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ON FOURIER RESTRICTION AND THE NEWTON POLYGON.

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Abstract. Local $L^p \to L^2$ bounds are proved for the restriction of the Fourier transform to analytic surfaces of the form $S = (x, f(x))$ in $\mathbb{R}^3$. It is found that the range of exponents are determined by the so-called distance of the Newton polygon, associated to f, except when the principal quasi-homogeneous part of $f(x)$ contains a factor of high multiplicity. The proofs are based on the method of Phong-Stein and Rychkov, adapted to scalar oscillatory integrals.

1. Introduction.

If $S \subset \mathbb{R}^{n+1}$ is a manifold a central question of harmonic analysis is to determine the pair of exponents $p, q \geq 1$ for which the Fourier restriction property holds on S, denoted by $R_S(p \to q)$, that is when one has a bound

$$\left(\int_S |\hat{\phi}|^q \, d\sigma \right)^{1/q} \leq C_{p,q,s} \|\phi\|_{L^p(\mathbb{R}^{n+1})} \tag{1.1}$$

where $d\sigma$ is a measure compactly supported on S. This problem is still open even case of the unit sphere $S^2 \subset \mathbb{R}^3$, however the case $q = 2$ for surfaces of everywhere non-vanishing curvature was answered by the classic works of Stein and Thomas, see [S], [T].

The aim of this note is bring into attention the relation between the range of exponents p, q for which $R_S(p \to q)$ holds and a numeric invariant, the so-called distance of the associated Newton polygon.

To be more precise, first note that the restriction property is local by nature, and is invariant under affine maps $x \to Ax+b$ (where $det\,A \neq 0$). Thus in principle it is enough to study the local problem: $S = (x, f(x))$ where $f : \mathbb{R}^n \to \mathbb{R}$ is a (germ of an) analytic function, such that $f(0) = \nabla f(0) = 0$. If $f(x) = \sum_{k \in \mathbf{Z}^n_+} a_k x^k$ is the Taylor expansion of $f(x)$ then its Newton polyhedron is defined by

$$N_f = Conv \left(\bigcup_{k \in A} k + \mathbb{R}^n_+ \right)$$

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where $A = \{k : \ a_k \neq 0\}$ is the support of the series. The smallest positive number $d = d_f$ such that the point $\mathbf{d} = (d, \ldots, d)$ is in N_f is called the distance of N_f, and its reciprocal $\delta_f = 1/d_f$ the Newton decay rate. The if the point $\mathbf{d_f} = (\mathbf{d_f}, \ldots, \mathbf{d_f})$ lies on a compact edge α, then α is called the principal edge of N_f.

First we give the following necessary condition

Theorem 1. *Let $S = (x, f(x))$ where $f : \mathbb{R}^n \to \mathbb{R}$ is an analytic function, such that $f(0) = \nabla f(0) = 0$ and let $d\sigma = \psi\, dx$ be a measure on S in local coordinates x where ψ is a smooth function of small support.*

If the local $R_S(p \to q)$ restriction property holds at (0,0) then one has

$$p' \geq q(1 + d_f) \quad \textit{where} \quad \frac{1}{p} + \frac{1}{p'} = 1 \tag{1.2}$$

and d_f is the distance of the Newton polyhedron N_f.

The idea of the proof of (1.2) is doing the well-known Knapp example, in the general case of an analytic hypersurface. In other words to test the restriction property (1.1) on functions ϕ_δ whose Fourier transform $\hat{\phi}_\delta$ is supported on a box B_δ best fitted to the surface S.

However the Knapp example is not always sharp (a fact hard to find in the literature), a simple example is the graph of the function: $f(x, y) = (y - x^2)^m$ where $m > 3$. Thus one needs additional conditions on $f(x, y)$. One such condition is the so-called $\mathbb{R}-$ *nondegeneracy* of Varchenko, see [AV]. However as we shall see that may be too restrictive especially when the surface has curvatures vanishing to a high order, that is when d_f is large. Our partial result of sufficiency is taking this into consideration.

Let $n = 2$ and let α be a compact edge of the boundary of the Newton polygon N_f. Define the quasi-homogeneous part of f by corresponding to α by

$$f_\alpha(x, y) = x^{-A'} y^{-B} \sum_{(k,l) \in \alpha} a_{kl}\, x^k y^l \tag{1.3}$$

if α is connecting the points (A, B) and (A', B') where $A' < A$ and $B < B'$. We define a number r_α, called the multiplicity of f_α.

Definition 1. *If α is a compact edge, then the multiplicity r_α of f_α is defined to be the minimum of the highest multiplicity of a real root of $\partial_y f_\alpha\,(1, y)$ and of the highest multiplicity of a real root of $\partial_x f_\alpha\,(x, 1)$. If α is an infinite edge, then we set $r_\alpha = 0$.*

If α is the principal edge of N_f, then f_α is called the principal part of f and we write r_f for r_α. By the above definition if the point $\mathbf{d_f}$ is on an infinite edge, then $r_f = 0$.

