Homework Assignment # 2
Due Tuesday, March 4

1. Prove the following *chain rule* for rational functions on $\mathbf{P}^1$: For all $f, g \in \mathbf{C}(z)$ and all $Q \in \mathbf{P}^1(\mathbf{C})$, we have
$$e_Q(fg) = e_{g(Q)}(f)e_Q(g).$$
Deduce in particular that if $\alpha, \beta \in \mathbf{C}(z)$ are Mobius transformations, then
$$e_Q(f \circ \alpha) = e_{\alpha(Q)}(f)$$
and
$$e_Q(\beta \circ f) = e_Q(f).$$
2. Let $f \in \mathbf{Z}[X_1, \ldots, X_n]$ be a nonzero polynomial, and let $\alpha_1, \ldots, \alpha_n \in \overline{\mathbf{Q}}$. Show that
$$h(f(\alpha_1, \ldots, \alpha_n)) \leq \log |f|_{L^1} + \sum_{j=1}^{n} (\deg_{X_j} f) h(\alpha_j),$$
where $|f|_{L^1}$ denotes the sum of the absolute values of the coefficients of f. Deduce in particular that $h(\alpha_1 + \ldots + \alpha_n) \leq h(\alpha_1) + \cdots + h(\alpha_n) + \log n$.
3. Suppose the nonzero polynomials $A(t), B(t), C(t) \in \mathbf{C}[t]$ are relatively prime, and that $h(A, B, C) = r(A, B, C) - 1$. Prove that the rational function
$$f(t) = \frac{A(t)}{C(t)}$$
is unramified outside $\{0, 1\infty\}$ (i.e., that f is unramified above P for all $P \in \mathbf{P}^1(\mathbf{C}) - \{0, 1, \infty\}$).
4. If n is a nonnegative integer, prove that there exists a unique symmetric normalized Belyi function f of degree $3n+1$. (See the ABC Conjecture, Part II notes for the definition of a symmetric normalized Belyi function. You may use the results from class concerning the polynomials $A_n(t), B_n(t), C_n(t)$).
5. Let K be a number field, let S be a finite set of places of K including all the archimedean places. Let $\Sigma_{K,S}$ be the set of $\lambda \in K$ such that E_λ has potential good reduction outside S, where E_λ is the elliptic curve $y^2 = x(x-1)(x-\lambda)$. Prove that $\Sigma_{S,K}$ is finite. [**Hint:** You may use the following result from Silverman's "Arithmetic of

Elliptic Curves" (Section VII, Proposition 5.5): E_λ has potential good reduction at a place v iff its j-invariant

$$j_\lambda = \frac{2^8(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$$

is integral at v.]

6. (Extra credit) Let K be a number field, and let $R_S \subset K$ be the ring of S-integers for some finite set S. Let $n \geq 2$ be an integer, and let $f \in K[x]$ be a polynomial of degree at least 3 with distinct roots in $\overline{K}$. Assuming the result stated in class about the finiteness of solutions to the S-unit equation, prove that there are only finitely many $(x, y) \in R_S \times R_S$ such that $y^n = f(x)$. (**Hint:** Find appropriate "Siegel identities" using the nth roots of unity instead of $\{\pm 1\}$.)