Homework Assignment # 1

Due Thursday, August 29

1. Prove that the following rings are not UFD's by explicitly finding two distinct factorizations of the same element.
   a. $\mathbf{Z}[\sqrt{-13}]$
   b. $\mathbf{Z}[\sqrt{-26}]$
   c. $\mathbf{Z}[\sqrt{10}]$ (Hint: Factor 6 in two different ways)
2. Prove that the following rings are Euclidean domains (and hence UFD's).
   a. $\mathbf{Z}[\sqrt{-2}]$ (Hint: For $x, y \in \mathbf{Z}[\sqrt{-2}]$ with $y \neq 0$, write $x/y = a + b\sqrt{-2}$ with $a, b \in \mathbf{Q}$, and choose $q = c + d\sqrt{-2} \in \mathbf{Z}[\sqrt{-2}]$ so that $|c-a| \leq 1/2, |d-b| \leq 1/2$)
   b. $\mathbf{Z}[\sqrt{2}]$ (Hint: Use the norm $\phi(a + b\sqrt{2}) = |a^2 - 2b^2|$)
   c. $\mathbf{Z}[\omega]$, where ω is a primitive third root of unity.
3. Find all integers x, y such that $x^3 - y^2 = 2$.
4. a. Determine all units of $\mathbf{Z}[\omega]$, where ω is a primitive third root of unity.
   b. Let $p \geq 5$ be a prime number, and let ζ_p be a primitive pth root of unity. Prove that $1 + \zeta_p$ is a unit in $\mathbf{Z}[\zeta_p]$ which is *not* a root of unity.