Theorem 2. *Let $S = (x, y, f(x, y))$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is an analytic function, such that $f(0, 0) = \nabla f(0, 0) = 0$. Then the local $R_S(p \to 2)$ restriction property holds at (0,0) for*

$$p' > 2(1 + \max(d_f, r_f + 1)) \tag{1.4}$$

Note that in case of $r_f < d_f$, (1.2) and (1.4) gives the sharp range of exponents p (up to the endpoint) for which the local $R_S(p \to 2)$ restriction property holds. This includes functions with factors of high multiplicity, which are $\mathbb{R}-$*degenerate* in the sense of Varchenko (which means that the principal part of f has distinct real factors). For example take: $f(x,y)=(y-a\,x^2)^m(y-b\,x^2)^m$ $(a\neq b)$ for any $m\geq 2$.

The proof of (1.4) is based on an oscillatory integral estimate

Lemma 1. *Let the function $f(x,y)$ be an analytic function such that $f(0,0)=\nabla f(0,0)=0$. Then for a smooth cut-off function $\psi(x,y)$ of sufficiently small support, one has the bound for the associated oscillatory integral*

$$|I(\lambda)|=|\int_{\mathbb{R}^2}e^{i\lambda f(x,y)}\psi(x,y)\,dx\,dy|\leq C_\varepsilon(1+|\lambda|)^{-\min(\delta_f,\frac{1}{r_f+1})+\varepsilon}\quad\forall\,\varepsilon>0 \tag{1.5}$$

This compares to the result of Varchenko in dimension 2, which shows: $|I(\lambda)|\leq C_\varepsilon(1+\lambda)^{-\delta_f+\varepsilon}$ under the so-called $\mathbb{R}-$*nondegeneracy* condition. The proof of (1.5) exploits a factorization, called the Puiseux product, similarly as was done for oscillatory integral operators in [PS1] and [R].

The route from from (1.5) to (1.4) is standard, but based on a highly nontrivial result of Karpoushkin, see [K], [PS3]; namely the local stability of the decay rate of oscillatory integrals with analytic phase in 2 dimensions. In fact it implies that for $\xi=\lambda(-1,u,v)$ the estimate

$$|\hat{d\sigma}(\xi)|=|\int_{\mathbb{R}^2}e^{i\lambda(f(x,y)-ux-vy)}\psi(x,y)\,dx\,dy|\leq C_\varepsilon(1+|\lambda|)^{-\min(\delta_f,\frac{1}{r_f+1})+\varepsilon}$$

for all $\varepsilon>0$ holds uniformly for (u,v) being in a small neighborhood of $(0,0)$, and then also for all (u,v), since outside the neighborhood the gradient of the phase: $|(\partial_x f-u,\partial_y f-v)|\geq C\,\lambda$ thus the integral is rapidly and uniformly decreasing.

Finally, by the classical argument of Thomas [T], a uniform bound of the form $|\hat{d\sigma}(\xi)|\preceq(1+|\xi|)^{-\beta}$ implies that the local $R_S(p\to 2)$ restriction property holds for $p'>2(1+\beta^{-1})$ (in fact by Greenleaf's theorem [G] the restriction property also holds for $p'\geq 2(1+\beta^{-1})$), which gives (1.4). Thus Theorem 2 follows from Lemma 1.

2. Necessary conditions.

In this section we prove Theorem 1. Let $f:\mathbb{R}^n\to\mathbb{R}$ is an analytic function, such that $f(0,0)=\nabla f(0,0)=0$, and let Γ_f be the boundary and V_f be the set of vertices of its Newton polyhedron. We use the notation $|x|^a=|x_1|^{a_1}\cdots|x_n|^{a_n}$ for $x\in\mathbb{R}^n$ and $a=(a_1,\dots,a_n)\in\mathbf{Z}^n$.

Proposition 1. *One has for x being in a sufficiently small neighborhood of the origin:*

$$|f(x)|\leq C\max_{b\in V_f}|x|^b \tag{2.1}$$

Proof. Let us introduce the ordering on $\mathbf{Z}^n_+$: $a\prec b$ if $b\in a+\mathbf{Z}^n_+$, $b\neq a$, and set of minimal elements: $A'=\{a\in A:\,\nexists\,b\in A,\ b\prec a\}$. It is easy to see by an induction on n that A' is finite, thus V_f is finite as well. Since the Taylor series of $f(x)$ converges one has for every $a\in A'$

$$\sum_{k\in A,\ a\preceq k}|a_k|\,|x|^k\leq C_a|x|^a$$

it follows that

$$|f(x)| \leq C \max_{a\in A'} |x|^a \tag{2.2}$$

Moreover for every $a \in A'$ there are vertices $b_1, \ldots, b_n$ (not necessarily distinct) such that for a convex for a combination of them: $b = \sum_i \delta_i b_i \preceq a$. Indeed every point on the boundary Γ_f is of this form. Thus

$$|x|^a \leq \prod_i |x|^{\delta_i b_i} \leq \max_i |x|^{b_i}$$

The Proposition follows from (2.2). □

Definition 2. *We say that* $a \in \mathbb{R}^n_+$ *is admissible if*

$$k \cdot a \geq 1 \quad \textit{for every} \quad k \in \Gamma_f \tag{2.3}$$

Proposition 2. *Let*

$$\delta = \min\{ \sum_{j=1}^n a_j \ : \ a = (a_1, \ldots, a_n) \ \ \textit{is admissible} \ \}$$

Then $\delta = \delta_f = 1/d_f$.

