem 3 *(Plancherel)*

(i) Let $f \in L^1 \cap L^2$ *then:* $(2\pi)^{-n}\|\hat{f}\|_2 = \|f\|_2$

(ii) The Fourier transform: $\mathcal{F}: \; f \to (2\pi)^{-n/2}\hat{f}$ *defined for* $f \in L^1 \cap L^2$ *extends to unitary operator* $\mathcal{F}: \; L^2 \to L^2$.

Proof. Note for the function $h(x)$ defined in Prop. 6., one has: $h(0) = \|f\|_2^2$ and $\hat{h}(\xi) = \hat{f}(\xi)\overline{\hat{f}}(xi) = |\hat{f}(\xi)|^2$ thus (i) is equivalent to

$$h(0) = (2\pi)^{-n} \int_{\mathbf{R}^n} \hat{h}(\xi)\, d\xi$$

which is true since $h(x)$ is continuous hence the inversion formula hold at every point.

Thus The mapping $\mathcal{F}$ is an isometry defined on the dense subspace $L^1 \cap L^2$ of L^2 and the extends uniquely to L^2. Finally we must show it is onto. Assume indirect that there is a function $g \neq 0 \in L^2$ such that: $(\hat{f}, g) = 0$ for every $f \in L^1 \cap L^2$. Here we used the fact the orthogonal complement of a proper subspace of L^2 contains a nonzero element g, which is true for every Hilbert space. Then

$$\int \hat{f}(\xi)\bar{g}(xi)\, d\xi = \int \int e^{-x\cdot\xi} f(x)\bar{g}(\xi)\, d\xi\, dx = \int f(x)\bar{\hat{g}}(x)\, dx$$

or in short: $(\hat{f}, g) = (f, \hat{g})$. Thus $\hat{g} = 0$ and hence $g = 0$ which contradicts our assumption. $\square$