Proof. If a is admissible then $a \cdot \mathbf{d_f} = d_f \sum_j a_j \geq 1$ hence $\delta \geq \delta_f$. On the other hand, let α be the face of N_f containing $\mathbf{d}_f$ and let $a \in \mathbb{R}^n$ be defined such that $a \cdot k = 1$ for every $k \in \alpha$. Then by the convexity of N_f the plane through α separates 0 and N_f, thus $a \cdot l \geq 1$ for every $l \in N_f$. In particular $a \in \mathbb{R}^n_+$ and a is admissible. Also $a \cdot \mathbf{d_f} = d_f \sum_j a_j = 1$ and thus $\delta \leq \delta_f$. □

Proof of Theorem 1. Let $\psi(x)$ be smooth cut-off function of small support and let $a = (a_1, \ldots, a_n)$ be admissible. For $0 \leq \tau < 1$ define the function $\phi_{a,\tau}$ such that

$$\hat{\phi}_{a,\tau}(x_1, \ldots, x_{n+1}) = \psi(\tau^{-a_1}x_1, \ldots, \tau^{-a_n}x_n, \tau^{-1}x_{n+1})$$

If $|x_j| \leq \tau^{a_j}$ for $1 \leq j \leq n$ then since a is admissible, one has by (2.1)

$$|f(x)| \leq C \max_{k\in V_f} \tau^{k\cdot a} \leq C\tau$$

and thus (choosing ψ appropriately), $|\hat{\phi}_{a,\tau}(x, f(x))| \geq c > 0$. Hence

$$\int_S |\hat{\phi}_{a,\tau}|^q \, d\sigma = \int_{\mathbb{R}^n} |\hat{\phi}_{a,\tau}(x, f(x))|^q \, dx \geq c\, \tau^{\delta_a} \tag{2.4}$$

where $\delta_a = \sum_j a_j$. On the other hand, by scaling, one has that $\|\phi_{a,\tau}\|_p \approx \tau^{(1+\delta_a)/p'}$.

If the $R_S(p \to q)$ restriction property holds, then one must have

$$\tau^{\delta_a/q} \leq C\, \tau^{(1+\delta_a)/p'}$$

for every $0 < \tau < 1$ and admissible a. Thus by Proposition 2

$$p' \geq \max_{a \ \textit{admissible}} q(1 + \delta_a^{-1}) = q(1 + d_f)$$

This proves Theorem 1. □

3. Sufficient conditions.

We will need the following Van der Corput type lemma for oscillatory integrals with polynomial type phases, proved in [PH], see Theorem. 1 there, and the remarks after it.

Lemma 2. *Let $I = [a, b]$ be an interval of length at most 1, $\psi(y) \in \mathcal{C}^1$ be a smooth function and let $f(y)$ be a real and $\mathcal{C}^2$. Assume that $|f'(y)| \geq \delta > 0$ for all $y \in I$, and $f''(y)$ has at most d roots in I. Then one has for $\lambda > 0$*

$$| \int_I e^{i\lambda f(y)} \psi(y)\, dy| \leq \lambda^{-1} \delta^{-1} (2d + 1)(\sup_I |\psi| + \sup_I |\psi'|) \tag{3.1}$$

We remark that inequality (3.1) remains true, with $d = 0$, when $f''(y)$ is constant 0 on the interval I, which is easy to see by performing an integration by parts, and using that $f'(y)$ is constant on I.

To apply the above lemma for a function of two variables $F(x, y)$, we invoke an immediate corollary of the Weierstrass Preparation Theorem.

Proposition 3. *Let $F(x, y)$ be a nonzero real and analytic function Then there is an $\eta > 0$ and a positive integer d, such that for every $0 < |x| < \eta$, the function $y \rightarrow F(x, y)$ has at most d roots in $[-\eta, \eta]$. Moreover, the same is true by interchanging the role of x and y.*

Proof. Let $K = min\{k : \ a_{kl} \neq 0$ where $F(x, y) = \sum_{k,l} a_{kl} x^k y^l$ and write $F(x, y) = x^K G(x, y)$. The function $G(0, y)$ is not identically zero, thus by the Weierstrass preparation theorem, see [H] Sec. 7.5:

$$G(x, y) = U(x, y)(c_d y^d + c_{d-1}(x) y^{d-1} + \ldots + c_0(x))$$

where $c_d \neq 0$, and $U(x, y) \neq 0$ for $|x| \leq \eta$, $|y| \leq \eta$, $\eta > 0$ being sufficiently small. Thus for any $|x| \leq \eta$, the function $y \rightarrow F(x, y)$ can have at most d roots. ☐

We describe the Newton polygon N_F associated to an analytic function $F(x, y)$ in more detail. If α runs through the compact edges of the boundary Γ_F, connecting the vertices (A_α, B_α) and (A'_α, B'_α) where $A_\alpha > A'_\alpha$ then put $n_\alpha = B'_\alpha - B_\alpha$ and $\gamma_\alpha = \frac{A_\alpha - A'_\alpha}{B'_\alpha - B_\alpha}$. Also let A be the x and B be the y coordinate of the vertical and horizontal infinite edges of Γ_F. The intersection of the bisector and line of the edge α is the point (d_α, d_α) where $d_\alpha = (A_\alpha + \gamma_\alpha B_\alpha)/(1 + \gamma_\alpha)$ By convexity $d_f \geq d_\alpha$ for all compact edges α, thus writing $\delta_\alpha = 1/d_\alpha = (1 + \gamma_\alpha)/(A_\alpha + \gamma_\alpha B_\alpha)$, it follows

$$\delta_f = \min_\alpha \ (1/A, 1/B, \ \delta_\alpha) \tag{3.2}$$

We introduce the ordering $\alpha \prec \beta$ if $\gamma_\alpha < \gamma_\beta$ and note that

$$A_\alpha = A + \sum_{\beta \preceq \alpha} n_\beta \gamma_\beta \quad \text{and} \quad B_\alpha = B + \sum_{\beta \succ \alpha} n_\beta \tag{3.3}$$

Also, N_F has some symmetry properties, if $G(x, y) = F(\pm x, \pm y)$ then $N_F = N_G$, if $G(x, y) = F(y, x)$ then N_G is obtained by reflecting N_F to the bisector $y = x$. The germ of an analytic function $F(x, y)$ admits a factorization called the Puiseux product, see [R], of the form

$$F(x,y) = U(x,y)x^A y^B \prod_{\alpha} \prod_{i=1}^{n_\alpha} (y - y_{\alpha i}(x)) \tag{3.4}$$

where $U(x,y)$ and $c_{\alpha i}$ are nonzero, and $y_{\alpha i}(x))$ is asymptotic to a fractional power series of the form: $c_{\alpha i}x^{\gamma_\alpha} + c_{\alpha' i}x^{\gamma_\alpha+\gamma} \dots$, as $x \to 0$ in particular for any given $\tau > 0$ one has

$$|y_{\alpha i}(x) - c_{\alpha i}x^{\gamma_\alpha}| \leq \tau x^{\gamma_\alpha} \tag{3.5}$$

for $x > 0$ small enough w.r.t τ. Moreover there is a fixed large constant $D > 0$ such that if $2^{-j-1} \leq x < 2^{-j}$ and $2^{-k-1} \leq y < 2^{-k}$ for $j, k \geq J$ large enough, then

$$|y - y_{\alpha i}(x)| \sim \begin{cases} 2^{-j\gamma_\alpha} & \text{if} \quad k \geq j\gamma_\alpha + D \\ 2^{-k} & \text{if} \quad k \leq j\gamma_\alpha - D \end{cases} \tag{3.6}$$

where $x \sim y$ means that $C^{-1}x \leq y \leq Cx$ for some constant $C > 0$whose value is unimportant. By the notation $k \gg j\gamma_\alpha$ it is meant that $k - j\gamma_\alpha \geq D$. We remark that both conditions can be achieved by choosing the support of $\psi(x,y)$ small enough.

One may use the Puiseux product to estimate the size of $F(x,y)$ on the dyadic rectangles $R_{jk} = [2^{-j-1}, 2^{-j}] \times [2^{-k-1}, 2^{-k}]$, which are "away" from the zero set of F.

Proposition 4. *Let $F(x,y)$ be an analytic function, such that $F(0,0) = 0$. Then one has the following size estimates*

(i) Let $\alpha \prec \alpha'$ be two consecutive compact edges of the Newton polygon N_F, and let (A_α, B_α) denote the common vertex of the edges α and α'. If $j\gamma_\alpha \ll k \ll \gamma_{\alpha'}$, then for $(x,y) \in R_{jk}$

$$|F(x,y)| \sim 2^{-jA_\alpha - kB_\alpha} \tag{3.7}$$

(ii) Let α_0 denote the minimal edge of N_F (i.e. $\gamma_{\alpha_0} \leq \gamma_\alpha \ \forall \ \alpha$), and let (A_0, B_0) denote the common vertex of the edge α_0 and the infinite vertical edge. If $k \ll j\gamma_{\alpha_0}$, then for $(x,y) \in R_{jk}$

$$|F(x,y)| \sim 2^{-jA_0 - kB_0} \tag{3.8}$$

(i3) Let α_s denote the minimal edge of N_F (i.e. $\gamma_{\alpha_s} \geq \gamma_\alpha \ \forall \ \alpha$), and let (A_s, B_s) denote the common vertex of the edges α_s and the infinite horizontal edge. If $k \gg j\gamma_{\alpha_s}$, then for $(x,y) \in R_{jk}$

$$|F(x,y)| \sim 2^{-jA_s - kB_s} \tag{3.9}$$

Proof. Assume first $j\gamma_\alpha \ll k \ll \gamma_{\alpha'}$. The by (3.6) for $(x,y) \in R_{jk}$ one has $|y - y_{\beta i}(x)| \sim 2^{-j\gamma_\beta}$ for $\beta \preceq \alpha$, and $|y - y_{\beta i}(x)| \sim 2^{-k}$ for $be \succ \alpha$. Thus by (3.3) and (3.4)

$$|F(x,y)| \sim 2^{-j(A + \sum_{\beta \preceq \alpha} n_\beta \gamma_\beta)}\, 2^{-k(B + \sum_{\beta \succ \alpha} n_\beta)} = 2^{-jA_\alpha - kB_\alpha}$$

The proof of estimates (3.7) and (3.8) proceeds the same way, noting that $A_0 = A$ and $B_s = B$. $\square$

In the proof of Lemma 1, we will apply the above size estimates to either $F = \partial_x f$ or $F = \partial_y f$ thus we will need some information on the Newton polygons associated to $\partial_x f$ and $\partial_y f$, which we will denote by N_x and N_y. Note that N_x is obtained from N_f by shifting it to the left by 1, and may be replacing the minimal edge α_0 by and edge α_0', a series of edges $\alpha_0' \succ \ldots \succ \alpha_{-r}'$, or an infinite edge. These replacements can only happen when the infinite vertical edge of N_f is the $y - axis$. An important observation here is that, by convexity, $\gamma_{\alpha_0'} \geq \gamma_{\alpha_0}$. The analysis of N_y is similar, here

N_f is shifted down by 1, and the maximal edge α_s. may be replaced by edges $\alpha'_s \prec \ldots \prec \alpha'_t$ such that $\gamma_{\alpha_s} \leq \gamma_{\alpha'_s}$, or the horizontal infinite edge.

Proof of Lemma 1.

Following [R], one decomposes the support of the integral in (1.5) into the four quadrants of $\mathbb{R}^2$ and note that each of the resulting integrals are treated the exactly the same way because the Newton polygon is invariant w.r.t. coordinate changes: $x \leftrightarrow -x$, $y \leftrightarrow -y$. Thus we assume that the integration is taking place for $x > 0$, $y > 0$ and define

$$I_{jk}(\lambda) = \int_{R_{jk}} e^{i\lambda f}\psi \tag{3.10}$$

Since only those rectangles R_{jk} intersecting the support of ψ contribute to the integral, one may assume $j, k \geq J$ for a sufficiently large J.

We start with the (rather) special case when N_f has no compact edges. Then the Puiseux product takes the form

$$f(x,y) = U(x,y)x^A y^B$$

where $A > 0$ or $B > 0$ by our assumption on f. Assume $B > 0$ and let $F = \partial_y f$. For $(x,y) \in R_{jk}$ one has

$$|F(x,y)| \sim x^A y^{B-1} \sim 2^{-jA-k(B-1)} \tag{3.11}$$

Thus applying Lemma 2. for fixed $2^{-j-1} \leq x \leq 2^{-j}$, and then integrating trivially in x, one obtains

$$|I_{jk}(\lambda)| \leq C\lambda^{-1} 2^{j(A-1)+k(B-1)} \tag{3.12}$$

Taking the geometric mean of this and the size estimate $|I_{jk}| \leq 2^{-j-k}$ gives

$$|I_{jk}(\lambda)| \leq C\lambda^{-\delta}\; 2^{j(\delta A-1)+k(\delta B-1)} \tag{3.13}$$

for any $0 \leq \delta \leq 1$. Choosing $\delta = \delta_f - \varepsilon$ where $\delta_f = \min(\frac{1}{A}, \frac{1}{B})$, and $\varepsilon > 0$ one has

$$|I(\lambda)| \leq \sum_{j,k \geq J} |I_{jk}(\lambda)| \leq C_\varepsilon \lambda^{-\delta_f + \varepsilon}$$

for any $\varepsilon > 0$. The case $A > 0$ is handled analogously, choosing $F = \partial_x f$.

Now assume that N_f has at least one compact edge, and for a fixed compact edge α define

$$I_\alpha(\lambda) = \sum_{j\gamma_\alpha \ll k \ll j\gamma_{\alpha'}} I_{jk}(\lambda) \quad \text{and} \quad I^\alpha(\lambda) = \sum_{|j\gamma_\alpha - k| < D} I_{jk}(\lambda) \tag{3.14}$$

where $\alpha \prec \alpha'$ are consecutive edges, moreover let

$$I_-(\lambda) = \sum_{k \ll j\gamma_{\alpha_0}} I_{jk}(\lambda) \quad \text{and} \quad I^+(\lambda) = \sum_{k \gg j\gamma_{\alpha_s}} I_{jk}(\lambda) \tag{3.15}$$

Next, we estimate the quantities $I^{\pm}(\lambda)$. Let α_0 be the minimal edge of N_f connecting the points (A_0, B_0) and (A', B') with $B_0 > B'$. Then $(A_0, B_0 - 1)$ is the highest vertex point of the Newton polygon N_y. We argue that $\gamma_{\alpha_0} \leq \gamma_{\alpha'_0}$, where α'_0 denotes the minimal edge of N_y. Indeed, if $B' > 0$

then α_0' is parallel to α_0 hence $\gamma_{\alpha_0} \leq \gamma_{\alpha_0'}$, otherwise N_f has only one compact edge hence $\alpha_0 = \alpha_s$ and the estimate follows from our previous analysis of N_y. Thus $k \ll j\gamma_{\alpha_0} \leq j\gamma_{\alpha_0'}$ and (3.7) applies for $F(x,y) = \partial_y f(x,y)$ giving

$$|F(x,y)| \sim 2^{-jA_0 - k(B_0-1)} \tag{3.16}$$

similarly as in (3.10). Proceeding as in (3.11)-(3.13) one obtains for $0 \leq \delta \leq 1$

$$|I_{jk}(\lambda)| \leq C\lambda^{-\delta}\; 2^{j(\delta A_0 - 1) + k(\delta B_0 - 1)} = 2^{k(\delta B_0 + A_0/\gamma_{\alpha_0}) - (1 + 1/\gamma_{\alpha_0})}\; 2^{r(\delta A_0 - 1)} \tag{3.17}$$

where the last equality was obtained by writing $j = k/\gamma_{\alpha_0} + r$. By our assumption $r \geq D' > 0$, $j \geq J$, thus choosing $\delta = \min(\delta_{\alpha_0}, 1/A_0) - \varepsilon \geq \delta_f - \varepsilon$ one gets

$$|I^-(\lambda)| \leq \sum_{j \geq J,\ r \geq D'} |I_{jk}(\lambda)| \leq C_\varepsilon \lambda^{-\delta_f + \varepsilon} \tag{3.18}$$

The quantity $I^+(\lambda)$ ie estimated analogously, by choosing $F = \partial_x f$ and $\delta = \min(\delta_{\alpha_s}, 1/B_s) - \varepsilon$.

To estimate $I_\alpha(\lambda)$ let $\alpha \prec \alpha'$ be consecutive edges of N_f with common vertex (A_α, B_α) and let $j\gamma_\alpha \ll k \ll j\gamma_{\alpha'}$. First assume that $A_\alpha \geq B_\alpha > 0$ and note that $d_f \geq B_\alpha$ by the convexity of N_f. The point $(A_\alpha, B_\alpha - 1)$ is the common vertex of two edges $\beta \prec \beta'$ of the Newton polygon N_y (where β' may be infinite), and by our previous analysis $\gamma_\beta = \gamma_\alpha$ and $\gamma_{\alpha'} \leq \gamma_{\beta'}$ (taking $\gamma_{\beta'} = +\infty$ if β' is infinite). Thus $j\gamma_\beta \ll k \ll j\gamma_{\beta'}$, and estimate (3.7) (or (3.9)) applies to $F(x,y) = \partial_y f(x,y)$ when $(x,y) \in R_{jk}$, which gives $|F(x,y)| \sim 2^{-jA_\alpha - k(B_\alpha - 1)}$. Proceeding as before and substituting $k = j\gamma_\alpha + r$, one has

$$|I_{jk}(\lambda)| \leq C\,\lambda^{-\delta}\; 2^{j(\delta(A_\alpha + \gamma_\alpha B_\alpha) - (1 + \gamma_\alpha))}\; 2^{r(\delta B_\alpha - 1)} \tag{3.19}$$

Choosing $\delta = \min(\delta_\alpha, 1/B_\alpha) - \varepsilon \geq \delta_f - \varepsilon$ and summing in $j \geq J$, $r \geq D$ one gets

$$|I_\alpha(\lambda)| \leq C_\varepsilon \lambda^{-\delta_f + \varepsilon} \tag{3.20}$$

The case $B_\alpha \geq A_\alpha$ is handled analogously, choosing $F = \partial_x f$, $\delta = \min(\delta_{\alpha'}, 1/A_\alpha) - \varepsilon \geq \delta_f - \varepsilon$, and substituting $j = k/\gamma_{\alpha'} + r$.

The multiplicity condition enters in the estimates for $I^\alpha(\lambda)$, that is when $|k - j\gamma_\alpha| \leq D$. Consider first a non-principal edge α of N_f. Then α lies completely below or above the bisector $y = x$. Assume the first case, that is when $A'_\alpha > B'_\alpha > 0$. Then the point $(A'_\alpha, B'_\alpha - 1)$ is a vertex of N_y. Let α' be the edge of N_y with upper vertex $(A'_\alpha, B'_\alpha - 1)$, then $\gamma_{\alpha'} \geq \gamma_\alpha$. If $\gamma_{\alpha'} > \gamma_\alpha$, then $k \ll j\gamma_{\alpha'}$, and our analysis reduces to the cases ($I_\alpha(\lambda)$ or $I^-(\lambda)$) considered earlier. In fact, it is easier here as one needs to sum only for $j \geq J$, using the fact that $k - j\gamma_\alpha$ is bounded. One obtains

$$|I^\alpha(\lambda)| \leq C \sum_{j \geq J} |I_{jk}(\lambda)| \leq C_\varepsilon \lambda^{-\delta_\alpha + \varepsilon} \tag{3.21}$$

So we will assume that $\gamma_{\alpha'} = \gamma_\alpha$ from now on.

Let $F(x,y) = \partial_y f(x,y)$ and consider its Puiseux product defined in (3.4). It is clear that the product

$$F_{\alpha'}(x,y) := x^{A'_{\alpha'}} y^{B_{\alpha'}} \prod_{i=1}^{n_{\alpha'}} (y - c_{\alpha' i}\, x^{\gamma_\alpha}) \tag{3.22}$$

is a quasi-homogenous polynomial whose support lies on the edge α'. It is well-known, and is not hard to see, that $F_{\alpha'}(x,y)$ is in fact the quasi-homogeneous part of $F(x,y)$ corresponding to α' defined in (1.3). Moreover, since $\gamma_{\alpha'} = \gamma_\alpha$, a point (k,l) is on α' if and only if the point $(k,l+1)$ is on α and hence: $F_{\alpha'} = (\partial_y f)_{\alpha'} = \partial_y(f_\alpha)$.

Let $\{c_1,\dots,c_t\}$ be the set of the real coefficients $c_{\alpha' i}$, and write

$$\partial_y\, f_\alpha\,(x,y) = F_{\alpha'}(x,y) = x^{A'_{\alpha'}} y^{B_{\alpha'}} \prod_{l=1}^{t}(y - c_l x^{\gamma_\alpha})^{r_l} \prod_{i:\, c_{\alpha' i}\ \text{complex}} (y - c_{\alpha' i}\, x^{\gamma_\alpha}) \tag{3.23}$$

and let $r_\alpha = \max_l\, r_l$. By the above remarks it is easy to see that r_α is the highest multiplicity of a real root of $\partial_y\, f_\alpha\,(1,y)$.

For each $1 \le l \le t$ and for each x, define the clusters: $\mathcal{C}_{x,l} = \{Re\, y_{\alpha i}(x) :\ c_{\alpha i} = c_l\}$, where $Re\, z$ denotes the real part of z. Observe that by (3.5) the diameter of each cluster is at most $\tau 2^{-j\gamma_\alpha}$, while the distance between them is at least $C\,2^{-j\gamma_\alpha}$, where $\tau > 0$ can be chosen sufficiently small.

For a fixed x we make use of the Whitney decomposition of the set: $[2^{-k-1}, 2^{-k}]/\mathcal{C}_x$ where $\mathcal{C}_x = \{Re\, y_{\alpha i}(x) :\ c_{\alpha' i} \text{ is real}\}$. It is a collection of intervals I_m of length $\sim 2^{-m}$ such that the distance of a point $y \in I_m$ from the set C_x is again $\sim 2^{-m}$. Clearly, one can assume $m \ge j\gamma_\alpha$, and for a given m there are at most $2n_{\alpha'}$ of intervals I_m. If $\mathcal{C}_l$ is the closest cluster to I_m then for $y \in I_m$ one has

$$|y - y_{\alpha' i}(x)| \ge \begin{cases} C\,2^{-m} & \text{if } \ c_{\alpha' i} \text{ is real, and } y_{\alpha' i}(x) \in\ \mathcal{C}_l \\ C\,2^{-j\gamma_\alpha} & \text{otherwise} \end{cases} \tag{3.24}$$

Note that the intervals I_m may depend on x the above estimates are uniform in $x \in [2^{-j-1}, 2^{-j}]$.

Thus one estimates the size of $F(x,y)$ for $y \in I_m$

$$\begin{aligned} |F(x,y)| &\ge C\,2^{-(jA+kB)} \prod_{\beta \prec \alpha'} 2^{-jn_\beta\gamma_\beta} \prod_{\beta \succ \alpha'} 2^{-kn_\beta}\, 2^{-mr_l} 2^{-(n_{\alpha'}-r_l)j\gamma_\alpha} \\ &\ge C\,2^{-j(A'_\alpha + \gamma_\alpha(B'_\alpha - 1))} 2^{r_\alpha(m - j\gamma_\alpha)} \end{aligned} \tag{3.25}$$

using the facts that $2^{-k} \sim 2^{-j\gamma_\alpha}$, $r_\alpha \ge r_l$, $m \ge j\gamma_\alpha$ and that $(A'_\alpha, B'_\alpha - 1)$ is the upper vertex of α'. Next, one uses by Lemma 2, which yields

$$|I_{jk,m}(x,\lambda)| := |\int_{I_m} e^{i\lambda f(x,y)} \psi(x,y)\, dy\,| \le C\,\lambda^{-1} 2^{jA'_\alpha + k(B'_\alpha - 1)} 2^{r_\alpha(m - j\gamma_\alpha)} \tag{3.26}$$

uniformly for $x \in [2^{-j-1}, 2^{-j}]$.

As in the previous case, this can be balanced against the trivial estimate $|I_{jk,m}| \le C\,2^{-m}$ to obtain that for any $\varepsilon > 0$

$$|I_{jk,m}(x,\lambda)| \le C\,\lambda^{-\frac{1}{r_\alpha + 1} + \varepsilon} 2^{-m\varepsilon} 2^{j\frac{A'_\alpha + \gamma_\alpha(B'_\alpha - 1) - r_\alpha\gamma_\alpha}{r_\alpha + 1}}\ 2^{-\varepsilon j(A'_\alpha + \gamma_\alpha(B'_\alpha - 1) - r_\alpha\gamma_\alpha)} \tag{3.27}$$

Summing in m and integrating trivially in x, gives

$$|I_{jk}(\lambda)| \leq C_\varepsilon\, \lambda^{-\frac{1}{r_\alpha+1}+\varepsilon}\; 2^{j(\frac{A'_\alpha+\gamma_\alpha(B'_\alpha-1)-r_\alpha\gamma_\alpha}{r_\alpha+1}-1)}\; 2^{-\varepsilon j(A'_\alpha+\gamma_\alpha(B'_\alpha-1)-r_\alpha\gamma_\alpha)} \tag{3.28}$$

Notice that the exponent: $\frac{A'_\alpha+\gamma_\alpha(B'_\alpha-1)-r_\alpha\gamma_\alpha}{r_\alpha+1} - 1$ is non-negative if and only if

$$r_\alpha + 1 \leq (A'_\alpha + \gamma_\alpha B'_\alpha)/(1+\gamma_\alpha) = d_\alpha \tag{3.29}$$

In this case, choosing $\delta = (r_\alpha+1)\delta_\alpha - 2\varepsilon$ to balance against the trivial estimate $|I_{jk}(\lambda)| \leq C\, 2^{-j(1+\gamma_\alpha)}$ to kill the j factors, a straightforward computation shows

$$|I^\alpha(\lambda)| \leq \sum_{|k-j\gamma_\alpha|\leq D} |I_{jk}(\lambda)| \leq C_\varepsilon \sum_{|k-j\gamma_\alpha|\leq D} 2^{-j\varepsilon}\lambda^{-\delta_\alpha+\varepsilon} \leq C_\varepsilon\, \lambda^{-\delta_\alpha+\varepsilon} \tag{3.30}$$

We argue that the multiplicity condition (3.29) holds when α is non-principal edge lying under bisector $y = x$. Indeed, then $r_\alpha + 1 \leq n_{\alpha'} + 1 \leq B'_{\alpha'} + 1 = B'_\alpha < d_\alpha$.

If α is a non-principal edge lying above the bisector, then one has to repeat the above analysis by reversing the role of x and y. More formally, let $\tilde{f}(x,y) := f(y,x)$. Note that $I_{jk}(\lambda, f) = I_{kj}(\lambda, \tilde{f})$ and $N_{\tilde{f}}$ is obtained by reflecting N_f to the bisector. If $\tilde{\alpha}$ is the edge of $N_{\tilde{f}}$ obtained by reflecting the edge α to the bisector, then $\delta_{\tilde{\alpha}} = \delta_\alpha$, $\gamma_\alpha = 1/\gamma_{\tilde{\alpha}}$ and estimate (3.30) applies to the quantity: $I^{\tilde{\alpha}}(\lambda) := \sum_{|j-k\gamma_{\tilde{\alpha}}|\leq\tilde{D}} I_{kj}(\lambda, \tilde{f})$. This gives (choosing $\tilde{D}$ large enough)

$$|I^\alpha(\lambda)| \leq |I^{\tilde{\alpha}}(\lambda)| \leq C_\varepsilon \lambda^{-\delta_{\tilde{\alpha}}+\varepsilon} = C_\varepsilon \lambda^{-\delta_\alpha+\varepsilon}$$

Finally, if α is the principal edge of N_f then estimate (3.29) holds assuming the multiplicity condition (3.29). Otherwise the exponent $\frac{A'_\alpha+\gamma_\alpha(B'_\alpha-1)-r_\alpha\gamma_\alpha}{r_\alpha+1} - 1$ is negative, and choosing $\varepsilon > 0$ small enough one can sum estimate (3.28) in j (also trivially in k) to obtain

$$|I^\alpha(\lambda)| \leq C_\varepsilon\, \lambda^{-\frac{1}{r_\alpha+1}+\varepsilon} \tag{3.31}$$

In this case one can do the analysis both for $f(x,y)$ and $\tilde{f}(x,y)$, in the latter case $r_{\tilde{\alpha}}$ will be the highest multiplicity of a real root of the function $\partial_x f_\alpha(x,1)$. Thus estimate (3.31) holds for r_α defined to be the minimum of the highest multiplicity of a non-zero real root of $\partial_y f_\alpha(1,y)$ and the highest multiplicity of a non-zero real root of $\partial_x f_\alpha(x,1)$. This finishes the proof of Lemma 1.

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